Algorithmic Game Theory
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Lecture - 25
Sperner's Lemma

Welcome. So, in the last class we started studying complexity of finding an MSNE in a strategic form game. So, we will continue that study in today's class.

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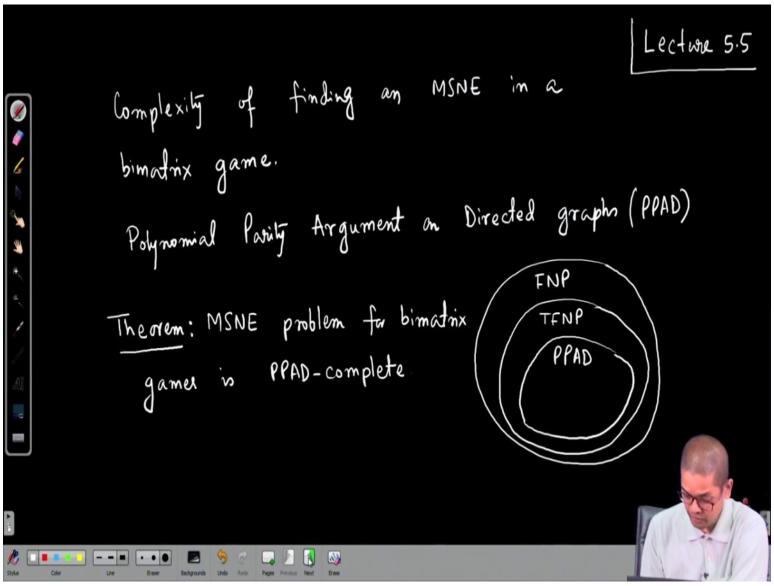

So, complexity of finding an MSNE in bimatrix game, towards that we observe that classical complexity theoretic framework of NP completeness is not useful and we introduced a complexity class called Polynomial Parity Argument on Directed graphs, PPAD in short.

And if you recall the journey was we started with NP and we defined a complexity class called FNP, functional NP where the job is to not only give yes no answer, but for the e s case we are supposed to give a certificate. Then we observed that; then we observe that the MSNE computation problem cannot be FNP complete, if it happens then we have NP equal to co-NP which is quite unlikely according to current understanding of our complexity theory.

Then we defined a subclass of it which we called total FNP or TFNP and we observed that TFNP is a semantic class which does not have any complete problem. Then we defined an appropriate subclass of TFNP which we called PPAD and this is the cornerstone result in algorithmic game theory whose proof is beyond the scope of this course, that MSNE problem for bimatrix games is PPAD complete.

And we gave a brief sketch there were intermediate problems like Brower's and Sperner's. And today we will prove Sperner's lemma which is a very beautiful combinatoric result and forms the heart of Nash's theorem. So, let us see Sperner's lemma.

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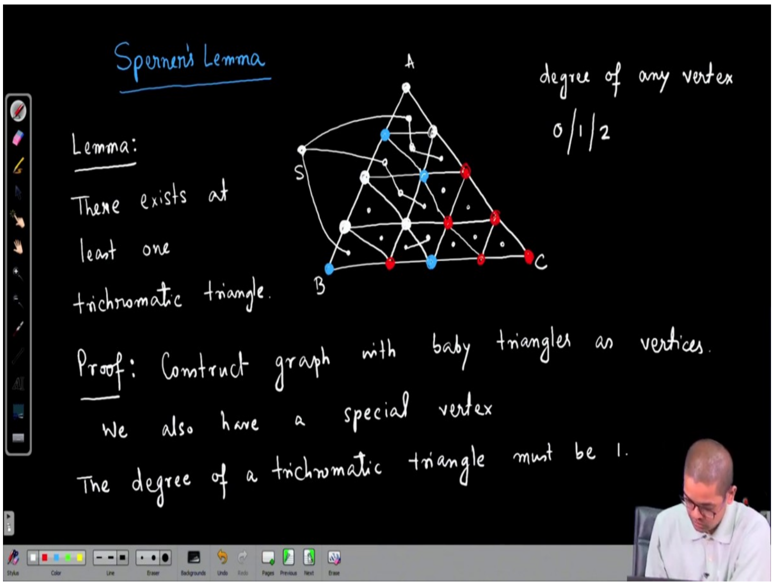

So, think of a equilateral triangle on a plane and what we do is that we subdivide this equilateral triangle. What do we mean by subdivide? We take midpoints of each of the three edges and join them.

So, from 1 triangle I get 4 triangles and I repeat the same process for these 4 triangles also and I keep doing this and stop at any arbitrary level. So, let us call the vertices say A, B, C. Next what I do is that I color the vertices the junctions, I color a with white color and B with blue colour and C with red color. Then the vertices on the outer edges of the triangle for example, the vertices on the edge A B I am allowed to color any color from white or blue, because its end points are white and blue. So, let me do any color.

This is just one way of coloring the vertices along A B, same thing I do for the other edges for B C the allowed colors are green and red and for C A the allowed colors are red and white. So, these are the; these are for the vertices on the outer edges. How about vertices inside the triangle? There I have complete freedom I can color the inside vertices with any color of our my choice either white blue or red.

So, let me perform any color. So, this is one legal coloring of this subdivided triangle and what Sperner's lemma states that, lemma: there exists at least one trichromatic baby triangle that that; that means, the triangles which does not contain any other triangle, trichromatic triangle. This is the nice combinatorial statement and it turns out it is it forms the heart of Nash's theorem and it is also used, its computational version is also used in the proof of the theorem that MSNE problem is PPAD complete.

How does the proof goes? So, there are various ways to prove it. So, let me present one way, what I do is that I first introduce an outs a vertex outside, let us call this vertex S and I draw an edge from between every white and blue edge. So, I draw an edge here ok and I introduced vertex for every triangle for every baby triangle. So, I am constructing a graph, the vertices are baby triangles and there is a special vertex outside. And how I add edges? I just add edges crossing the white and blue edge.

So, here is an edge, here is another edge, so another edge, here is another edge, here is another edge, so another edge, another edge. Here is another edge, is there any white blue edge ok? So, it seems that these are the all white blue edges, so these are my graph. So proof: construct graph with baby triangles as vertices, you also have a special vertex.

Now, we ask what could be the degrees of the vertices? Degree is the number of edges incident on a vertex, degree of any vertex could be 0 or 1 or 2, it cannot be 3 because for a triangle it cannot have all 3 edges have a red blue edge sorry white blue edge. So, we have an edge for every white blue edge. So, a triangle cannot have all three edges white blue, so its degree cannot be 3. So, its degree can be 0, 1 or 2.

Now, what is the degree of a trichromatic triangle? Trichromatic triangle has only one edge which is colored white and blue. So, the degree of a trichromatic triangle must be 1, it is a easy observation because a trichromatic triangle can have at is or have exactly one blue white edge.

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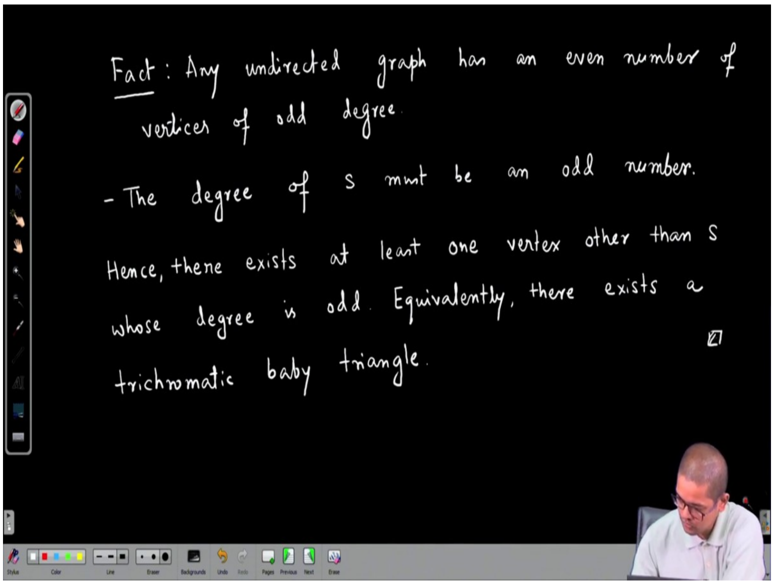

Good, now here is a standard result of graph theory, let me state. Fact: whose proof is not difficult any undirected graph has an even number of vertices of odd degree ok.

So, that we will apply here, but before that what is the degree of S? You know look at the A, B edge and its the color of A is white, color of B is blue, it is a plain observation that the number of blue white edges across that edge A B must be an odd, because the color of A is white and color of B is blue. So, the degree of S must be an odd ok, odd integer. The degree of S must be an odd number, but this fact says that the number of odd degree vertices must be even and I have already found a vertex namely S whose degree is odd.

Hence, the number of odd degree vertices cannot be 0, it must be at least 2, hence there exists at least one vertex other than S whose degree is odd. And odd degree vertices are exactly trichromatic baby triangles, equivalently same statement there exists trichromatic baby triangle which completes the proof.

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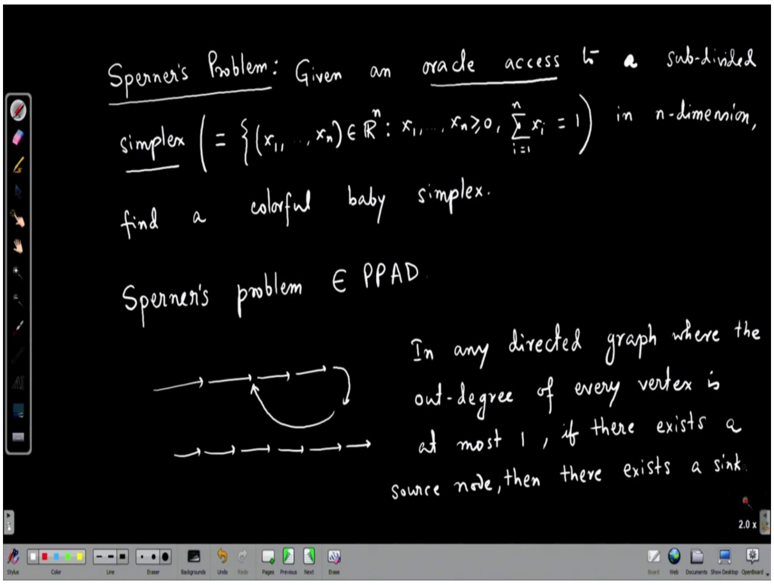

Now, we can find, we can write a computational problem around this basic Sperner's lemma which is called Sperner's problem. So, given an oracle access to subdivided simplex. What is sub divided simplex? It is just a fancy term for equilateral triangles or a triangle for 2-dimension for 2-dimension. For higher dimension you can think of this as 3-dimensional objects and so on. And so, what is simplex let me write, simplex in n dimension?

So, simplex is the set of all points these points x one to x n in n dimension n dimensional simplex is this in $\Re^n$, such that $(x_1, \ldots, x_n)$ is greater than equal to 0 and $\sum_{i=1}^{n} x_i = 1$. So, in higher dimension also you can think you can subdivide that simplex. And what do you mean by oracle axis this it means that this simplex is not given explicitly, you can query a sub divided simplex and get to know its color.

So, that is a succinct representation of the input. So, given an oracle axis to a subdivided simplex in n-dimension find a colorful. So, in n-dimension you use n colors and you find a colorful baby simplex and here by efficiency we mean you need to be efficient in n. So, you the a simple algorithm of simply iterating over all baby simplex is not simply efficient enough.

So, one thing is easy to see that not very easy, but we I we will explain, Sperner's problem belongs to the class PPAD, why? At first look it does not look like that, because recall what is the what are the problems in PPAD, it is like we are searching for a sink node or another source node in a very large directed graph. So, I have a directed graph whose out degrees are one, but you know this graphs can have cycles.

But the it is a straight fact from graph theory is that if there is a source node, so I am given one source node and I and my task is to find either a sink node or another source node. So, it is a graph theoretic property that, if in any directed graph where the out degree of every vertex is at most 1, if there exists a source, source node. A node is called source node if it is in degree is 0, if there exists a source node then there exists a sink node.

Now, this particular perspective may not be immediately clear, but recall in PPAD any problem in PPAD has two algorithms. One is that given a given a solution or it will either the algorithm either gives the next solution or outputs that it is locally optimal. So, this graph you can think of as a graph on solution space, but still it is not clear why Sperner's problem belongs to PPAD.

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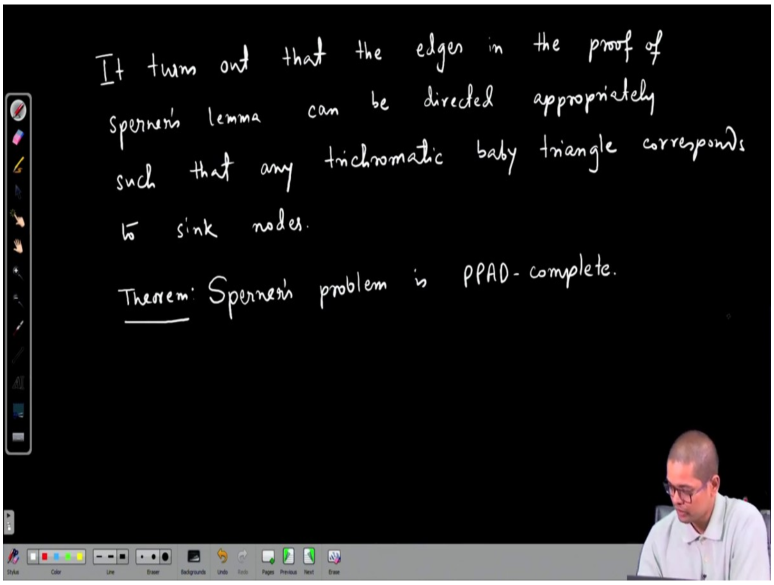

And the reason is that it turns out that just go to the proof of Sperner's lemma that we have just done, it turns out that the edges in the proof of Sperner's lemma can be directed appropriately such that such that any trichromatic baby triangle corresponds to sink nodes.

So, we will not go into it and we need slight change, but yeah almost same construction works only edges needs to be directed. So, this shows that the Sperner's problem belongs to PPAD. And it turns out that Sperner's problem is also PPAD complete and this also is a very important step in the proof of the theorem that the that MSNE problem is PPAD complete. So, again we will not see the proof of PPAD heartiness, but let me state Sperner's problem is PPAD complete. So, we will conclude here today.

Thank you.