Algorithmic Game Theory
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Lecture - 04
Equilibrium Concepts

Ok. So, in the last class we have study, we have started investigating the question that how each player will play and these two questions are mixed together, like how each player is play and what will be the system outcome, what each all the players will play. So, towards that we introduced the notion of strongly dominant strategy and the corresponding notion of strongly dominant strategy equilibrium.

And we have seen that the confessing is a strongly dominant strategy in the prisoner's dilemma game. But if you see the other examples of the normal form games like Battle of Sexes coordination game, then Tragedy of commons, Matching Pennies, rock–paper-scissor, none of the games has any strongly dominant strategy. So, although this strongly dominant strategy this concept is very appealing and indeed when a player has a strongly dominant strategy the player has no reason to deviate from it.

It should not play any other action than its strongly dominant strategy, but the problem the downside is that you know most real-world examples, most games does not have a strongly dominant strategy.

And if a player does not have a strongly dominant strategy, even if one player does not have a strongly dominant strategy then the game does not have a strongly dominant strategy equilibrium. So, to take care of this problem what we do is that we weaken the notion of strongly dominant strategy and what we define is called weakly dominant strategy; weakly dominant strategy.

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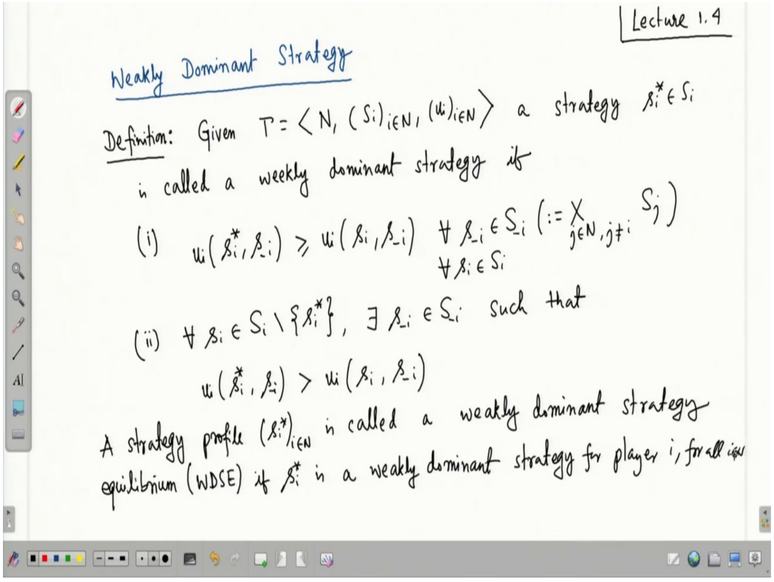

So, what is a weakly dominant strategy? Again, suppose given a normal form game; given $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, given a normal form game a strategy $s_i^* \in S_i$ is called a weakly dominant strategy, if two conditions must get satisfied the utility should be as good as any other strategy for irrespective of what other players are playing.

That means $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$, this should happen for all strategy profile of other players. Recall what is what was S_{-i}; S_{-i} is the strategy profile of all other players $\times_{j \in N, j \neq i} S_j$ this will happen for all the strategy profile of other players and for all $s_i \in S_i$. So, irrespective of what other players are playing, playing s_i^* is as good as any other strategy that is first condition. And the second condition is that you know this strategy s_i^* is in some sense better than any other strategy.

And what do we mean by that? It is that for all $s_i \in S_i \setminus \{s_i^*\}$ if I look at any other, any strategy other than $s_i^* \in S_i$, then for all these there exist a strategy profile of other players. $s_{-i} \in S_{-i}$ such that ui of utility when other players are playing s_{-i} and player i plays s_i^* this is strictly greater than when player i is playing s_i and other players are playing s_{-i}.

So, this is called a weakly dominant strategy and the corresponding equilibrium is called a weakly dominant strategy equilibrium. A strategy profile $(s_i^*)_{i \in N}$ is called a weakly dominant strategy equilibrium WDSE, in short if each s_i^* is a weakly dominant strategy; if s_i^* is a weekly dominant strategy for player i, for all i ok.

So, this is called a weakly dominant strategy and weakly dominant strategy equilibrium and we will see that the second price auction has a very beautiful weakly dominant strategy equilibrium. But before that there is another notion, which is even weaker that weakly dominant strategy equilibrium and that is called very weakly dominant strategy.

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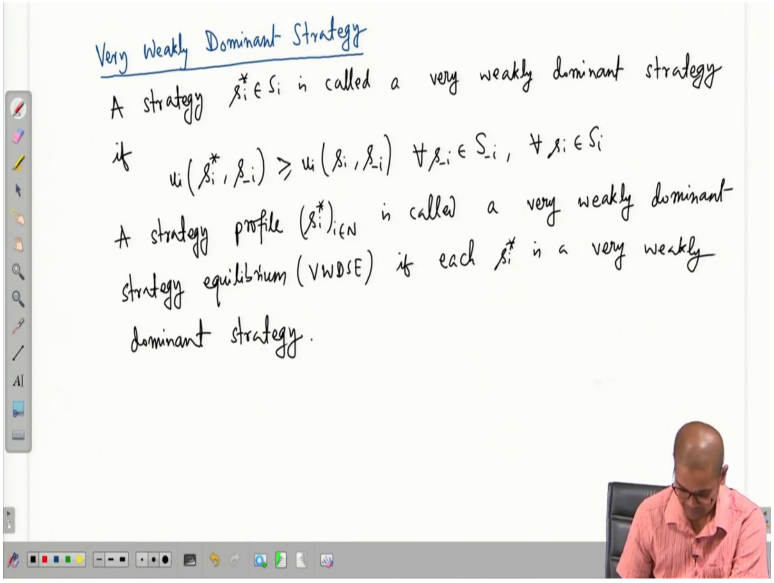

So, what is it? A strategy $s_i^* \in S_i$ is called a very weakly dominant strategy, if it always performs as good as any other strategy irrespective of other players, that is it. It does not need to bid the other strategies as was required in weakly dominant strategy. If utility of player i by playing this strategy s_i^* when other players are playing s_{-i}, this is as high as utility of playing any strategy s_i when other players are playing s_{-i}.

This should hold for all $s_{-i} \in S_{-i}$ for all small $s_i \in S_i$, this is called a very weakly dominant strategy and a corresponding equilibrium is called a very weakly dominant strategy equilibrium. A strategy profile $(s_i^*)_{i \in N}$ is called a very weakly dominant strategy equilibrium. In short VWDSE, if each s_i^* is a very weakly dominant strategy ok.

So, these are the some of the equilibrium concepts and we will let us see their use. And next let us prove the first theorem of our course; what is the theorem? That the second price auction has a weakly dominant strategy equilibrium and it is nothing but the simple action of bidding your valuation, each player knows its valuation of the item and simply bid that valuation as simple as that.

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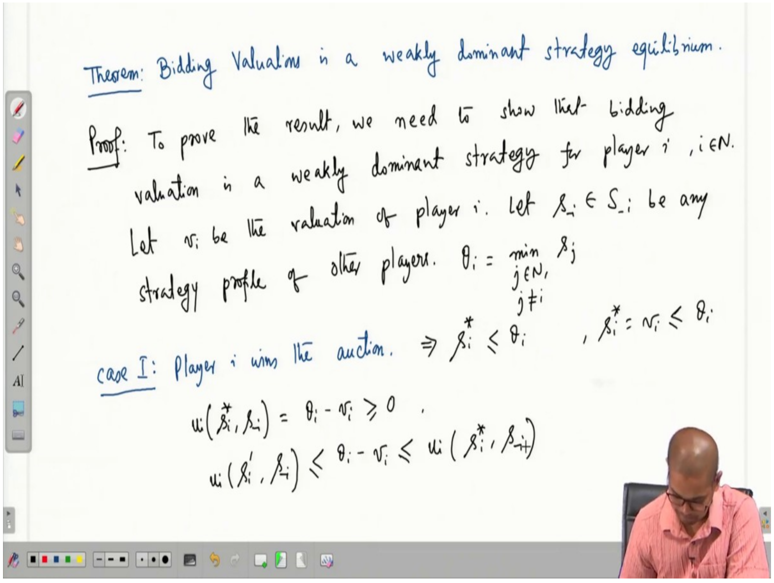

So, theorem bidding valuations is a weakly dominant strategy equilibrium proof and this is sort of one of the main reason of using the second price auction. It is an easy exercise that you know bidding true valuation is not at all a forget weakly, it is not even a very weakly dominant strategy for the player for any player. So, in some sense fast price auction is more complex from the players point of view and because players it is not clear what to bid and how to bid.

And because, if players from players if players are not clear about what to bid, then it is it becomes difficult like what to difficult to predict what will be the outcome of the system and you know these somethings sometimes called truthful bidding. So, truthful bidding is what is truthful bidding? It is nothing but bidding your valuations and truthful bidding is sometimes a desirable design desiderata.

And you know designing or coming up with a auction a non-trivial auction where bidding true valuations is the best strategy for all the players was sort of a very challenging thing and a very interesting thing and this auction was discovered by Myerson. And this is one of the main contributions of Myerson because of which he was awarded Nobel Prize, this is such a grand thing ok.

So, to prove this to prove the result, to prove the result to prove the we need to show that bidding valuation is a weakly dominant strategy for each player, for say player i $i \in N$. So, in the remaining proof let us fix a player i and let us prove that biding its true valuation is the is a weakly dominant strategy for player i.

So, let v i be the valuation of player i, of player i ok. And what we need to show? To show that it is a weakly dominant strategy equilibrium by definition we need to show two properties, two conditions let us prove them one by one. The first one is that we need to show that for any strategy profile of other players bidding this $s_i = v_i$ is maximizes the utility of player i.

So, towards that so let us fix any strategy profile of other players, let S_{-i} be any strategy profile of other players ok and we break this proof into two cases, depending on θ_i. So, define θ_i equal to minimum over $j \in N, j \neq i$. N is the set of players this is the minimum of s_j, this is the second highest bid in some sense. Not second highest bid, this is the; this is the minimum bid of all the players except the i-th player. So, in some sense if player i bids less than equal to θ_i, then player i wins otherwise it does not win.

So, our first case I: is player i wins the auction. So, in that case we know that what is s_i? s_i is in that case this implies that s_i must be less than equal to θ_i ok. And what is the utility of player i in this case? u_i of s_i^*, let s_i^* suppose player i is playing s_i^*. So, (s_i^*, s_{-i}). So, s_i^* and s_i^* is v_i, we want to show that this particular strategy that $s_i^* = v_i$ is the best strategy.

So, what is this? This is valuation minus payment. Now, what is payment? Because of second price auction by definition payment is θ_i. So, sorry this is payment minus valuation $\theta_i - v_i$ and we see that you know because v_i is at most θ_i, v_i is then at most θ_i, $\theta_i - v_i$ this is greater than equal to 0. Now, if player i plays some other strategy say $s_i^{'}$ then you know $u_i(s_i^{'}, s_{-i})$.

By playing $s_i^{'}$ player i can continue to win. So, in that case its utility will continue to be $\theta_i - v_i$ or it can lose and in that case, utility will be 0. So, its utility in $(s_i^{'}, s_{-i})$ is always less than equal to $\theta_i - v_i$, again why? Because if player i wins, then its utility is $\theta_i - v_i$

which is greater than equal to 0. And if player i loses in this strategy profile $(s_i^{'}, s_{-i})$, then player i its utility is 0.

So, in both the cases, its utility is always less than equal to $\theta_i - v_i$ this the utility that player i enjoys he by playing s_i^*, by truthful bidding when it is winning. So, when it is winning playing s_i^* is obviously, a better strategy. So, what is second case? The second case is player i loses the auction.

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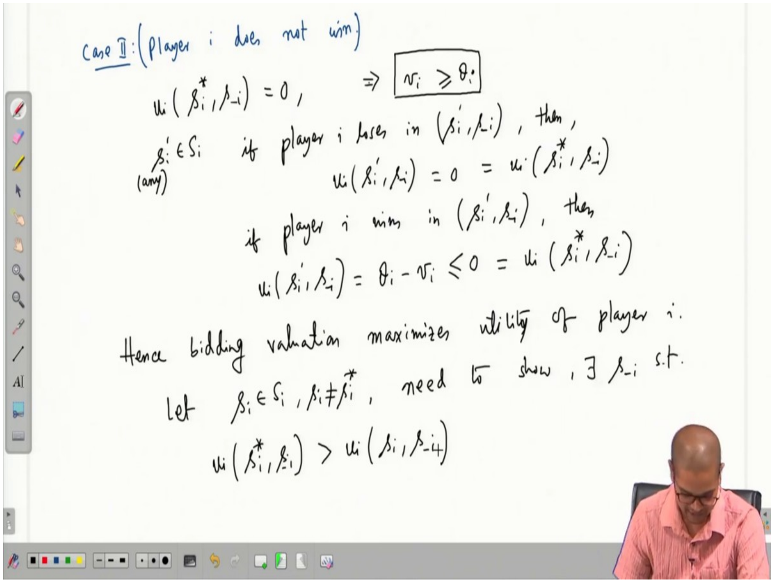

Case II: player i does not win ok. So, what is its utility then? In this case the utility of player i is zero. So, $u_i(s_i^*, s_{-i})$ is 0 and $s_i^* = v_i$. So, in this case we have v_i is greater than equal to θ_i, because v_i is the bid of player i. And if player i bids less than θ_i then it should have won, but it has lost; that means, v_i is greater than equal to θ_i.

So, consider any $s_i, s_i^{'} \in S_i$ is any, we need to show that the utility of player i in $s_i^{'}, s_{-i}$ is not more than utility of player i in (s_i^*, s_{-i}). What is its utility? If player i loses, if player i loses in $(s_i^{'}, s_{-i})$, then what do we have? Then $u_i(s_i^{'}, s_{-i})$ is 0, which is same as $u_i(s_i^*, s_{-i})$.

So, in this case by playing $s_i^{'}$ player i does not get more utility. If, let us see if player i wins what happens if player i wins in $(s_i^{'}, s_{-i})$, then let us see what happens. Then

$u_i(s_i^{'}, s_{-i})$ is its payment is θ_i and its utility is v_i. Now, you see that $\theta_i - v_i$ is less than equal to 0, this is less than equal to 0 and 0 is equal to $u_i(s_i^*, s_{-i})$.

So, in this case also you see that its utility can decrease compared to $u_i(s_i^*, s_{-i})$, but it cannot increase. So, this shows that it is if player i loses also, playing or bidding its valuation is always gives the best utility for player i, it always maximizes the utility of player i.

Now, we need to show the second case. So, this shows that; so hence let me write, hence bidding valuation maximizes utility of player i. The if we finish this proof here this only shows that bidding valuations is a very weakly dominant strategy equilibrium, but we need to show the second condition that it bids all other strategies in some sense. Now what do it mean? That means, so, let us $s_i \in S_i, s_i \neq s_i^*$, let us pick any other strategy.

So, we need to show let me write what we need to show. Need to show that there exist s minus i such that $u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$; how do you prove? So, we need to given s_i, I need to construct a strategy profile s_{-i}. So, let us construct.

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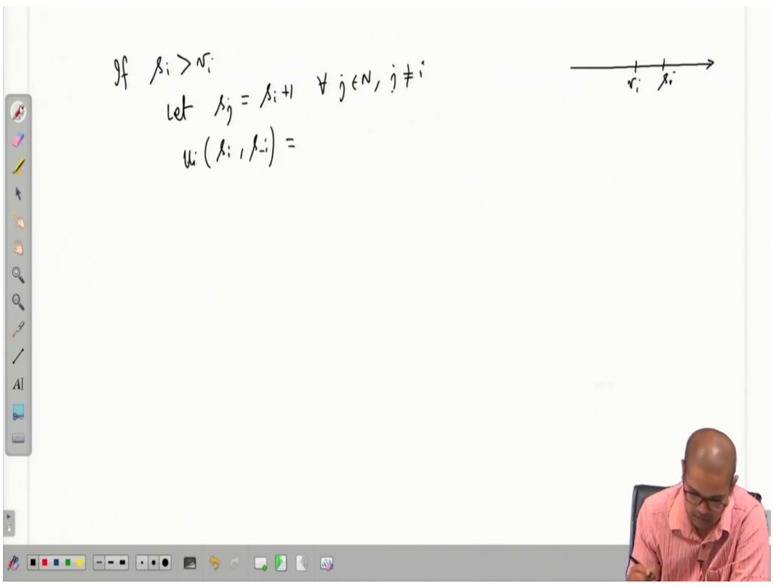

So, there are two cases, case 1: if $s_i \neq s_i^*$; that means, $s_i \neq v_i$. So, it must be greater than v_i or less than v_i. First let us consider it is greater than v_i. So, in the real line here is v_i and here is s_i. So, let s_j is; so, let us define s_j to be s_{i+1} for all $j \in N, j \neq i$ ok. So, all other players are bidding s_{i+1} so, ok. So, player i wins in this case. So, what is its utility? $u_i(s_i, s_{-i})$ is how much, sorry if s_i is greater than v_i yes. So, then let us see let us, what is eraser ok.

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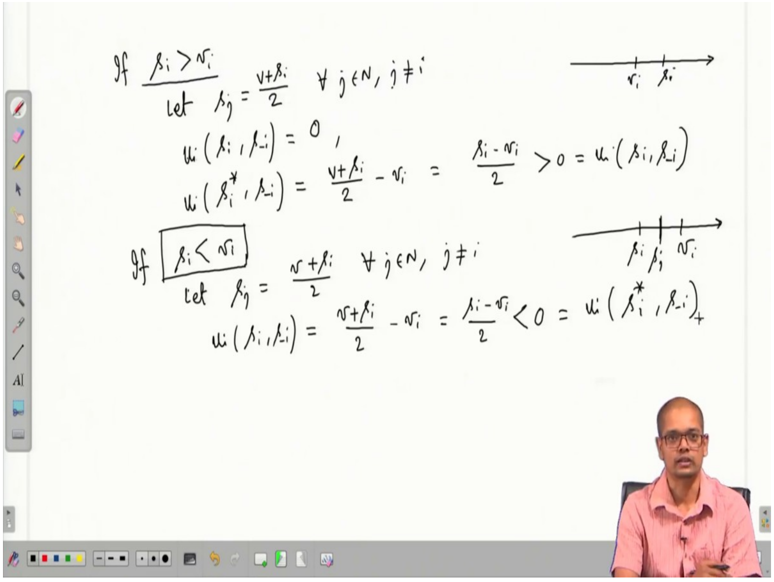

So, let us consider every player is bidding in between v and s_i. In that case player i loses, player i loses because it is the maximum bid it bids maximum. So, its utility is 0. On the other hand, if it could bid s_i^* , then its bid will be minimum and in that case it will win and what will be its utility. The second highest bid is $\frac{v+s_i}{2}$. So, its utility, its payment is $\frac{v+s_i}{2}$ and its valuation is v_i.

So, what is it? This is $\frac{s_i - v}{2}$ which is strictly more than 0, because s_i is greater than v_i and 0 is $u_i(s_i, s_{-i})$. So, I have exhibited a profile s_{-i} where playing s_i^* gives strictly more utility than playing s_i. Symmetrically, if s_i is less than v_i then; that means, in real line it looks like v_i is here and s_i is here.

So, in that case let us consider a strategy profile; let consider a strategy profile s_j, such that s_j is again $\frac{v+s_i}{2}$, this is for all $j \in N, j \neq i$, let us see what does its utility. Then $u_i(s_i, s_{-i})$. So, here is all the s_js, here is s_j in between in the midpoint of s_i and v_i in that case player i wins. But how much it gets paid? It gets paid the s_j, s_j is $\frac{v+s_i}{2}$ and its valuation is v_i.

So, this is again $\frac{s_i - v}{2}$, but in this case s_i is less than, strictly less than v_i is strictly less than 0, this should be star. Now, if it would have bid s_i^* which is v_i then it would have then it would have lost and its utility will be 0. Sorry, fine this is s_i^*, s_{-i} ok. So, this concludes the proof ok.

Thank you.