Algorithmic Game Theory
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Lecture - 34
Braess's paradox and Pigou's Network

Welcome. So, in the last class, we have finished the first sort of an important part of the course. We begin with the important question of, the main question of Game Theory. One of the central question is how to predict, and towards that we studied various equilibrium concepts starting with strongly dominant strategy equilibrium, weakly domain strategy equilibrium, and so on to mixed strategy NASH equilibrium, then correlated equilibrium, and quartz correlated equilibrium.

And then we end up yesterday in last class that how there can exist natural algorithms to discover a correlated equilibrium or quartz coded equilibrium. So, that players can discover this this equilibrium concepts and play according to that. So, now, we will having resolved the first question to some extent, we will ask some questions around it. And today, towards that we will study what is called Price of Anarchy.

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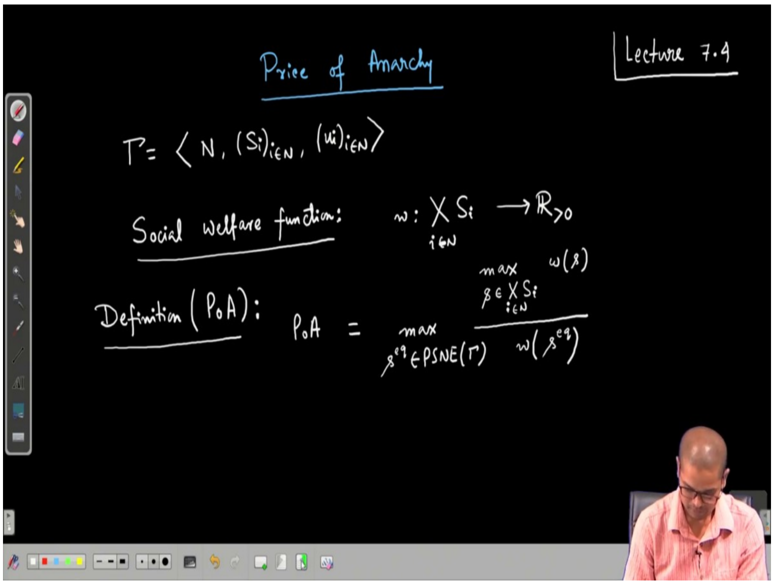

So, what is price of anarchy? So, you see that players have full freedom. They can play any strategy from their set of strategies and accordingly they will get the payoff. But what is system? A system is a set of players each playing an action; and so each player's action collectively decide the outcome for the system and for some person outside of the system, say government or someone, they may desire certain outcomes, they may desire systems to behave in such certain ways.

And so it need not be that the way system behaves with full freedom to the players each player is allowed to play any strategy from their strategy set in the resulting outcome, may not be the most desirable outcome of the system level.

And how bad that outcome can be from the best outcome or optimal outcome is what is called price of anarchy, what price players or we are paying to have freedom, what is the price in terms of performance to have freedom so, towards that. So, as usual we have a strategic form game gamma equal to $\Gamma=\langle N,(S_i)_{i\in N},(u_i)_{i\in N}\rangle$. These are strategic form game.

And there is a concept called social welfare function. It is a function from strategy profiles to positive real numbers, sort of quantifying the goodness of each strategy profile thereby quantifying the goodness of each outcome. So, the higher the value of a social choice social welfare function, the better the strategy profile and the resulting outcome is.

Now, what is price of anarchy? So, definition, definition price of anarchy often abbreviated as PoA. It is the price of anarchy is you first find what which strategy profile maximizes at the social welfare, what is the best outcome from society's perspective, the outcome which has the highest social welfare. So, it is max is in the strategy profiles w of s, and then you divide by the outcome the social welfare of the equilibrium outcome.

Assume that the equilibrium exist. And you know there could be multiple equilibriums. For example, pure strategy NASH equilibrium, mixed strategy NASH equilibrium, correlated equilibrium or quartz code equilibrium, this equilibriums is may not be unique there may be more than one such strategy profiles which satisfy this equilibrium constraints.

So, which equilibrium I will take? I will take the equilibrium which sort of maximizes this ratio. So, this is max. And you know because we are applying social welfare function to the pure strategy profiles, so let us assume that pure price of anarchy with respect to pure strategy NASH equilibriums. So, s equilibrium belongs to the set of all pure strategy NASH equilibriums of the game Γ, ok. So, this is called price of anarchy.

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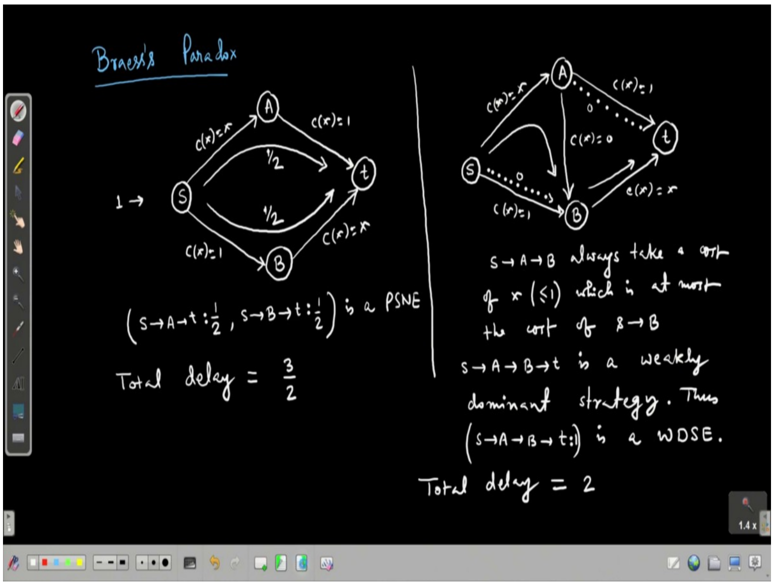

So, let us see an example. An example we have already seen of Braess paradox. It talks about a network s, t, and here is A, B. It is a network congestion game and the cost functions are like $c(x)=x$ and here $c(x)=1$. So, 1 unit of traffic is being sent from s to t. Here $c(x)=1$, $c(x)=x$. So, what is the pure strategy NASH equilibrium here? s to A to t, this path half amount of traffic follows the top path and half amount of traffic uses the bottom path.

Then we say that, ok, so let us add a high speed network bit from A to B. The cost of this edge A to B is 0, irrespective of the traffic. And this basically makes the A and B point identical because it is not identical. If you if you want to go to B you can go to A that is also enough.

So, now you see that we have 3 paths s to A to t, s to B to t, and s to A to B to t. Now, you see from s to B, irrespective of the traffic, s to A to B always take a cost of x which is at most 1, which is at most the cost of s to b. So, no traffic will follow will use this path s to B. And all the traffic will follow the s to A to B.

And now from B to t, or A to B to t this path has cost x whereas, A to t will have no; will have cost of 1. So, the traffic this dotted path will have 0 because this is this path is strongly or weakly dominated and same with this path, A to t. This is also the traffic will be 0, because it strongly dominated. So, all traffic we will follow this path. And indeed, s to A to B to t is a weekly dominant strategy. Thus s to A to B to t all traffic using this path is a WDSE, weekly dominant strategy equilibrium.

But what are the delays? So, the delay in the first case total delay, total delay is how much? Half fraction of traffic follows the top path, and the half fraction of traffic follows the bottom path, so the cost of the top path is 1 plus half, so half times 1 plus half and the cost plus the cost of the bottom path is half times 1 plus half. , so it is $\frac{3}{2}$.

On other hand, here the total delay is all the traffic are following s to A to B to t, so total delay is 2.

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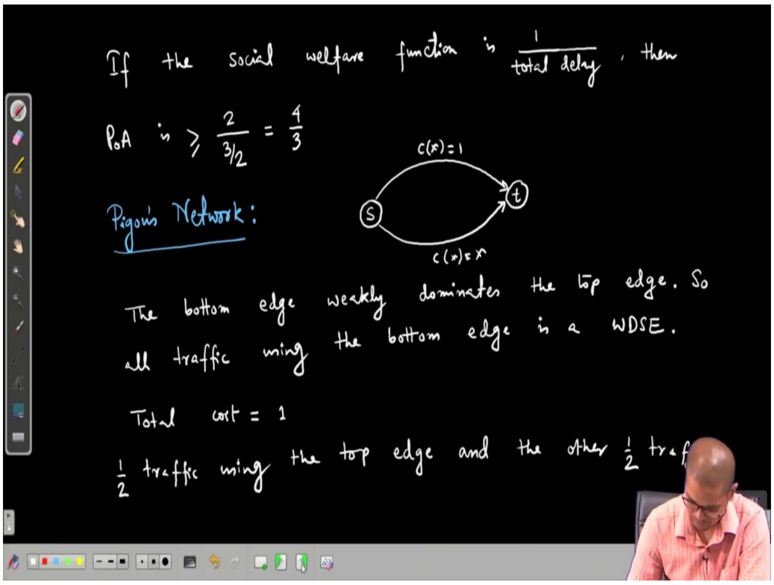

So, if the social welfare function is 1 by total delay, then price of anarchy is $\frac{2}{3/2}$ which is price of anarchy is at least $\frac{2}{3/2}$ which is $\frac{4}{3}$. And it is an easy exercise to show that the price of anarchy for Braess paradox is exactly $\frac{4}{3}$. Let me not show it here. You can take it as an exercise.

So, let us see another example even simpler which is called Pigou's network. How does the graph look like here? It is only two vertices, s and t. And it has only two edges, parallel edges. The cost of top edge $c(x)=1$, cost of bottom edge $c(x)=x$. So, the bottom edge weakly dominates the top edge. So, all traffic using the bottom edge is a weekly dominant strategy equilibrium.

What is total delay? Total cost is 1. But, which strategy profile max minimizes total cost?

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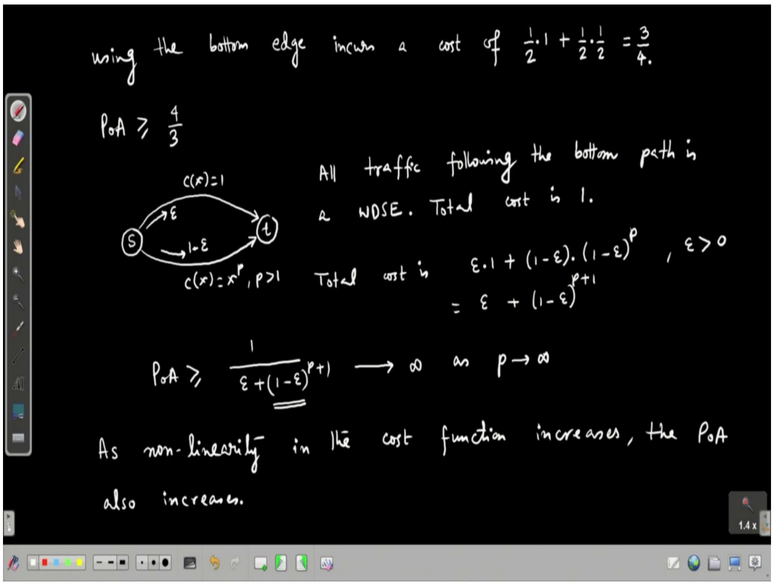

So, half traffic using the top edge and the other half traffic using the bottom edge incurs a cost of; how much cost? The top edge the cost is 1 and the bottom edge the cost is $\frac{1}{2}$, so it is $\frac{3}{2}$. So, let us see. The half fraction of traffic they use the top edge and their cost is 1 and the other half fraction of traffic is the bottom edge and their cost is $\frac{1}{2}$. So, it is half plus one-fourth, $\frac{3}{4}$.

So, the price of anarchy, again the social welfare function is 1 over cost, the price of anarchy is, the price of anarchy is sorry this is greater than equal to $\frac{1}{3/4}$ which is at least

$\frac{4}{3}$. And again, here also a careful analysis it can be shown that the price of anarchy is tied, it is actually equal to $\frac{4}{3}$ for this Pigou's network.

Now, let us change the Pigou's network slightly and at some non-linearity. And let us see how price of anarchy changes. The cost of top edge remains the same and the cost of bottom edge is x^p, for some $p>1$, ok. Again, the bottom path always dominates the top path and the all traffic following the bottom path is a weekly dominant strategy equilibrium. Total cost is 1 because x equal to 1 in the bottom, in the bottom path, bottom edge.

On the other hand, let us see what is the best way to split the traffic. Suppose epsilon fraction of the traffic I send along the top path, and remaining $1-\epsilon$ on the bottom path. So, here in this total cost is $\epsilon+(1-\epsilon)(1-\epsilon)^p$, which is like $\epsilon+(1-\epsilon)^{p+1}$. So, price of anarchy is >= $\frac{1}{\epsilon+(1-\epsilon)^{p+1}}$.

Now, you see that $1-\epsilon$ is a fraction because ϵ is greater than strictly greater than 0, so $1-\epsilon$ is strictly less than 1. And so if we keep increasing p, if the non-linearity of the bottom path increases then the cost decreases, the this term $(1-\epsilon)^{p+1}$ goes to 0. So, this goes to infinity as p goes to infinity.

So, what is the corollary? The observation here is in the simple network that as non-linearity in the cost function increases, the price of anarchy also increases. So, intuitively speaking, price of anarchy is then proportional to the amount of non-linearity of the cost function. And at least for this simple Pigou's network.

But you know real life real life networks are not so simple network. It is a very complicated network. If you take any road network of any city even a small city like Kharagpur or even IIT, Kharagpur campus then also it is it so many edges, so many connections and so on. And if we need to do so much hard work to find price of anarchy of this simple Pigou's network, then how will we going to compute the price of anarchy of a complex network. And that is sort of our next topic, Price of Anarchy for Selfish Networks.

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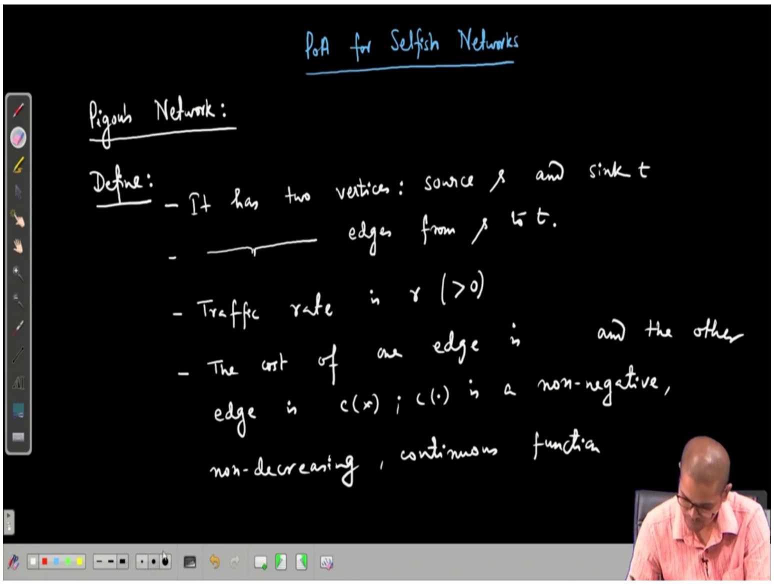

Selfish, what is selfish network? Network is static, but the players are selfish, players are rational. And we have seen that the amount of non; with the with the increase of non-linearity the price of anarchy increases. So, we need to sort of bound the non-linearity.

But the, but the striking or the crown jewel of this line of research is that you know, we only need to look at the amount of non-linearity in the cost function. We do not need to look at to the complexity of the network structure. Price of anarchy only depends on the amount of non-linearity in the cost functions involved. It sort of does not give a, does not depend so much on the network structure how complex it is.

And what will show is that you know Pigou's network is, the price of anarchy of Pigou's network is sort of the maximum price of anarchy of any network, if the cost function the amount of non-linearity involved in the cost function is same. So, towards there, towards that we need to formally define what is Pigou's network; that we do it now.

Definition, formal definition, and we will prove a theorem like on a high level it says that the price of anarchy is any network is at most the price of anarchy of Pigou's network. Now, to prove that statement to state that result and prove it formally, we need to formally define what is Pigou's network.

So, formally let us define it has exactly two vertices, source s and sink t. It has two edges from s to t. Traffic rate from s to t we have assumed 1, let us call it r, to make it more general. Traffic rate is r which is strictly greater than 0.

The cost of one edge is 1, irrespective of the traffic and the other edge is $c(x)$, and this $c(x)$ c is a non-negative, non-decreasing continuous function, ok. And sorry cost of other edge is not 1, the cost of other edge is $c(r)$, cost of one edge is $c(x)$ and the cost of other edge is $c(r)$.

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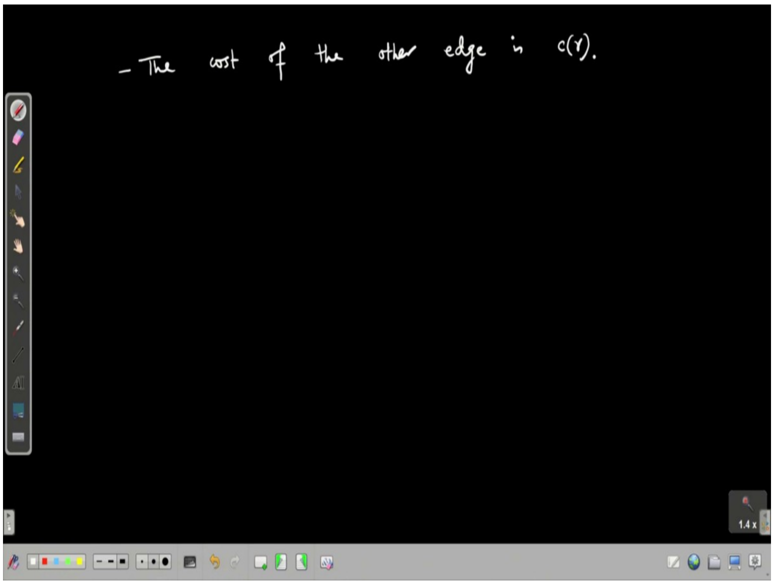

So, the edge with cost function $c(x)$, always weakly dominates the other edge whose cost function is cost is fixed which is $c(r)$ because c is a non-decreasing function, ok. So, this is Pigou’s network.

So, in the next class we will do some more preliminaries, and then state the result which at the high level says that the price of anarchy of any network is say is at most the price of anarchy of Pigou’s network. And we will prove that result, ok.