Algorithmic Game Theory
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Lecture - 18
Best Response Dynamics

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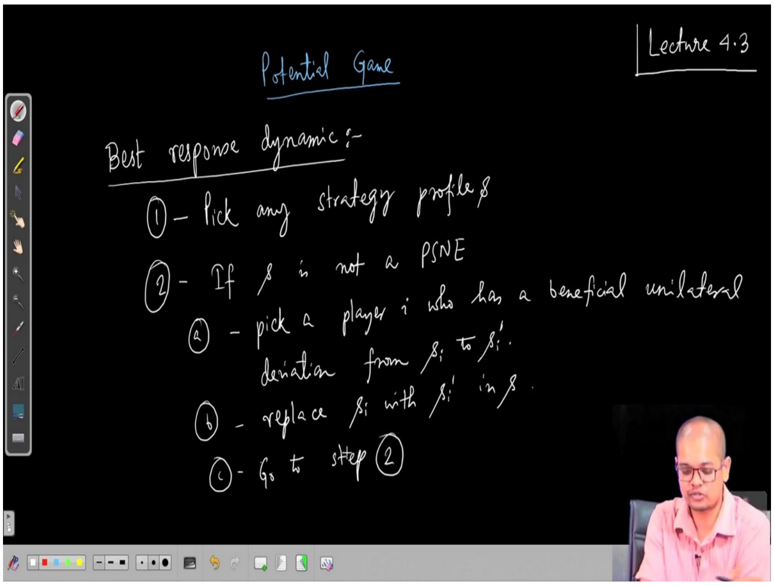

Welcome to today's lecture. In the last class we started studying Potential Games. This is an important class of game because, every potential game has a pure strategy Nash equilibrium there exists a pure strategy Nash equilibrium, every finite potential game. And, the very reason is because it admits a potential function. So, what is a potential function? Let us recall. It is the it is a function from set of strategy profiles to real numbers such that if any player makes a unilateral deviation, then change in utility; change in utility of that player is exactly the change of potential.

Now, if the game is finite, then it has it has a maximum value and if you look at the maximum value, the corresponding strategy profile is a is a pure strategy Nash equilibrium, because there cannot exist any unilateral deviation which can increase the utility. If you look from cost perspective, then the minimizer the minimum value of the potential function the corresponding strategy profile which gives minimum value is a pure strategy Nash equilibrium.

Now, today we will see a very natural algorithm based on that proof technique which is called best response dynamics. So, what is the best response dynamic? It is a iterative algorithm, which for potential games for a finite potential games it will converge to a pure strategy Nash equilibrium. So, let us write down the algorithms. Pick any strategy profile s; strategy profile s.

If s is not a PSNE, then there must exist a player i for which there is a beneficial unilateral move. Pick a player i, who has a beneficial unilateral deviation from s_i to $s_i^{'}$. And then simply replace s_i with $s_i^{'} \in S$ and so let me write this. This is step 1, step 2, 2 a, b and c, c is go to step 2. So, basically you iterate you keep changing your current strategy profile until it is not a PSNE.

If it is not a p a until it is a PSNE and if it is a PSNE you have found it, you can output it and if it is not a PSNE then there must exist a player i, where is the beneficial unilateral deviation unilateral move from s_i to $s_i^{'}$ and then you replace s_i with $s_i^{'} \in S$ and then you repeat the entire process until you find a PSNE.

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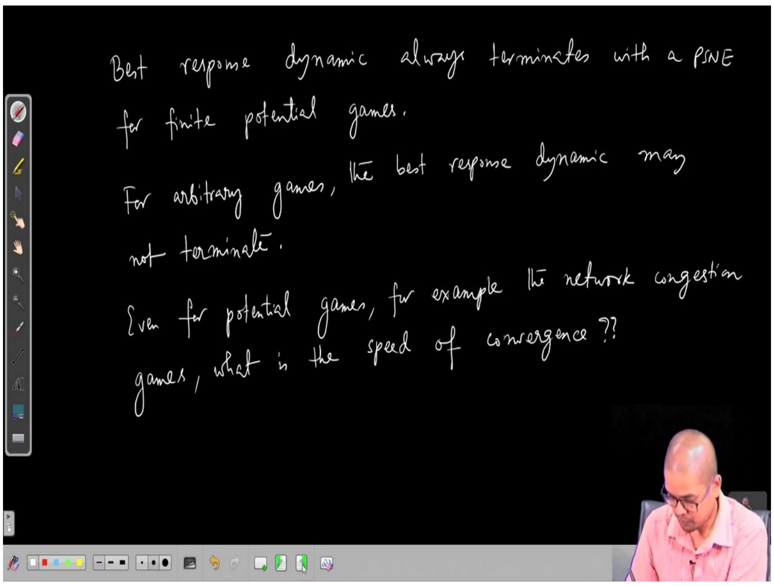

Now, why this process should terminate? This not this let me write. Best response dynamics, always terminates with PSNE for finite potential games. Why? Because, the potential value, if you are looking if you are working with cost function the potential value is always decreasing strictly decreasing and that is why you will never encounter two encounter the same strategy profile twice.

But of course, you know for non-potential games, for arbitrary games the best response dynamic may not terminate, ok. Not only that you know even for potential games, what is the speed of convergence? So, even for potential games, for example, the network congestion games which is a potential games we have proved network. What is the speed of convergence? That this is a very important question and we will come back to it. So, but you know there are certain easy cases. So, if the potential function happens to take only polynomially many values.

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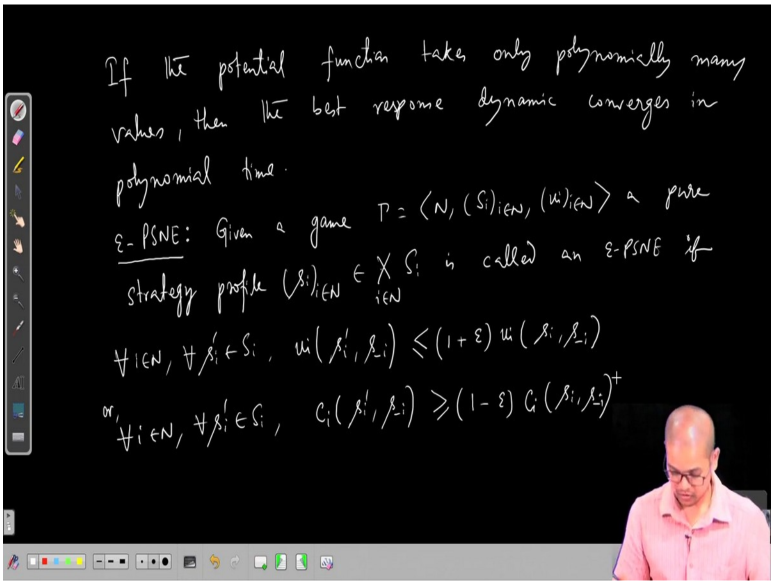

If the potential function takes only polynomially many values; polynomially many values, because potential value is always strictly decreasing you do not even revisit same value twice, then the best response dynamic. Of course, converges in polynomial time. However, if the potential function happens to take infinitely many values, they or exponentially many values cannot take infinitely many values if it takes exponentially many values, then it is possible that the best response dynamic takes exponential time to converge.

And thus, exponential time to find the PSNE because you know many many times many potential games like for example, network congestion games are succinct games. So, exponential time is not allowed there. Because, the input is not that big, input is represented succinctly. What we will show is that for network congestion game, you know this best response dynamic converges quickly under some mild assumptions. So, towards that, so let me define few notions.

So, we define what is called epsilon PSNE, so again given a game in normal form. Pure strategy profile $(s_i)_{i \in N} \in \times_{i \in N} S_i$, a pure strategy profile is called an epsilon PSNE. If for all player $i \in N$, for all $s_i, s_i' \in S_i$, the utility of player i when it unilaterally deviates from s_i to s_i' it can increase, but it cannot it should not increase by more than $1+\epsilon$ times, ok.

For, if we if you are working with cost, so or equivalently or if you are working with costs then cost functions, then the condition that we write is for all $i \in N$, for all unilateral deviation to $s_i' \in S_i$ cost of i in (s_i', s_{-i}) is greater than it can drop, but it should not drop by more than $1-\epsilon$ times its current cost. This is what is called an epsilon PSNE.

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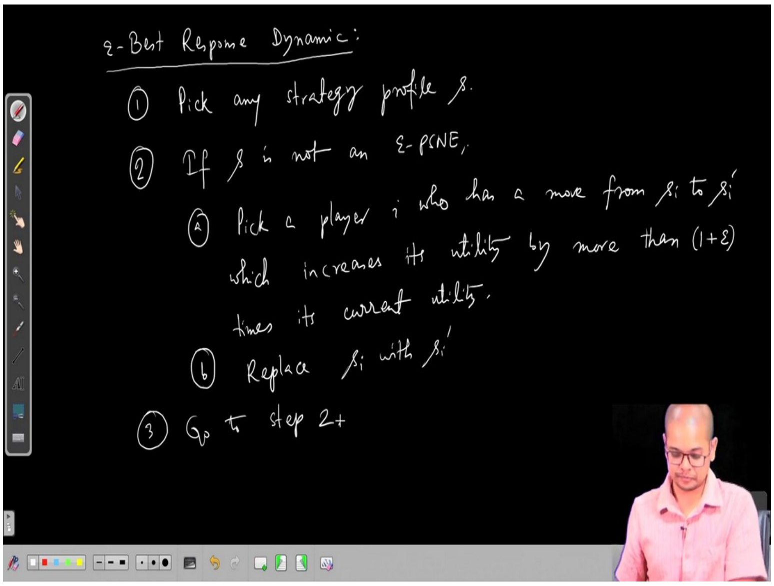

Now, we sort of slightly modify the basic version of epsilon best response dynamic, sorry, best response dynamic and we call it epsilon best response dynamic. As usual, the first step is pick any strategy profile. If s is not an epsilon PSNE, then what should we do? We pick a player i who has a move from s_i to s_i' , which increases its utility by more than $1+\epsilon$ times its current utility. Such a player and such a move is guaranteed if s is not a PSNE, sorry, not 2. Then, what you do? Is that you replace s_i with s_i', not c, go to step 2.

So, you keep iterating, ok. So, here is a theorem. The fast convergence of epsilon best response dynamics for network congestion games under some mild conditions.

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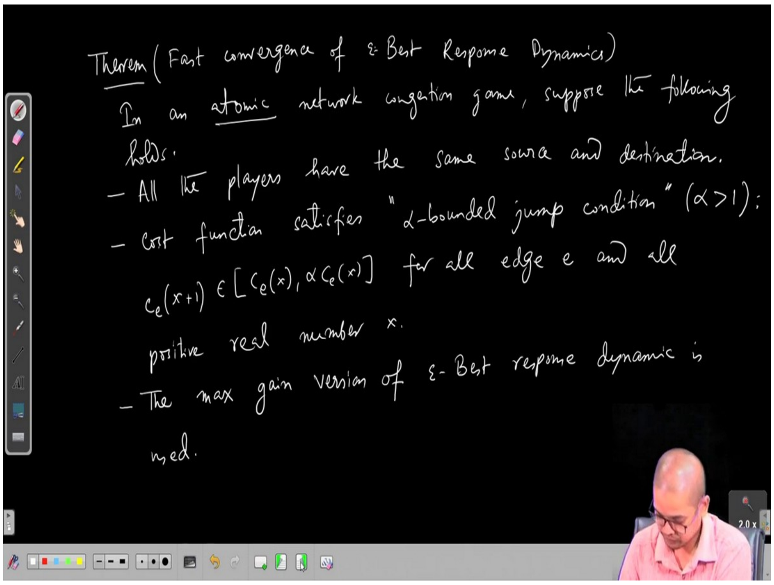

Theorem: This is can call Fast Convergence of Epsilon Best Response Dynamics: So, in a network congestion games, in an atomic, I will explain what is atomic, atomic network congestion game. Suppose the following holds suppose the following holds. All players have the same source and destination all the players have the same source and destination, ok. Cost function is does not make arbitrary jump. So, cost function satisfies. What is called alpha bounded jump condition? Where alpha is a alpha is a real number greater than 1.

So, what do you mean by that? So, if you look at the cost function of any edge $c_e(x+1)$, the traffic increases by 1. Then this is the cost function increases by at most alpha times. So, it cost function must be non-decreasing. So, it cannot be less than $c_e(x)$ it should be at least $c_e(x)$, but at most $\alpha \times c_e(x)$. This should hold for all edge e and all positive real number x, ok. And the best response dynamics is another version of best response dynamics which is called the max gain version; the max gain version of ϵ best response dynamic is used. What is ϵ best response dynamics?

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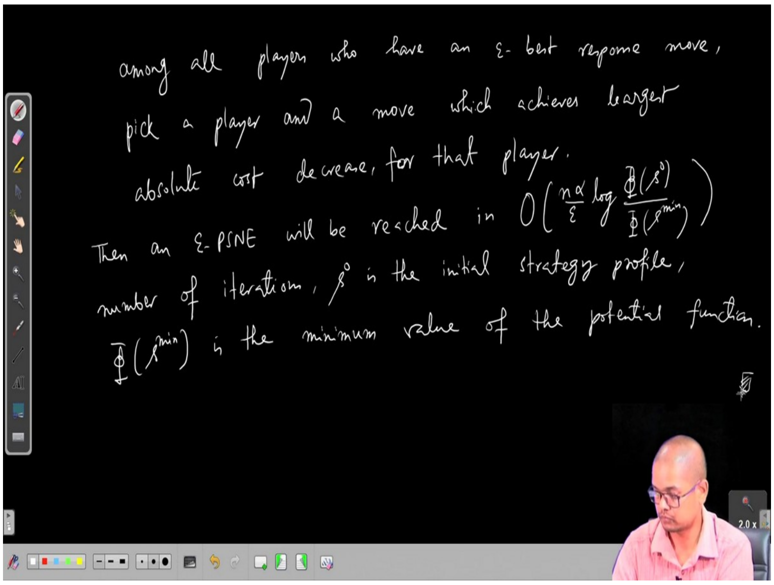

Is that among, all players who has an epsilon move and epsilon best response move. That means what? The current strategy profile is not a epsilon PSNE and there exist so that means, there exist a player for which there exist a move from strategy s_i to $s_i^{'}$ which increases its utility or decreases its cost by more than $1-\epsilon$ times its current cost.

Among those players, who has an epsilon best response those moves let us call it epsilon best response move, pick a player and a move which achieve; which achieves largest absolute cost decrease. So, in the ϵ best response move you know in 2 a in step 2 a, we are just picking a player who has a epsilon best response move. But we are saying that you know among all players you pick the player i which has a which should have a ϵ best response move.

But, it also among all players which has the epsilon best response move pick the player and a move which decreases the utility which decreases the cost by highest absolute amount by highest absolute value for that player of course, for that player. There is no potential. This best response dynamics is defined for any arbitrary game decrease for that player of course.

Then, what do we say? What is the conclusion? Then, an epsilon PSNE will be reached in $O(\frac{n\alpha}{\epsilon}\log(\frac{\phi(s^0)}{\phi(s^{min})}))$ of iterations of ϵ best response dynamics where, ϵ is the parameter we are finding epsilon PSNE, alpha is the is the parameter for the α

bounded jump and s^0 is the s^0 is the initial strategy profile. What do you mean by initial strategy profile?

You know in in both best response dynamics or ϵ best response dynamic, we pick we start with the arbitrary strategy profile. So, let s^0 be that arbitrary strategy profile and s^{min} is an $\phi(s^{min})$ is the minimum value of the potential function. Yeah. So, you see that we will need more iterations if the number of players is more. And if we, the further we start from the minimum the further our strategy profile are from the minimum the more number of iterations I need to reach the to reach up PSNE an epsilon PSNE.

So, let me conclude here. So, we will prove we will see the proof in the next class, yeah. So, we will see the proof in the next class.