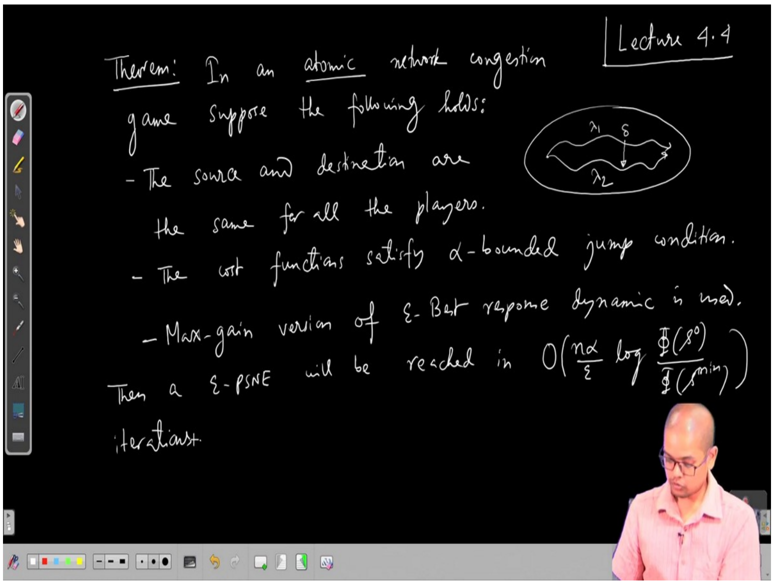
Algorithmic Game Theory
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 19
Fast Convergence of Best Response Dynamics

Ok, welcome. In the last class, we were studying Best Response Dynamic and epsilon best response dynamics and potential games. And at the end, we stated a theorem which states that under mild assumptions the atomic network congestion games always for atomic network congestion games epsilon based response dynamics always converges to a pure epsilon PSNE quite fast. Now, we will see its proof today.

(Refer Slide Time: 00:58)

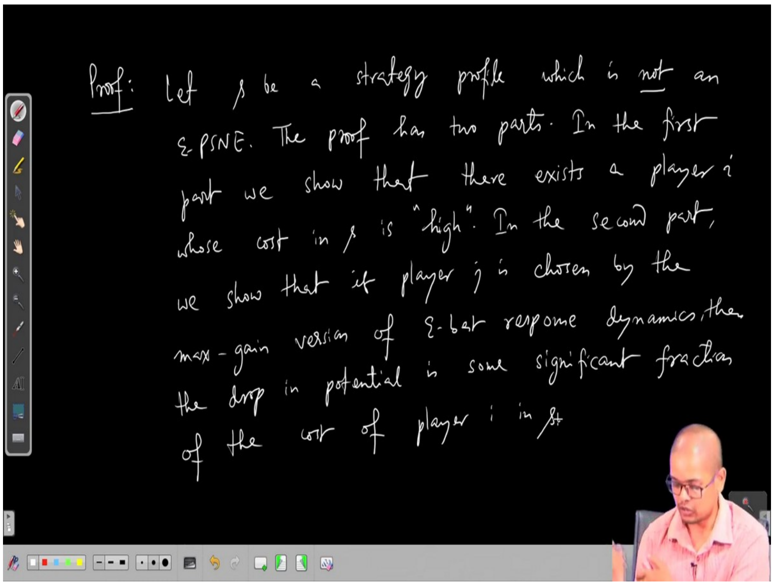


So, let us briefly recall what was the theorem, in an atomic. So, what does the word atomic means? Atomic means, it means that the number of players is. So, high that between every two paths every pair of paths and every small and for all small numbers, let me write here what does atomic mean. So, suppose here is a path and here is another path.

And suppose λ_1 is the amount of fraction of total traffic that is following this path the top path, and λ_2 is the fraction of total traffic which follow which are following the bottom path. It says that for any arbitrarily small positive integer say δ, it is possible to shift a delta fraction of traffic from say top path to bottom path that is what the what atomic means.

In an atomic network congestion game, suppose the following holds. So, what are the conditions? The source and destination are the same for all the players; are the same for all the players. The cost function satisfy alpha bounded jump, the cost functions satisfy alpha bounded jump condition. And max gain version of version of epsilon best response dynamic is used. Then, and epsilon PSNE will be reached in $O\left(\frac{n\alpha}{\epsilon}\log\left(\frac{\phi(s^0)}{\phi(s^{min})}\right)\right)$ number of iterations, this in this iterations ok. So, we will see up see its proof today.

(Refer Slide Time: 05:21)



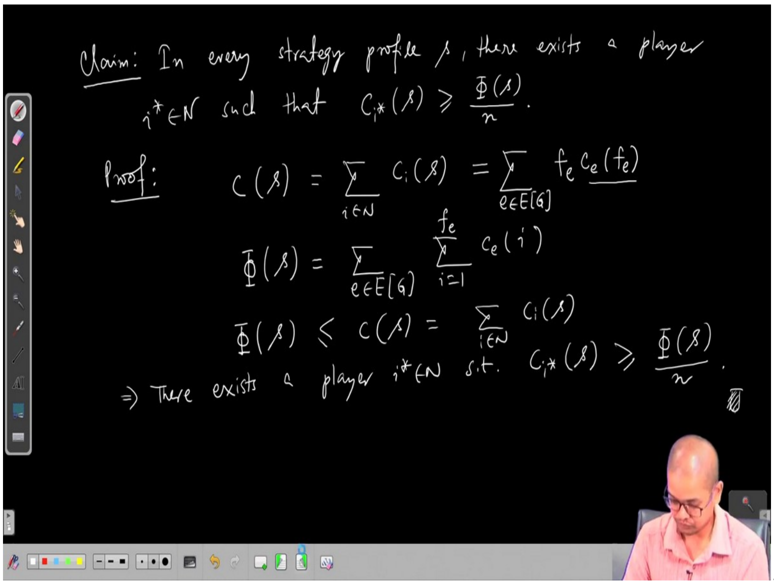
Proof ok. So, let s be a strategy profile which is not an epsilon PSNE. What is the high level idea of the proof? We will show that you know in every iteration the drop in potential is significant and which makes sense. We are starting at $\phi(s^0)$ and if we have to reach at $\phi(s^{min})$ then at every, and if we have to reach fast, Then, every step we should be able to reduce the potential by significant amount and that is the spirit of the proof. So, the proof has two parts.

Logically, in the first part we show that there exists player i there exist a player whose cost in the strategy profile s is high, that is the first part. With this, in the second part in the second part, we show that you know if player j is chosen by the max gain version of epsilon best response dynamics, then the drop in potential is some significant fraction of the cost of.

So, in the first part we are showing that there exist a player say i whose cost is high; and the second part will show that if player j is chosen and in the chosen in the max gain version of the epsilon best response dynamics, then the drop in potential is some good fraction of the cost of player i in s. So, that is the high level structure. So, first part there exist a player i whose current cost is high and in the second part will show that if player j is chosen and the move that is chosen by the max gain version.

Then, the drop in potential which is same as the drop in cost for player j is some significant fraction of player i. And because player is cost is high that implies that are the dropping potential is also high, and from there we will we will be able to conclude.

(Refer Slide Time: 10:35)



So, first part there exist a player i whose cost is high. So, to make the proof modular, I have broken it down into few claims. So, in every strategy profile s there exists a player. Let us call it $i^* \in N$, such that $C_{i^*}(s) \geq \frac{\phi(s)}{n}$; that is what we mean by the cost is high. Proof: very easy proof. So, we define $C(s)$ a function on the strategy profiles, it is not a potential function. It is simply the sum of costs. So, now, what is the sum of costs?

From edge wise, the cost of player i let us recall in the network congestion game the cost of player i is see its path and you add the cost of all the edges in that path. Now, you look at the same sum from edge perspective.

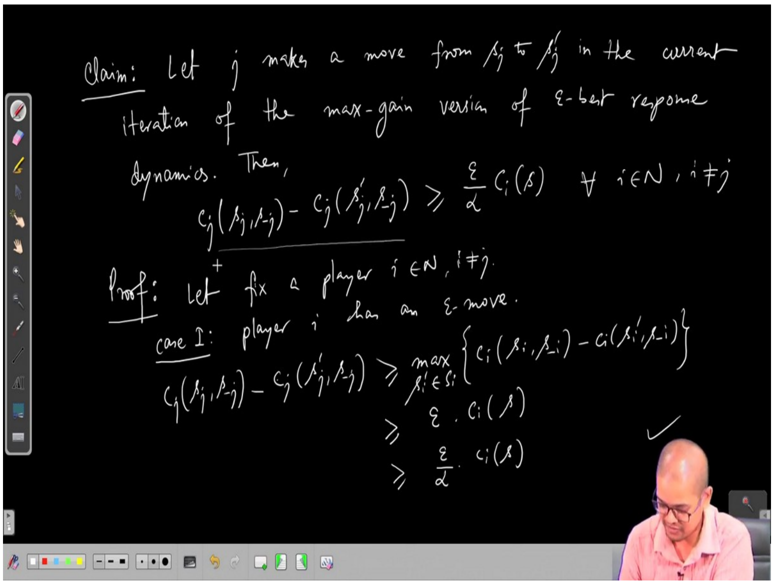
So, you go over all edges $e \in E[G]$. Now, what is the cost of this edge? $c_e(f_e)$; and how in to how many players this must this particular cost $c_e(f_e)$ is getting contributed that f_e many players, because that is the flow. Now, we will compare this with potential function. What is potential function? What was Rosenthal's potential for network congestion game? Let us recall, $\phi(s)$ is $\sum_{e \in E[G]} \sum_{i=1}^{f_e} c_e(f_e)$ ok.

Now, I would again request you to pause this video and you try to see your proof that $\phi(s)$ is less than equal to $C(s)$. Please pause the video and try to prove it yourself, it is not a difficult proof ok. So, let me explain. So, you see that you see this these two quantities $\phi(s)$ and $C(s)$ from edge perspective and this. So, focus on a focus on an edge e and the contribution of this edge e to $C(s)$ is $f_e \times C(s)$ you can see $c_e(f_e)$.

You can think of we are I am adding $c_e(f_e)$, f_e many times. But, for potential function I am adding $c_e(f_e)$ only once and I am adding $c_e(f_e - 1)$ and then I am adding $c_e(f_e - 2)$ and so on. Because, c_e is a cost is a non decreasing function, then the contribution of this edge e is always at contribution of this edge e in the potential function is at most the its contribution in $C(s)$.

And this holds for all the edges and so we have this inequality. Now, what is $C(s)$? $C(s)$ is nothing but sum of costs. So, if some of this n numbers is at least phi of s and simply by averaging principle, if sum of n numbers is at least something then the average there exists at least one term whose value is at least average then there exist implies there exists a player i. Let us call i star in N such that C i star of s is greater than equal to $\frac{\phi(s)}{n}$ ok. So, this concludes the proof of the claim ok.

(Refer Slide Time: 16:49)



So, this is the first part. There exists a player i namely i^* whose cost is high means at least $\frac{\phi(s)}{n}$. Now, we prove the second part, claim: So, let j makes a move from s_j to $s_j^{'}$ in the max in the current iteration of the max gain version of epsilon based response dynamics. Then, we want to show that that change in potential is significant.

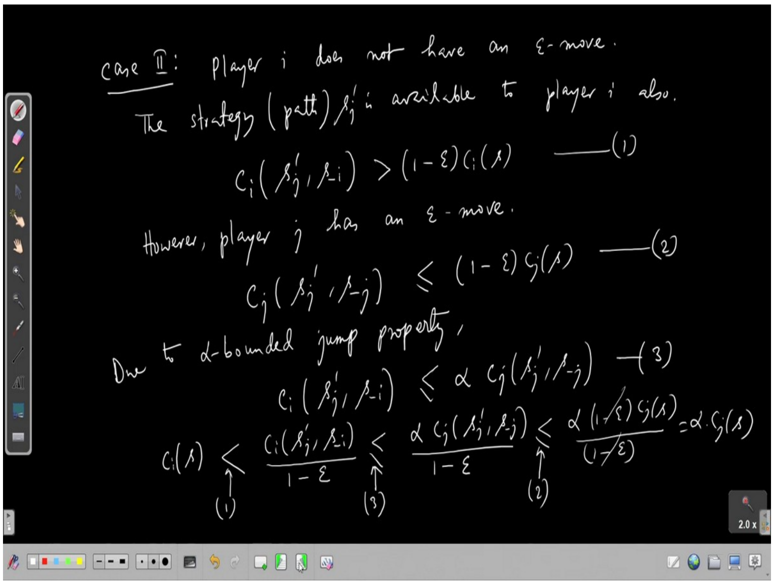
Now, change in potential is same as change in cost of j; that means, $C_j(s_j, s_{-j}) - C_j(s_j^{'}, s_{-j})$. This is greater than equal to epsilon by alpha times $C_i(s)$ for all player $i \in N$. In particular, for that player i^* whose cost is high. So, this shows that this is this is greater than equal to for all $C(s) \alpha \epsilon$ by α times the cost of all the players.

So, proof: So, two cases case 1. So, let us fix a player i. So, let fix a player $i \in N$ for which we will show this result and they say this is an arbitrary player. So, first case, case 1 is suppose player i has an epsilon move; now because player i has an epsilon move, but player i has not been. So, we can assume $i \neq j$. So, $i \neq j$. So, although player i has an epsilon move, but player i still has not been chosen by the epsilon best response dynamics the max gain version of epsilon best response dynamics.

That means, the drop in cost of player j s is at least the maximum drop possible in the cost of player i by unilateral deviation ok good, but you know player i has an ϵ move; that means, it has a unilateral deviation which can reduce its cost by ϵ fraction its current cost. So, this is and if I am taking max; that means, is greater than equal to $\epsilon \times C_i(s)$.

And because α is greater than one this is again greater than equal to $\frac{\epsilon}{\alpha} \times C_i(s)$ which is what we need to prove in the claim. So, for case 1 we are done. So, for case one when player i has an ϵ move it is very easy to prove.

(Refer Slide Time: 22:27)



Case 2: we just need to do little bit more work. Player i does not have an epsilon move ok. Now, you see till now we have not used so many assumptions in the statement for example, alpha bounded jump; for example, all players have the same source and destination. Now, we are we will use the our assumption that all players have the same source and destination.

Because player i and player j has the same source of destination, the strategy which is nothing but a path for network congestion game is available the strategy s_j the strategy $s_j^{'}$ is available to player i also, but it is not reducing its cost by at least epsilon fraction its current cost. So, you have that cost of i when it deviates from s_i to $s_j^{'}$ unilaterally, this is more than $1-\epsilon$ times its current cost is equation 1. On other hand, this particular strategy $s_j^{'}$ reduces the cost by at least epsilon fraction of its current cost for player j.

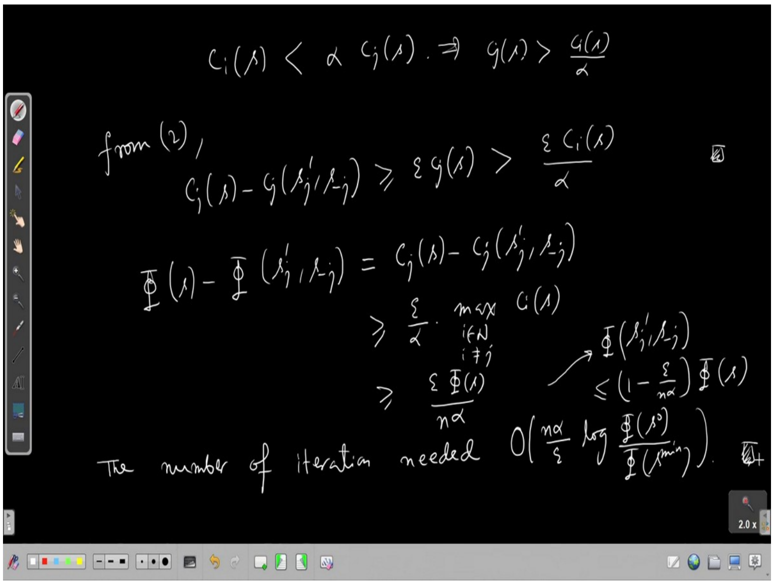
So, C_j let me write; however, player j has an ϵ move; that means, it has a strategy it has a unilateral deviation which reduces its cost by at least ϵ fraction its current cost and we are taking the max gain version. So, $C_j(s_j^{'}, s_{-j}) \leq (1-\epsilon) \times C_j(s)$ is equation 2. Now, you see, now we will use the alpha bounded jump property.

So, from 1. So, what we had from 1? Not from 1, due to α bounded jump property due to α bounded jump property you know you compare with the cost $C_i(s_j^{'}, s_{-i})$ with this cost $C_j(s_j^{'}, s_{-j})$. You see that you know for each edge, the traffic can the load can change by at most 1. So, this is less than equal to α times this. So, the cost of each edge can differ by at most 1. This is less than equal to α times this equation 3.

Now, you see we will compare $C_i(s)$ and $\alpha C_j(s)$, let us see. So, what is $C_i(s)$? So, from one $C_i(s)$ is greater than $C_i(s_j^{'}, s_{-i})$ by $1-\epsilon$, but $C_i(s_j^{'}, s_{-i})$, sorry this is not greater than this less than; this is from equation 1. Now, you we apply equation 3, and this is less than equal to $C_i(s_j^{'}, s_{-i})$ is less than equal to $\alpha C_j(s_j^{'}, s_{-j})$ by $1-\epsilon$. So, this inequality is due to equation 3, inequality 3.

And now we apply inequality 2 this is less than equal to $C_j(s_j^{'}, s_{-j})$ is $(1-\epsilon) \times C_j(s)$ by $1-\epsilon$ this $1-\epsilon$ cancels and what we have is $\alpha C_j(s)$. So, what I have is this is equation 2 $C_i(s) < \alpha C_j(s)$ very good.

(Refer Slide Time: 29:25)



And now we also have this chain of inequality. So, let me see yeah previous page. So, from equation 2 you see the drop. So, what we need to show? We need to show that

$C_j(s_j^{'}, s_{-j})$. So, let us write this quantity; by equation 2, this quantity is at least $\epsilon \times C_j(s)$ by equation 2. So, from 2, $C_j(s) - C_j(s_j^{'}, s_{-j})$ is at least $\epsilon C_j(s)$; and $C_j(s)$ is. So, from here $C_j(s)$ is strictly greater than $C_i(s)$ by α .

So, you put it here is strictly greater than $\frac{\epsilon C_i(s)}{\alpha}$. This is exactly what we need to show in that claim. So, what we have is that is that the drop in potential $\phi(s) - \phi(s_j^{'}, s_{-j})$ is what? It is a drop in cost of player j $C_j(s) - C_j(s_j^{'}, s_{-j})$. And is what? This is at least epsilon sorry epsilon by alpha times max over C i of s because that this holds for all $(C_i)_{i \in N, i \neq j}$.

And we have shown that there exist a i namely i^* for which $C_i(s) \geq \frac{\phi(s)}{n}$. So, we get $\frac{\epsilon \phi(s)}{n \alpha}$. So, the current potential current value of potential drops by $\frac{\epsilon}{n \alpha}$ fractions. So, the number of iterations needed to find an ϵ PSNE. Now, what is this? Is let us see just need some more calculation.

So, from here we see that $\phi(s_j^{'}, s_{-j})$ is less than equal to $1 - \frac{\epsilon}{n \alpha \phi(s)}$, correct. Now, from here I let you check that the number of iterations needed is $O\left(\frac{n\alpha}{\epsilon} \log\left(\frac{\phi(s^0)}{\phi(s^{min})}\right)\right)$ which proves the theorem ok.

Thank you.