Algorithmic Game Theory
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Lecture - 38
BNE of First Price Auction

Welcome, in the last class we have started studying games beyond the strategic form game and in particular we have studied started studying Bayesian game we will continue that study here.

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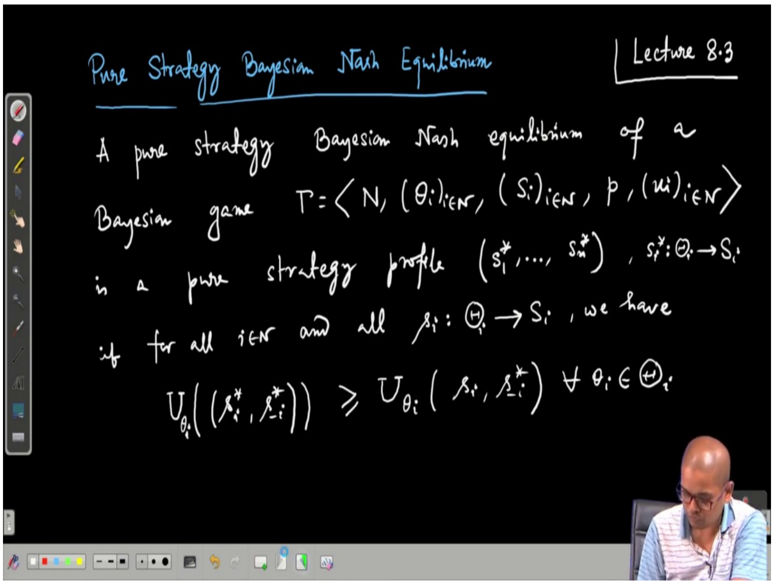

In this class we begin with defining what is a Pure Strategy Bayesian- Nash Equilibrium pure strategy Bayesian-Nash equilibrium? So, a pure strategy Bayesian-Nash equilibrium of a Bayesian game $\Gamma = \langle N, (\Theta_i)_{i \in N}, (S_i)_{i \in N}, p, (u_i)_{i \in N} \rangle$ is a pure strategy profile; is a pure strategy profile $(s_1^*, \ldots, s_n^*)$. So, let me first intuitively tell what does it mean.

A pure strategy Nash equilibrium for a Bayesian game pure strategy Bayesian-Nash equilibrium for a Bayesian game is a pure strategy Nash equilibrium for the corresponding Selton game. Now, what do you mean by pure strategy Nash equilibrium, how does a pure strategy profile in the corresponding Selton game look like? Recall in the Selton game for each player in the Bayesian game I look at the type set of that player which is Θ_i and for each of the type in that type set I have a player in the Selton game corresponding Selton game.

Now, I have to tell the strategy from S_i the set of strategies for player i in the Bayesian game for each of the players of small each of the players corresponding to the type small theta i of the Bayesian game. So, really speaking a strategy profile as a strategy $(s_1^*, \dots, s_n^*)$ where you know all these are $s_i^* : \Theta_i \rightarrow S_i$.

And you can also think of this directly in the Bayesian game perspective it is like the strategy of player i is deciding beforehand for what type I will play what action. So, that entire plan is its strategy s_i^*. So, a pure strategy profile $(s_1^*, \dots, s_n^*)$ is a Bayesian-Nash equilibrium if the same definition that there does not exist any unilateral deviation to any other strategy profile for any player.

If for all $i \in N$ and all other strategy $s_i : \Theta_i \rightarrow S_i$ we have the following, we have what, $U_{\theta_i}(s_i^*, s_{-i}^*) \geq U_{\theta_i}(s_i, s_{-i}^*)$ for all type theta i in capital theta i ok.

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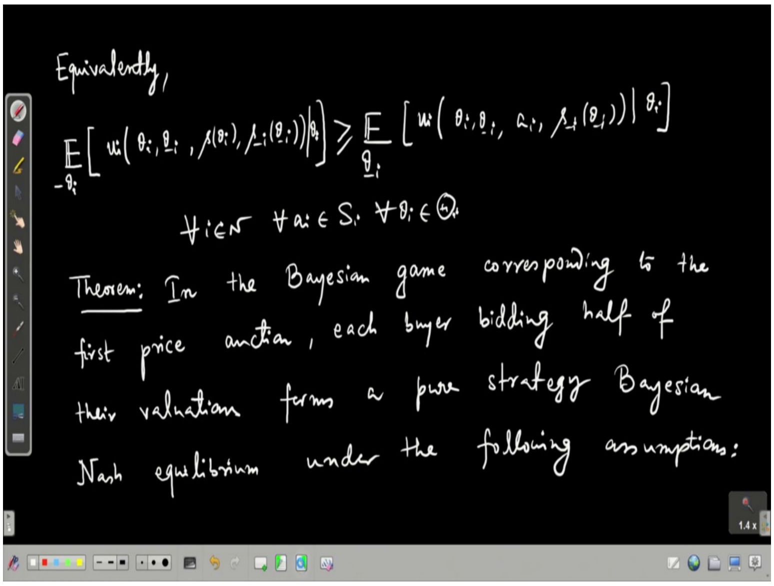

Or if you directly want to write for Bayesian in Bayesian game you can write equivalently expected $u_i(\theta_i, \theta_{-i}, s(\theta_i), s_{-i}(\theta_{-i})) \geq u_i(\theta_i, \theta_{-i}, a, s_{-i}(\theta_{-i}))$. This should hold for all player $i \in N$ and for all $a_i \in S_i$ and for all type $\theta_i \in \Theta_i$ this is given θ_i; given θ_i ok.

So, having defined the concept of pure strategy Bayesian-Nash equilibrium we next prove a very important result that in the first price auction what is a; what is a pure strategy Bayesian-Nash equilibrium, indeed there exist a pure strategy Bayesian-Nash equilibrium for the first price auction.

Recall for the second price auction we have a much stronger notion of equilibrium which is which exists there we have a weakly dominant strategy equilibrium, for second price auction where reporting or bidding your true valuation is a weakly dominant strategy equilibrium. But this is not the case for the first price auction that is why first price auction is more complex from both the players and the organizers point of view.

So, let me state the theorem first. So, theorem in the Bayesian game corresponding to the first price auction, recall in the last lecture we have seen how auctions can be suitably modeled using a Bayesian game. So, it says that if you model the first price auction as a Bayesian game each buyer bidding half their valuation half of their valuation forms a pure strategy Bayesian-Nash equilibrium under the following assumptions we have some mild assumptions under the following assumptions.

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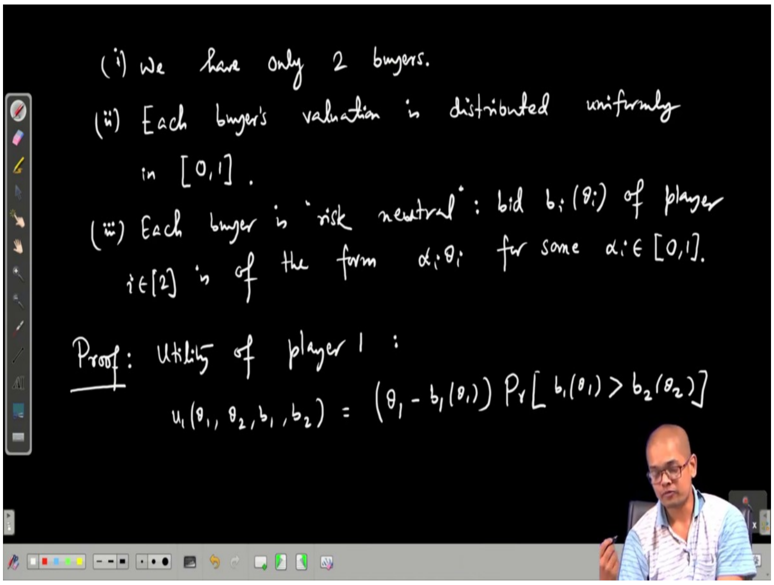

What are the assumptions? First assumption is we have only 2 buyers although the result can be extended to more than 2 buyers, but then it would not be half of the; half of the true valuations. If we have n number of buyers the pure strategy Nash equilibrium will be $\frac{1}{n}$ times the true valuation that will be form that will form a pure strategy Bayesian-Nash equilibrium. We have only 2 buyers, each buyer's valuation is distributed uniformly in the real interval $[0,1]$ and each buyer is risk neutral.

Now, a risk neutral intuitively it means that. So, a player is risk neutral if the player is indifferent between say on the one option where he gets 100 rupees with probability 1 and in the other option where he gets 0 rupees with probability half and 200 rupees with probability half, in both cases the expected utility is 100 a risk neutral buyer will be indifferent among these two options.

Now, in the context of first price auction a risk neutral buyer means that the bid b_i which is a function of θ_i of the players of player i is of the form $\alpha_i \theta_i$ for some $\alpha_i \in [0,1]$ ok. Proof: so, what is the utility of player i in a particular strategy profile and type profile? Utility of player 1 let us write; $u_1(\theta_1, \theta_2, b_1, b_2)$ is its utilities $\theta_1 - b_1(\theta_1)$. If it wins and the probability it wins is probability that it bids higher $b_1(\theta_1) \geq b_2(\theta_2)$.

So, if it wins its utility is $\theta_1 - b_1(\theta_1)$ is how much the player values and bid because that much amount he has to pay because we are in the first price auction. Valuation minus payment an and this is the this is when if the player wins and that happens when its bid is higher.

Observe that because you know bids these thetas are this probability is of is over theta 1 and theta 2 and they are continuous random variables probability that b 1 of theta 1 equal to b 2 of theta 2 is 0 because valuations are distributed uniformly and their risk neutral bids are proportional to valuation ok.

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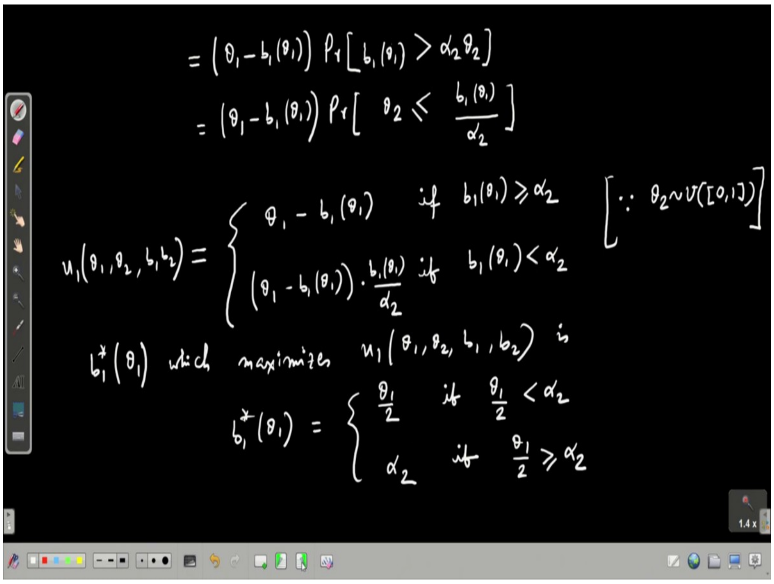

So, theta 1 minus b 1 of theta 1 now again now we apply risk neutrality and because of risk neutrality b 2 of b 1 of theta 1 is alpha 1 theta 1 should be greater than equal to alpha 2 theta 2. So, this is what, this is theta 1 minus b 1 theta 1. Now, here you see that theta 2 is varying theta 1 is fixed because we are looking for an expression in terms of theta 1 this is probability that theta 2 is less than equal to or let me keep it here.

Let me not put alpha 1 theta 1, we will see just in a moment let me keep it bid b 1 of theta 1; that means, theta 2 should be less than equal to b 1 of theta 1 by alpha 2 ok. Now, you see that there are two possibilities that if. So, there are two possibilities if b 1 of theta 1 by alpha 2 is greater than equal to 1 then this probability is always 1 because theta 2 is less than equal to 1.

So, if two cases if b 1 of theta 1 is greater than equal to alpha 2 then this probability is 1 and in that case its value is theta 1 minus b 1 of theta 1. On other hand if it is not; that means, b 1 theta 1 is strictly less than alpha 2 then what is its probability, this will come this will remain theta 1 minus b 1 theta 1 and because theta 2 is distributed uniformly in between 0 and 1. So, this will be b 1 of theta 1 by alpha 2 ok.

So, this is since theta 2 is distributed uniformly in between 0 and 1. So, what is its utility? So, this was utility let us write u 1 of theta 1, theta 2, b 1, b 2. Now, what value of b 1 will maximize this utility? The straightforward application of calculus we can gives us that b 1 of theta 1 b 1 star of theta 1 star which maximizes player 1's utility u 1 of theta 1, theta 2, b 1, b 2 is.

What is it, is b 1 star of theta 1 what you do is that, you take first derivative equal it to 0. Take second derivative and take the first derivative make first derivative equal to 0 and solve it and then what is the solution you, that solution you put on the second derivative and see its value is if it is negative then it is a maximizer. So, using that you can you do and you see that b 1 star equal to this value that theta 1 by 2 if theta 1 by 2 is less than alpha 2 and this is alpha 2 if theta 1 by 2 is greater than equal to alpha 2.

This particular b 1 star of theta 1 maximizes the utility of player 1. So, player 1 will play accordingly; that means, for player 1 alpha equal to half.

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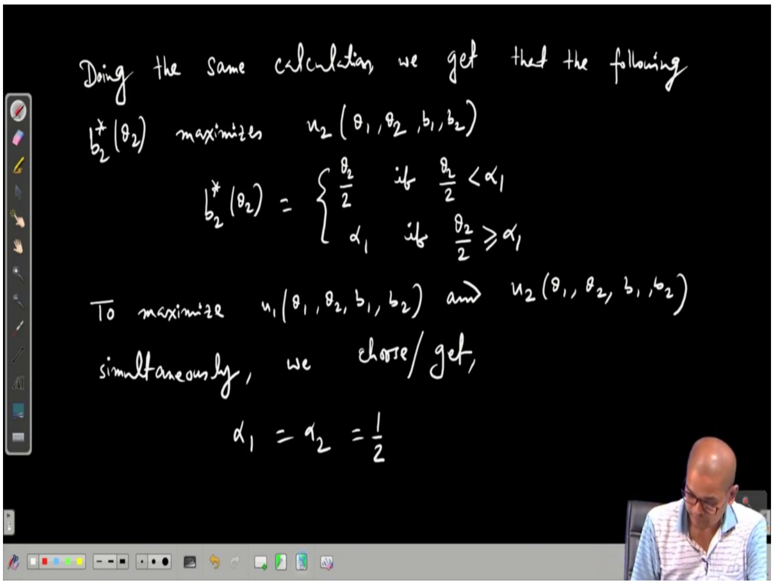

Now, we making the same argument, so doing the same calculation we get that the following b 2 star of theta 2 maximizes the utility of player 2, u 2 of theta 1, theta 2, b 1, b 2, what is it? Symmetric b 2 star of theta 2 is theta 2 by 2 if theta 2 by 2 is less than alpha and less than alpha 1 and this is alpha 1 if theta 2 by 2 is greater than equal to alpha 1.

Now, to maximizes both u 1 utility of player 1 and utility of player 2 what value of alpha 1 alpha 2 you should choose? So, to because this if conditions also needs to be satisfied if theta 2 by 2 is less than alpha 1 to maximize u 1 of theta 1, theta 2, b 1, b 2 and u 2 theta 1, theta 2, b 1, b 2 simultaneously, we choose the only value choose or get alpha 1 equal to alpha 2 equal to half. Let you put these values alpha 1 alpha 2 and see all these conditions are satisfied.

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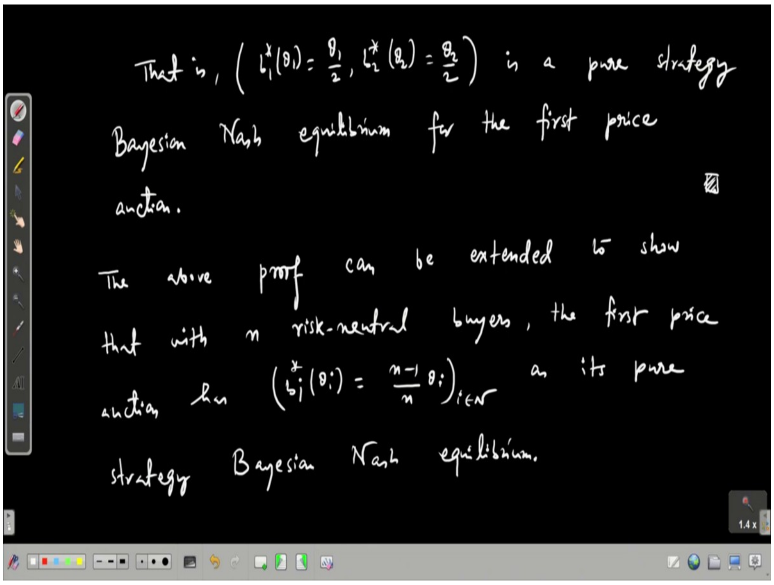

And that is if both the utilities are maximized that is b 1 star of theta 1 equal to theta 1 by 2 and b 2 star of theta 2 equal to theta 2 by 2 this is a pure strategy Bayesian-Nash equilibrium for the first price auction. So, as we have mentioned before the same proof technique can be generalized to n players.

So, we have three conditions, the first condition is we have only 2 players and that is not so much of a serious condition it just makes our calculation easier that is it. The second and third conclusion is more important, the second conclusion is also not so much of severe, it is like; it is like we do not have much information of the players all we know is that the valuations can be arbitrary and that arbitrary assuming valuations can be arbitrary is equivalent to assuming that the valuation lie in between 0 and 1.

In real life what really matters or what really dampens or reduces the predictability of this theorem is the third assumption that you know often people are not at not risk neutral there are risk sometimes risk averse or there are some people who are risk loving people are very very seldom they are risk neutral.

So, let me just write the above proof can be extended to show that with n risk neutral buyers, the first price auction has b 1 b i star of theta i equal to n minus 1 by n theta i i in N, this as its pure strategy Bayesian-Nash equilibrium. So, its n minus 1 times theta i by n it also shows that the more number of players we have; that means, that amounts to more competition and then players tend to bid more closer and closer to its true valuations.

So, we will stop here today, in the next lecture we will study extensive form game which is relaxing the simultaneous move nature of strategic form game ok.

Thank you.