Algorithmic Game Theory
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Lecture - 27
Correlated Equilibrium

Welcome. So, in today's lecture we will study some more Equilibrium concepts.

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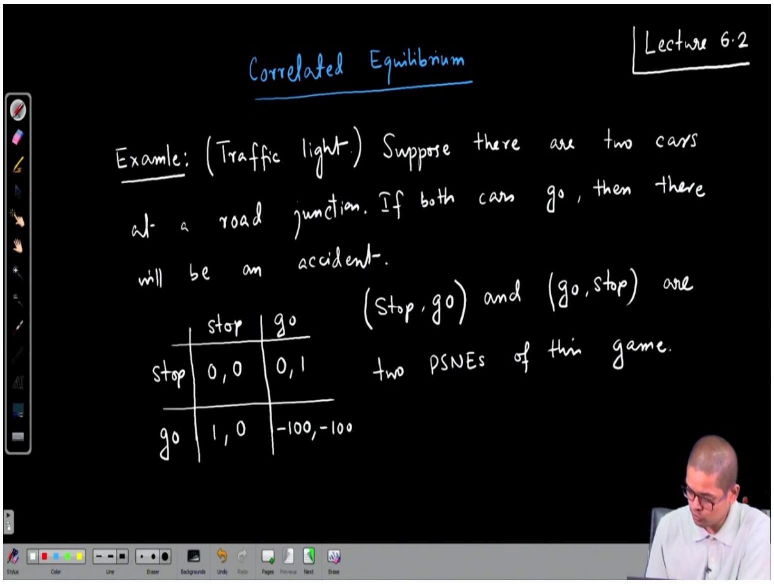

So, we will start with Correlated Equilibrium. So, let us understand this correlated equilibrium with an example. So, example is an example of traffic light. So, what is the scenario? Scenario is suppose there are two cars at a road junction, somehow for some reason the traffic light is not working, and they are crossing each other. It is like if both cars go then there will be an accident.

Of course, if only one of them go then it is fine they can go. So, what are the utility matrices? Both the players have two options, either stop or go; stop, go. If both players stop, then they get a utility of 0 comma 0. If player 1 stops and player 2 goes, then player 1's utility is 0, player 2's utility is 1.

Similarly, if player 1 goes and player 2 stops, then player 1's utility is 0, player 1's utility is 1 and player 2's utility 0; but if both of them go together then there will be accident and both of their utilities minus 100, ok. So, is there any pure-strategy Nash equilibrium? Yes.

So, stop go; that means, first player playing stop and second player going and go, stop are two PSNEs of this game. But you know again we have two PSNEs, which sort of substantially reduces the predictive power of this concept in this particular game, and also its bit unfair. If both players does not matter, if even if both players follow some PSNE play according to P one PSNE then it is still unfair.

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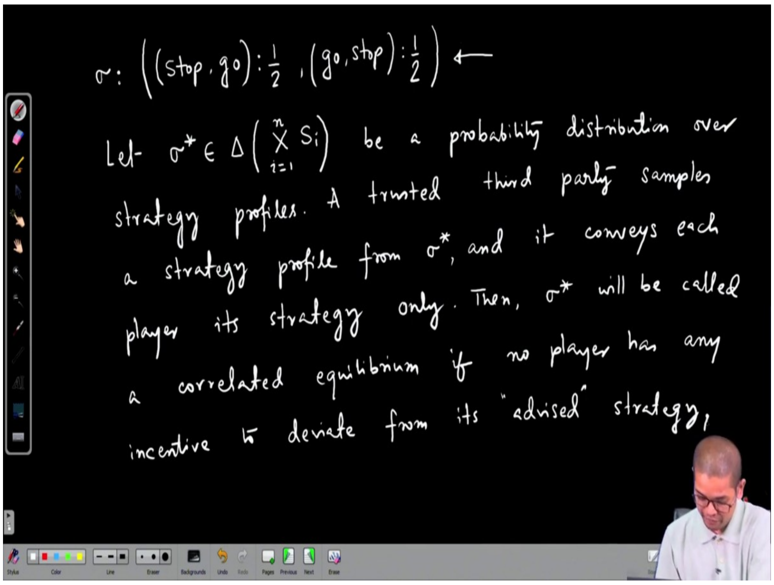

So, here comes the notion of correlated equilibrium. How about this idea, that look at this strategy profile stop, go. There is a trusted third party who comes and see tells that you know, I will pick this strategy profiles stop comma go with probability half and the strategy profile go comma stop with probability half. So, observe that then both; so, what is the setting?

So, a trusted third party will either pick stop comma go with probability half or pick go comma stop with probability half and tell each player their strategy. For example, if stop comma go is sampled, then player 1 is asked to stop and player 2 is asked to go. It player 1 only listens to what? Only listens to its part that means, it does not know that player 2 is told to go.

It just listens that player that player 1 that means, itself is asked to stop. So, what are the things it knows before playing? It knows that the trusted third party samples according to this probability distribution over the strategy profiles. And it knows its strategy. From the sampled strategy profile, it know it knows the strategy that it is supposed to play. It does not know the other strategy.

Of course, it can sometimes be inferred from this probability distribution sigma, which is in this case. In this case the knowledge of this probability distribution enables player 1 to know that player 2 will be asked to go, if player 1 is asked to stop. Similarly, player 2 will be asked to stop and player 1 is ask to go. So, let me write.

So, let sigma let us call this, let us sigma star be a probability distribution over strategy profiles ok. So, at the time of play, each player ok; this is a trusted third party samples a strategy profile from σ^*. And it conveys each player its strategy only. Then σ^* intuitively speaking will be called a correlated equilibrium.

If no player has any incentive to deviate from its advice strategy; think of the trusted third party is advising each player a strategy, its "advised" strategy. So, σ which is with probability of stop comma go and with probability of go comma stop is a correlated equilibrium. Let us see why.

Suppose, this suppose, the trusted third parties samples and it comes stop comma go. And it says stop to player 1, then player 1 knows that the trusted third party will tell go to player 2 and it is the best response of player 1 to stick to stop, to stick to the advice it is given. Same with player 2, because player 2 is asked to go; it knows that it is asked to the expert or the trusted third party has asked the player 1 to stop.

Because, it knows σ and hence, it is in the best response of player 2 to follow the advice; of course, assuming advice assuming every other player is following the advice.

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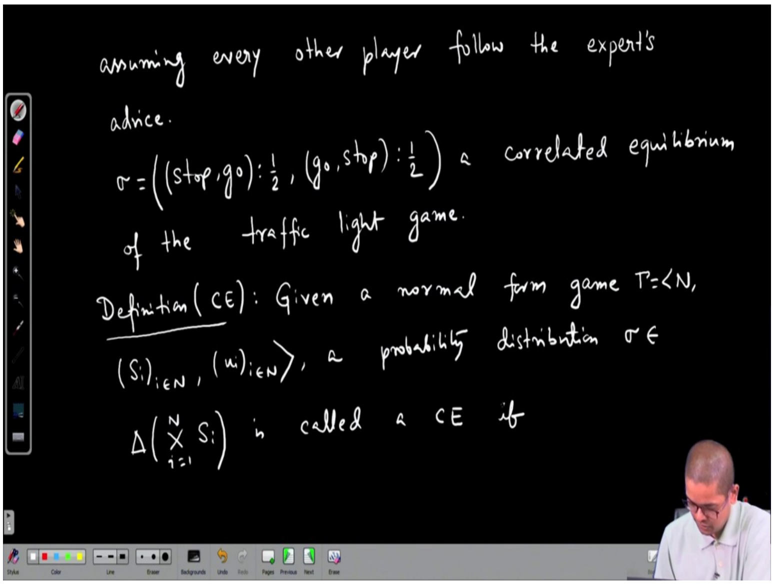

Assuming every other player follow the expert's advice; the that trusted third party in the jargon of correlated equilibrium is called an expert, follow the expert's advice ok. So, what we have observed is that this σ equal to stop comma go with probability half, and go comma stop with probability half.

This is a correlated equilibrium of the traffics light game. So, what is the definition of a correlated equilibrium? Formal definition: Given a normal form game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a probability distribution σ in given a normal form game gamma a probability distribution σ is called a correlated equilibrium, if the following condition hold.

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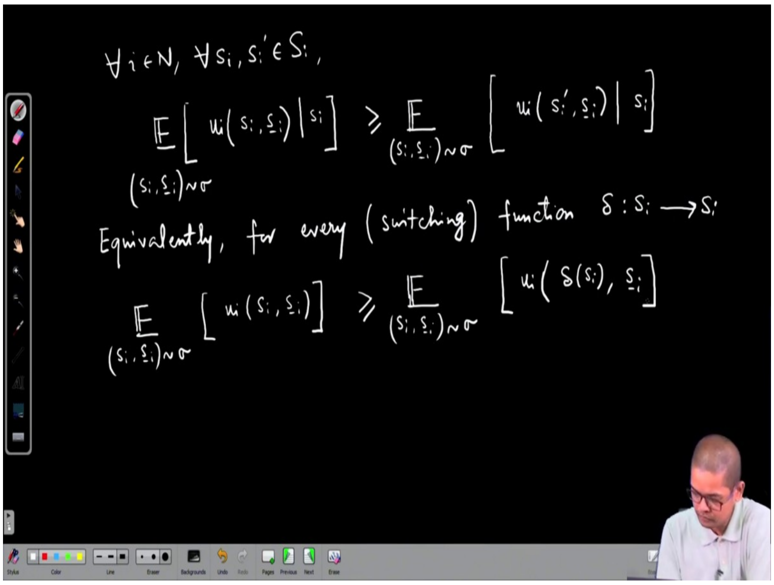

If for all player $i \in N$, for all strategy $s_i, s_i' \in S_i$, the expected utility of player i in (s_i, s_{-i}), where this strategy profile (s_i, s_{-i}) is sampled from σ given s_i; that means, player i knows that the sample is knows that the sample strategy profile has s_i.

This is greater than equal to expected utility of u sample (s_i, s_{-i}) from σ utility of player i, but player i prefers suppose, it does not play s_i it plays (s_i', s_{-i}) given s_i. Why given s_i, because player i at the time of playing knows s_i. And you know and so, in the right-hand side after knowing its knowing s_i and the fact that the strategy profile is sampled from the distribution sigma, it still prefers s_i'.

And so, this is the expected utility and this expected utility cannot be more than what it gets by playing or playing according to the expert's advice. So, this is one thing, this is this involves conditional expectation. There exist another equivalent definition, which does not involve conditional expectation; equivalently for every this every function, but this functions in this context are called switching functions.

For every switching function is just a function, but in this context it is called switching function. $\delta : S_i \to S_i$ equivalently for every switching function this, the expected utility of expected utility when (s_i, s_{-i}) is sampled from σ $u_i(s_i, s_{-i})$. This is this should be greater than equal to expected utility of (s_i, s_{-i}) sampled from σ .

But instead of playing s_i it plays $\delta(s_i)$; whereas, other players continue to follow the advice ok. So, I would leave the proof of equivalence to you as a homework.

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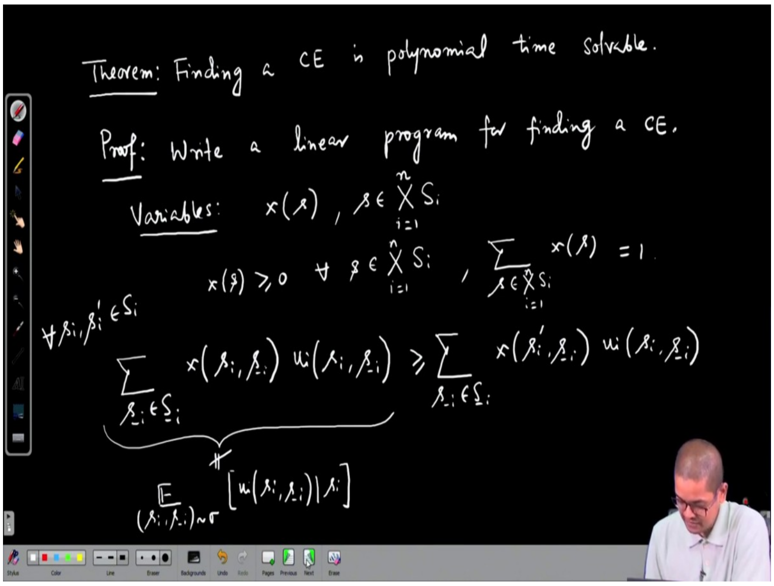

And so, let us move on and to prove why this is polynomial time solvable. So, theorem: Finding a CE is polynomial time solvable. Proof: What we will do is that we will write a linear program for finding a CE. Recall, what was linear program; this object this concept we have we used, while proving min max theorem in matrix games.

So, it is a maximization or minimization of a linear function or linear function subject to linear constraints ok. Now, what are the linear program? Let us write. So, variables are this. So, what are we searching? We are searching for a correlated equilibrium. And what is the correlated equilibrium? It is a probability distribution over the strategy profiles. So, the corresponding probability values are my variables. x_s, where $s \in \times_{i=1}^{n} S_i$.

Now, we want to write the, so this must be a probability distribution. So, this constraints of course, have $x_s \geq 0$, for all $s \in \times_{i=1}^{n} S_i$, this one constraint and of course, summation equal to 1. But how do we write the constraints for say this constraint, this conditional probability? If you see I let you verify, but this is the left-hand side. So, fix an s_i, for all $s_i, s_i' \in S_i, s_{-i} \in S_{-i}$.

What is the probability that (s_i, s_{-i}) is chosen is picked or is sampled is this probability, $x(s_i, s_{-i})$. Now, and what is its utility. Its utility is $u_i(s_i, s_{-i})$. This should be greater than equal to $s_{-i} \in S_{-i}$.

$x(s_i, s_{-i})$, but instead of s_i it is now playing $s_i^{'}$ and its utility is $u_i(s_i, s_{-i})$. Recall this is not a, but do not be mistaken. This is not equal to expectation of (s_i, s_{-i}) sampled from $\sigma(u_i(s_i, s_{-i}) | s_i)$. Please note that; but if you write this, then there will be some other terms on both sides, which will gets cancelled and you will get this equations.

So, there is a linear program with this many variables number of strategy profiles, and you know you cannot do better than that, because you know the output size is this. The output may need writing down this many probability values. So, this is a linear program and it is a feasibility linear program.

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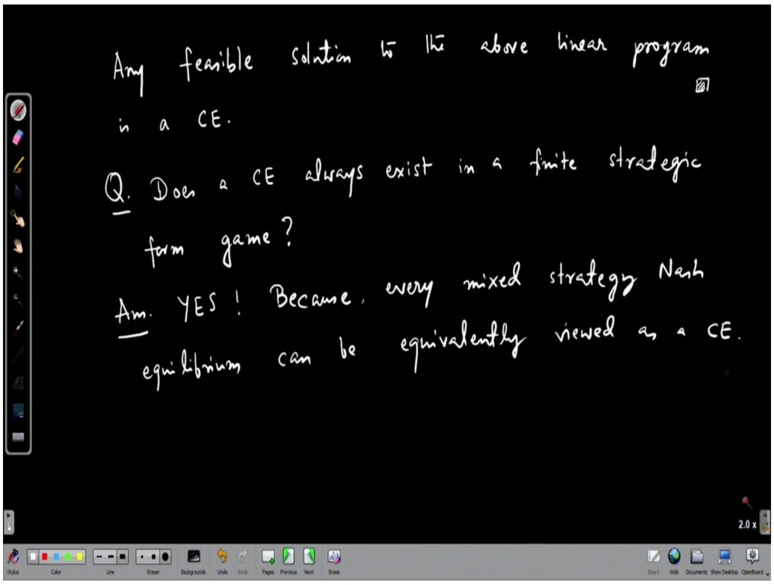

So, if or any feasible solution to the above linear program is a correlated equilibrium. So, that concludes the proof of the theorem, because linear programs can be solved in polynomial time ok. So, does a correlated equilibrium always exist? Very important question. Is the linear program always satisfiable, does a correlated equilibrium always exist in a finite strategic form game? The answer is yes.

How? Because the or let me write this way, every mixed-strategy Nash equilibrium can be equivalently viewed as correlated equilibrium. So, we will elaborate this a more in the next class. Ok.

Thank you.