Algorithmic Game Theory
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Lecture - 43
Properties of Social Choice Function

Welcome. So, in the last lecture we have studied a very important principle called revelation principle which in some sense makes the job of mechanism designer very easy. If mechanism designer is given a social choice function and it is asked to design a mechanism to implement the social choice function it has no job it can simply use a direct mechanism and if the direct mechanism implements it in dominant strategy incentive compatible or domain strategy equilibrium, then the in then we got the solution.

But if more importantly if the direct mechanism does not implement the social choice function in dominant strategy equilibrium then there cannot exist an indirect mechanism which can implement it in dominant strategy equilibrium.

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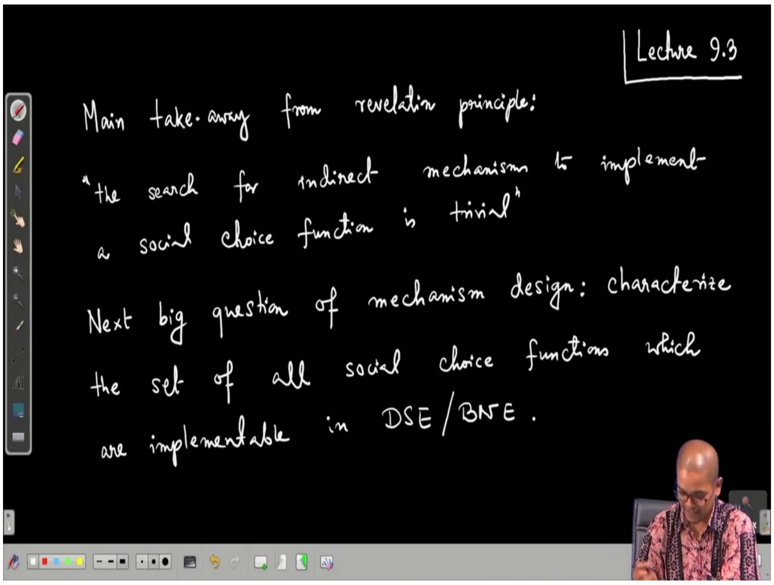

So, let me write. So, the message or the main take away from main take away from revelation principle. The search of the search for indirect mechanisms to implement a social choice function is let me write trivial. We do not have to search you just need to check whether the direct mechanism implements the social choice function in whatever kind of equilibrium you want. So, what is the next big question of mechanism design?

So, the next big question of mechanism design. You see that the revelation principle makes checking whether a given social choice function is implementable or not very easy you just need to check the direct mechanism. So, the next big question that we ask is that; characterize the set of all social choice functions which are implementable in either in say dominant strategy equilibrium or Bayesian Nash equilibrium.

Characteristics means in some sense find can you succinctly describe the set of all social choice functions which are implementable in dominant strategy incentive compatible dominant strategy equilibrium or Bayesian Nash equilibrium and that is sort of our next driving force for this lecture and next.

So, what we will do? The plan is that we will describe some properties of social choice functions we will see that most of them are desirable and in terms of those properties we will characterize the social choice functions that are implementable in say dominant strategy equilibrium or Bayesian Nash equilibrium.

On a high level may be the theorem may look like social choice function is implementable on dominant strategy equilibrium if and only if it has it satisfies these properties and so on.

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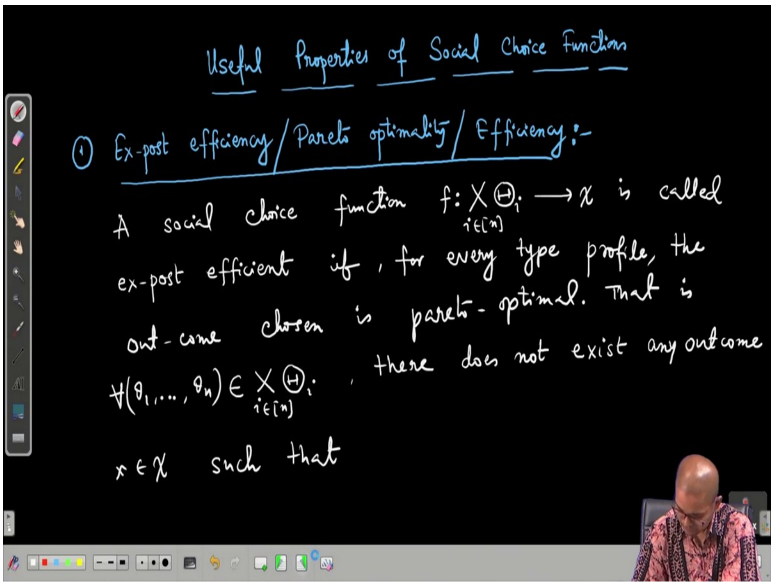

So, next we look for some useful properties of social choice functions. Let us see our first property is ex post efficiency it is also called pareto optimality or simply efficiency. What is it? Let us see a social choice function is called ex post efficient if. Let me write it intuitively in plain English first and then we will write it mathematically. If the if for every type profile the outcome chosen is pareto optimal.

What do you mean by that? So, let us take a type profile say $(\theta_1,\ldots,\theta_n)$ and the social choice function chooses the outcome $f(\theta_1,\ldots,\theta_n)$ to be the outcome at that particular type profile and it is pareto optimal. That means, there is no other outcome which makes everyone if all the n players at least as happy as $f(\theta_1,\ldots,\theta_n)$ and the at least 1 player who is strictly happier.

So, that is for all $(\theta_1,\ldots,\theta_n)$ all type profile and for all outcome $x\in X$ do not for all for all type profile there does not exist there does not exist any outcome $x\in X$.

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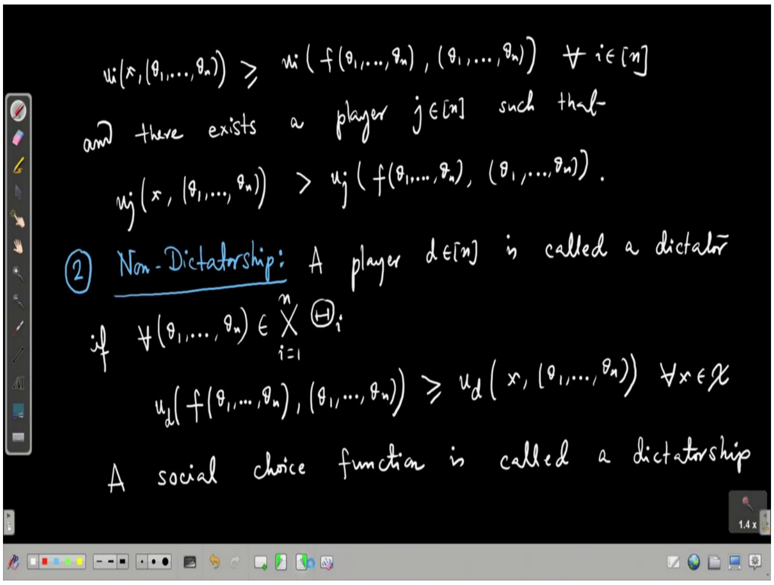

Such that $u_i(x)$ utility of player i which depends on the outcome and the type profile $(\theta_1,\ldots,\theta_n)$. This is greater than equal to $u_i(f(\theta_1,\ldots,\theta_n))$ and the type profile is $(\theta_1,\ldots,\theta_n)$. For all player it cannot happen. So, for all $i\in[n]$ and there exists an outcome there exists a player $j\in[n]$ such that player j utility from x when the type profile is $(\theta_1,\ldots,\theta_n)$ this is strictly more than player j is strictly happier in the outcome x then from theta $f(\theta_1, \ldots, \theta_n)$ when the type profile are $(\theta_1, \ldots, \theta_n)$. So, such an x should not exist.

So, see if this is the case then the f the social choice function is called the pareto optimal social choice function or the social choice function is called efficient or pareto efficient these are the terms used interchangeably. Our next property is non dictatorship. Let us see second property is non dictatorship. So, to define non dictatorship what we will do is that we will define dictatorship and a social choice function which is not dictatorship will be called which is which is not dictatorship will be called a non dictatorship social choice function.

So, a player to define a dictatorship we need to first define a dictator a player d is called a dictator if for all type profile $(\theta_1, \ldots, \theta_n)$. A player d is called a detector if for all type profile the utility of player i in the outcome chosen by social choice function f which is $f(\theta_1, \ldots, \theta_n)$ when the type profile is $(\theta_1, \ldots, \theta_n)$ this is greater than equal to this is the best possible outcome from user from player d's point of view.

So, this is not $u_d(x, \theta_1, \ldots, \theta_n)$. See if such a player exists the such that whose at every type profile the outcome chosen by the social choice function is the best among all possible outcomes. So, this is for all $x \in X$. Then such a player is called a dictator. A social choice function is called a dictatorship if there exist a dictator.

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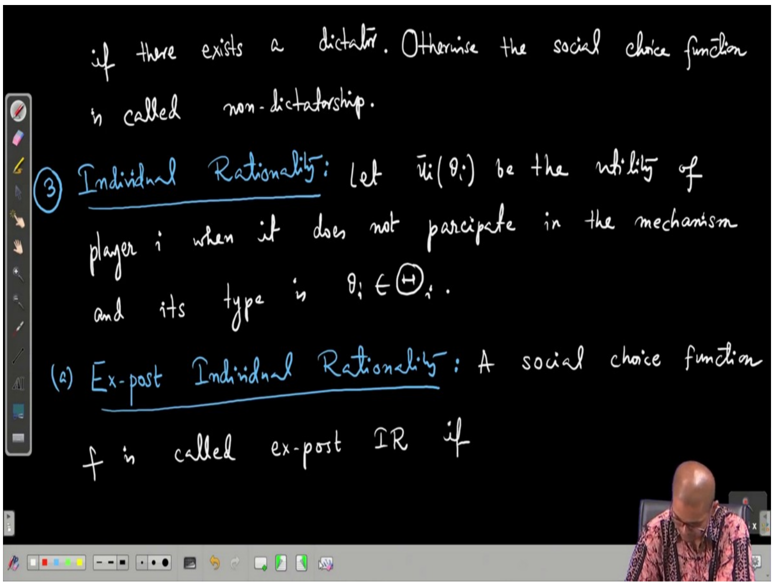

Otherwise the social choice function is called non directorship. Our third and last important property is what is called individual rationality. So, the property of individual rationality is very fundamental it basically justifies why players at all will participate in these social choice function business. So, the underlying assumption is that if player do not participate then player have a certain kind of utility.

So,let $\bar{u}_i(\theta_i)$ be the utility of player i when it does not participate in the mechanism and its type is $\theta_i \in \Theta_i$. Now we have three kinds of individual rationality. The first one is called ex post individual rationality and this is the most common type of individual rationality. If we say only individual rationality and does not use any adjective then by default it means ex post individual rationality.

So, what is it? So, what is ex post individual rationality? Ex post individual rationality. So, it simply says that for all type profile it is always in the best sort of interest of all the players to participate in the mechanism are then refraining from it and getting a utility of $\bar{u}_i(\theta_i)$. So, a social choice function f is called ex post individually rational IR.

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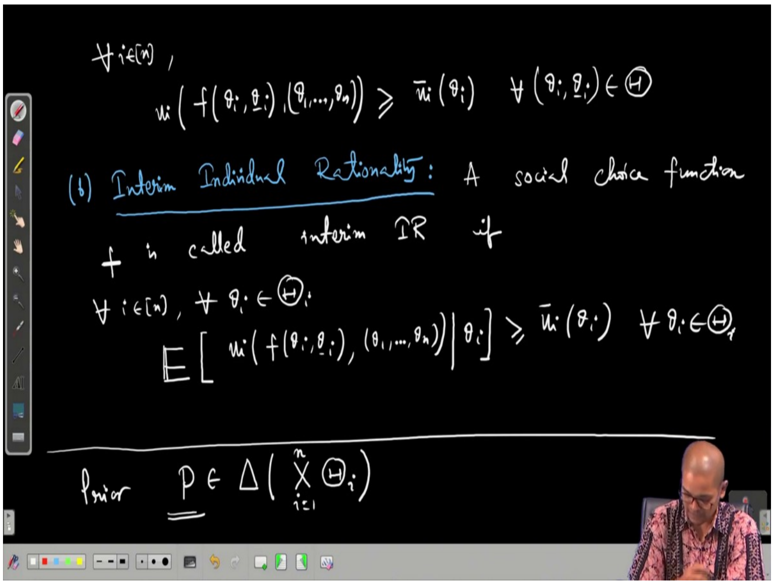

If for all player $i \in [n]$ utility of player i when it participates (θ_i, θ_{-i}) when it participates and all players are playing according to say $(\theta_1, \ldots, \theta_n)$. So, this is the utility that player i gets this should be greater than equal to $\bar{u}_i(\theta_i)$ this should hold for all $(\theta_i, \theta_{-i}) \in \Theta$ ok. So, this is called ex post individual rationality.

The second one is called interim individual rationality ok. So, social choice function is called function f is called interim individual rational if for all player $i \in [n]$ and for all type $\theta_i \in \Theta_i$. Now u_i you know there is some you know probability prior is coming here. You see that whenever we talk about mechanism there are many things other than this social choice function and players.

For example there are players each player has a strategy there is a prior distribution recall in the definition of mechanism there are so many things in the tuple and we said that we will not we will drop it because most of the cases it is immediate from the context. But now we need to bring it up specially the prior distribution. If you recall in the definition of individual the definition of mechanism there is a notion of prior probability distribution P.

There was a notion of prior P which is a probability distribution over type profiles and P is a common knowledge. Ex post individual rationality says that players even after knowing the type of each players $(\theta_1, \ldots, \theta_n)$, but suppose player i has come to know the type of each player $(\theta_1, \ldots, \theta_n)$ then also it participates in the mechanism voluntarily or saying in this in the same thing in a different manner. Suppose the mechanism is over and it was say dominant strategy incentive compatible mechanism.

And. So, after the mechanism all the true types $(\theta_1, \ldots, \theta_n)$ are revealed all players are know that then if the mechanism is ex post efficient then no player regrets participating in the mechanism irrespective of what is the type profile. Interim individual rationality is at the intermediate level where player i knows its type θ_i, but it does not know the type profile of other players.

So, it can only talk about or use the expected utility of expected utility over θ_{-i} given its type is θ_i. So, expectation and where is the underlying probability distribution that is the P, the common prior. $u_i(f(\theta_1, \ldots, \theta_n))$ this is a utility, but condition that conditioned on theta i. This should be more than or equal to $\bar{u}_i(\theta_i)$ and this should hold for all $\theta_i \in \Theta_i$. These are the intermediate level.

Suppose and it can be depicted using a using a example of an auction suppose an item is being auctioned and after looking at the item each player knows its own type its own valuation. And once the auction is over all the bids are revealed then all players knows all players solutions. So, at the interim level interim individual rationality implies that if a social choice function is interim IR then players will participate in the direct mechanism or a mechanism implementing that social choice function.

Even after at the intermediate stage where it only knows its type θ_i it does not know the type of other players.

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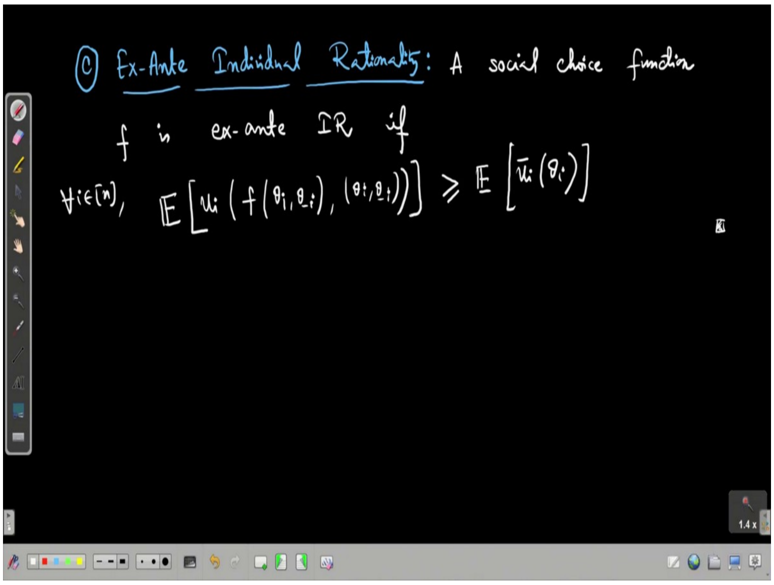

And the last kind of individual rationality is ex ante individual rationality. It says that suppose an announcement of auction has been made and in and that is all may be some description of the item is given, but I have not seen the item no player has not seen the item.

So, player does not know it is true type also at that time will it if you want to decide whether I will eventually go and participate in the individual mechanism or not then the individual ex ante individual rationality comes into picture. So, a social choice function f is ex ante IR if for all player $i \in [n]$ $u_i(f(\theta_1, \ldots, \theta_n))$.

But you know player does not know even player i does not know θ_i . So, only thing that makes sense is the expected utility of player i where expectation over the underlying the common prior distribution P this should be greater than equal to $\bar{u}_i(\theta_i)$, but again player i does not know θ_i . So, it can we can only talk about the expected utility ok. So, this holds then we call the social choice function ex ante individual rational.

So, this comes to conclusion that you know these are the main properties of social choice function that we want to talk in this lecture. So, in the next lecture we will characterize the social choice functions which can be implementable in dominant strategy incentive compatible or dominant strategy equilibrium. So, we will stop here today.