Algorithmic Game Theory
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Lecture - 22
PSNE for Symmetric Congestion Games

Welcome. In the last class, we have proved that the PSNE problem for congestion games is PLS complete.

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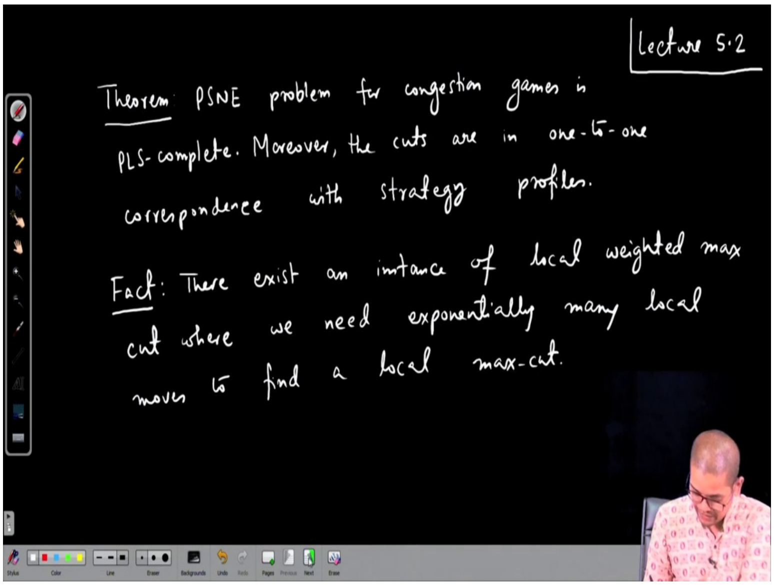

So, what did you prove? That this theorem we proved PSNE problem; that means, given a game find a PSNE. PSNE problem for congestion games is PLS complete. Not only that we showed that moreover, we showed that moreover the cuts are in one to one correspondence with strategy profiles and we will use that.

So, one result is known that for weighted local max cut, the local if we keep performing local move; we may need to perform exponentially many local moves before finding a local max cut. So, let us write this as a fact. So, there exist an instance of local weighted max cut, where we need exponentially many local moves to find a local max cut. Now, because max cuts are in one to one correspondence with strategy profiles, not only that local moves are also one to one correspondence with unilateral deviations.

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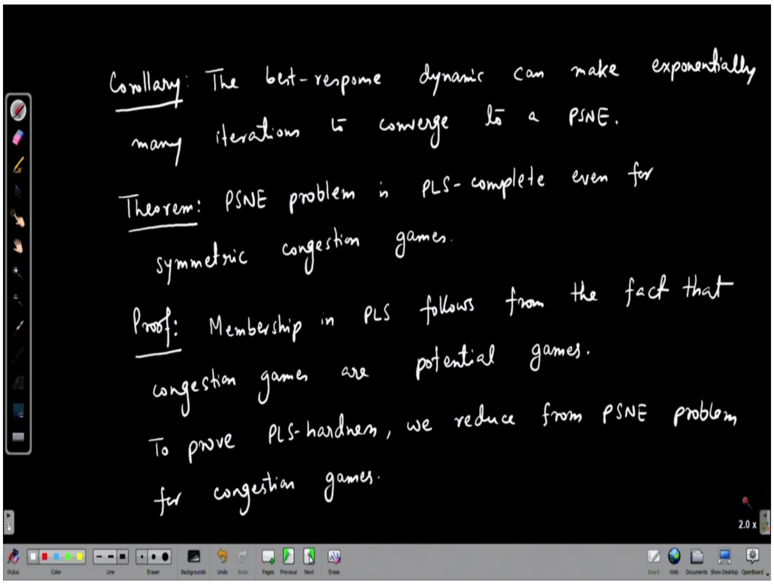

We get that; so, here is a claim or corollary of the above theorem and the fact that the best response dynamics can make exponentially many iterations to find to converge to a PSNE ok. But, still it is so, so definitely best response random does not work and it should not work. It we do not expect it to work, because we have proved that the PSNE problem for congestion game is PLS complete.

If we have an algorithm and in particular if the best response dynamic able to find a local able to find a PSNE for congestion game in polynomial time. Then, we can use that algorithm to find a to find a local max cut, local weighted max cut in polynomial time which would be a breakthrough. But, you see so, we ask the next special next easier question ok.

So, for congestion games you cannot find, how about for symmetric congestion games? So, our next theorem shows that even for symmetric congestion games finding a PSNE is PLS complete; that means, we do not expect to have a polynomial time algorithm for finding a PSNE even for symmetric games. So, what is the theorem? PSNE problem is PLS complete even for symmetric congestion games.

Proof. So, again for PLS completeness we need to show two things. One is membership in the complexity class PLS and other is PLS hardness. So, membership in PLS follows from the fact that congestion game congestion games are potential games, every congestion game you know is a potential game. And, hence to find the PSNE you it is enough to find a local maxima or local minima of the potential function. So, that is why it is a; it is an abstract search problem and hence it belongs to PLS. To prove PLS hardness, we reduce from PSNE problem for congestion games ok.

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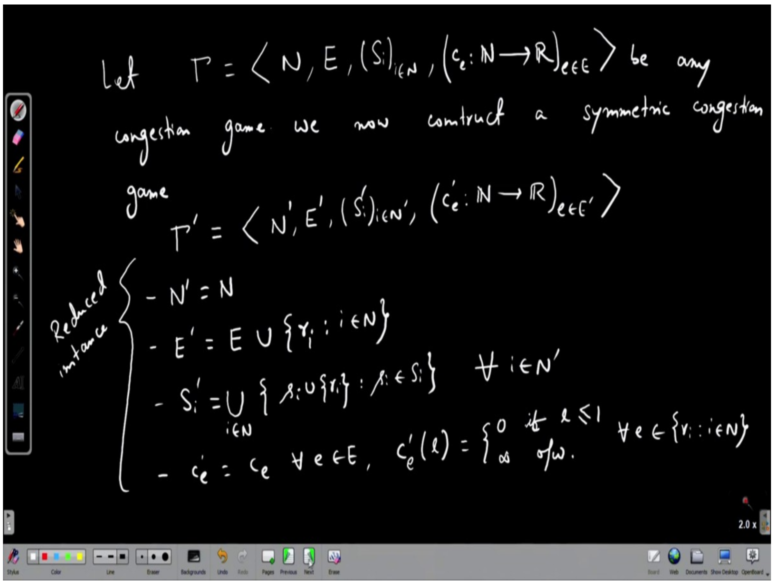

So, let Γ equal to N set of players, E set of underlying resources, then strategy set S_i of the players, i in N and a cost function from set of resources. Now, for each resource there is a cost function from the load to real numbers. This cost function we have for each set of resources. So, this be any congestion game ok. So, we construct a we now construct a symmetric, we now construct a symmetric congestion game $\Gamma' = \langle N, E, (S_i)_{i\in N}, c_e : N \to R \rangle$.

So, set of players will change, E will change, E', N', $e \in E'$. What is N'? N' remains same, the set of players remains same. What is the set of resources? E'. The original set of resources remains plus for each player i, we introduce a resource r_i for $i \in N$ ok. So, S'_i and what is S'_i? S'_i is you look at the strategy s_i of any player i and you augment r_i. What is S_i? $S_i \subseteq E$ and you include r_i there for all $s_i \in S_i$.

So, for with every strategy of player i you put the strategy, you put the resource r_i and you know it is a symmetric congestion game. So, the strategy set should be same for all the players. So, S'_i should be same and so, you take the union $i \in N$. So, this is for all

$i \in N^{'}$. So, all the players have the same set of strategies. Recall, in the symmetric congestion games, all players must have the same set of strategies.

And, cost function of c let call it $c_e^{'}$, $c_e^{'} = c_e$ for all $e \in E$. The original set of resources their cost remain same. And, the new resources which is which are r_i; so, $c_e^{'}$ of this load l, this is 0 if l is less than equal to 1 and infinity otherwise; that means, this resources this should hold for all resource $e \in r_i, i \in N$. So, for those newly introduced resources at most one player if is allowed to use, if two players use the same resource then the cost becomes infinite.

So, this is our reduced instance, reduced instance is this is first algorithm, algorithm a. The second algorithm I need to construct a solution of the solution or PSNE for the congestion PSNE for the congestion game from the PSNE of network congestion game.

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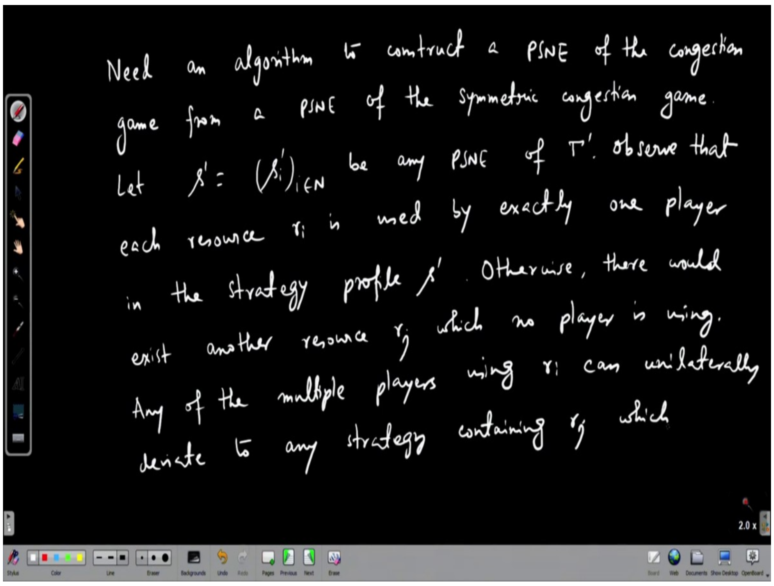

So, now let me write need an algorithm to construct a PSNE of the congestion game from PSNE of the symmetric congestion game. So, you take a PSNE and first observe that; so, let $s^{'}$ equal to $s_i^{'}, i \in N$ be any PSNE of $\Gamma^{'}$. First observe that each resource must be used at most once by at most one player. So, otherwise it cannot be the it cannot be a local minima for the potential function and in particular there will exist a player i for which there is a beneficial move.

If two players are using one resource r_i; that means, there exist another resource r_j which no player is using. And, one of the two players who are using r_i, one of them they can move to any other strategy involving r_j reducing the cost. In particular, there exists a beneficial move for that player which contradicts our assumption that is a PSNE.

So, first observe that each resource r_i is used by exactly one player in the strategy profile $\bar{s}$. Otherwise, there otherwise there would exist another resource r_i which no player is using. Now, any player any of the multiple players using the resource r_i can unilaterally deviate to any strategy containing r_i which reduces the cost of the deviating player. However, this contradicts our assumption that s prime is a PSNE.

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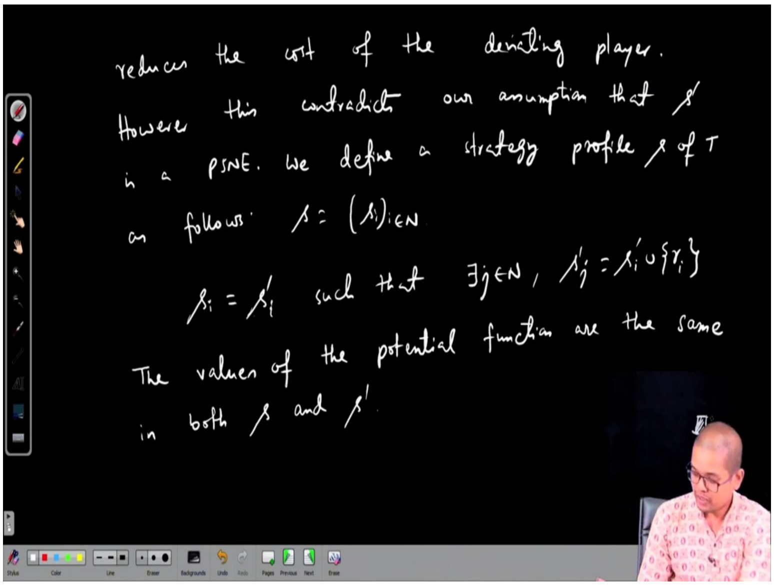

So, all the strategies all the resources r_i's must be used by exactly one player. Now, how will I define, how will I define the strategy profile of Γ from s'? So, we define a strategy profile s of Γ as follows. So, is like $(s_i)_{i \in N}$. So, what is s_i? s_i equal to you know $s_i = s_i'$ such that there exist a $j \in N$ with s_j' is nothing, but $s_i \cup \{r_i\}$. So, this should be this.

So, because each resource say r_i must be used by exactly one player. So, suppose the jth player is using that resource and $s_j' = s_i \cup \{r_i\}$, define then $s_i = s_i'$ where $s_j' = s_i \cup \{r_i\}$ ok. And, we observed that the value of the potential function values of the potential function are the same in both s and s'. So, here again the values of the potential function are in one to one correspondence. So, if s prime is a local optimal, then s is also a local optimum and that concludes the proof of PLS hardness.

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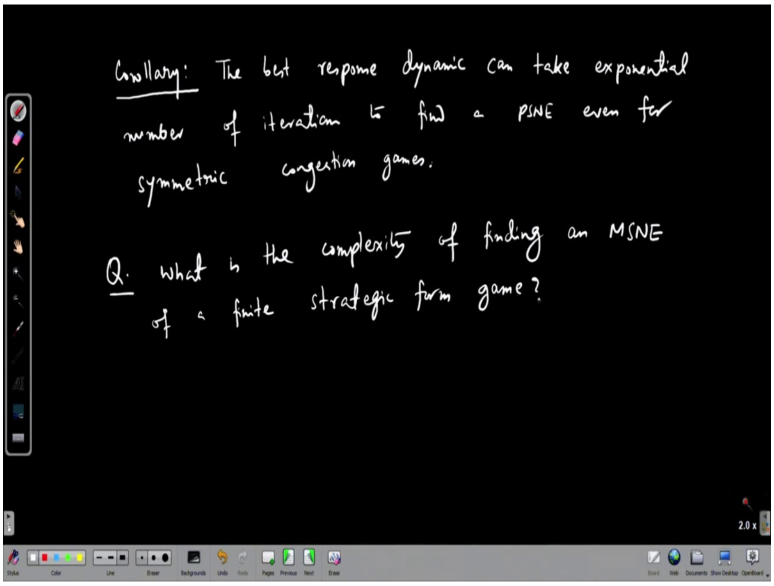

So, what in particular we show it shows because it is in one to one correspondence, the strategy profiles are in one to one correspondence, the potential values are also in one to one correspondence; we get the following corollary. Recall, we know that the best response dynamic can take exponentially exponential number of steps to reach a PSNE for potential games.

And, because of the because of the reduction above the strategy profiles of the potential games, the strategy profiles of congestion game and symmetry congestion games the reduced symmetric congestion games are in one to one correspondence; we can say that the best response dynamic can take exponential number of iterations to find a PSNE even for symmetric congestion games ok.

So, this sort of answers our question of why we do not expect to have a fast or efficient algorithm for finding a PSNE in congestion game why we do not expect to relax. Along, similar lines we can prove that we do not we cannot relax the our relax our assumptions under which the network congestion for network congestion gains the max gain version of ϵ best response dynamic converges fast.

So, but we have still not hm; so, this is about PSNE, but from in the next lecture we will focus on MSNE. So, what is the main focus? So, question, what is the computational complexity of finding an Mixed Strategy Nash Equilibrium: MSNE, MSNE of a finite strategic form game? See that you know, now again we cannot convert this problem into a decision problem and maybe we cannot ask like ok, what is the complexity of finding if a finite strategic form game has a MSNE or not?

Because, the answer is trivially yes, you do not need to do any computation and that is because of the Nash theorem. So, again we will define further complexity classes, some specialization of this or strict subset of the complexity class PLS which are suitable for studying MSNE. And, this we will continue in our next class, that will be the topic of our next class.

Thank you.