Algorithmic Game Theory

Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 17
Potential Games

Welcome to today's lecture. We were doing computing equilibrium and in the last class we studied the question of finding a PSNE. And, we observed that it for a general game it is trivial that the most easy algorithm runs in polynomial time and that is because of the huge input size. And, then we were studying succinct games.

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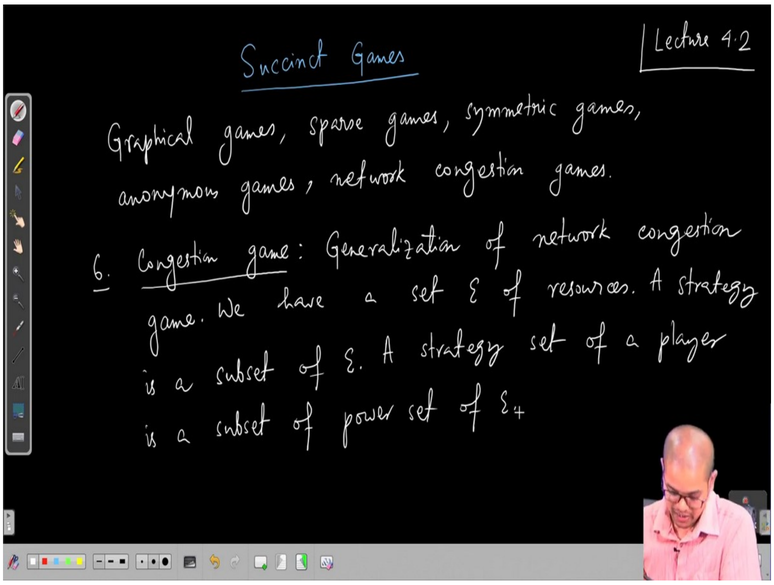

And, in the last class we have studied various succinct games, namely let us recall briefly we studied what is called ok. Let me write succinct games. And, in the last class we had seen graphical games, sparse games, symmetric games, anonymous games and network congestion games. So, we will continue and we saw 5 games.

So, number 6 is only congestion game. This is a generalization of network congestion game, generalization of network congestion game. So, what is it? Let us see. Here we have a ground set, have a set E of resources. These are like set of edges in the network congestion games ok. A strategy is a subset of E ok.

And so, each player has a strategy profile, each player is a set of strategies. So, a strategy set is a power set of E, a subset of power set of E. A strategy set of a player is a subset of power set of E. So, it is not that each player can play any subset of E, its some given list of subsets are given. So, think of network in network congestion game it is a set of all paths from s_i to t_i. And paths are nothing, but a specific kind of subsets of edges ok.

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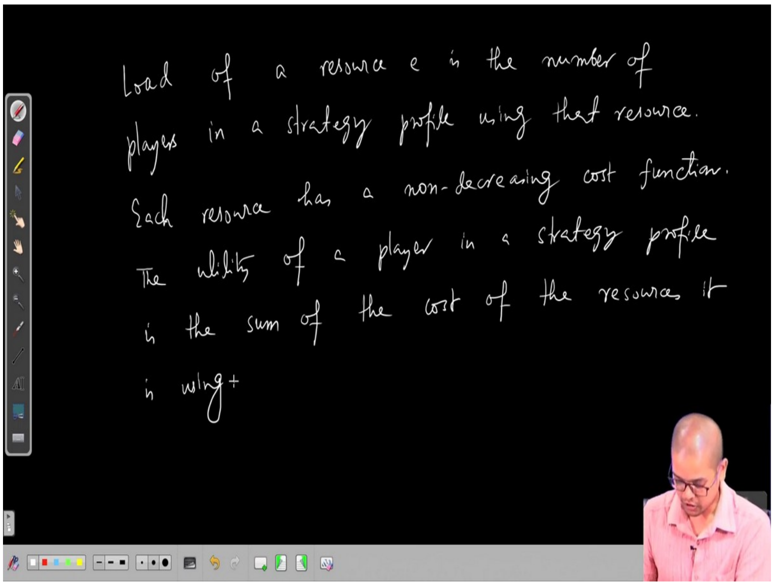

And again of course, we have a load function as usual, load of a resource in a strategy profile of course; resource e is the number of players, number of players in a strategy profile using that resource. So, for network congestion game, the load of a resource or load of edge e is the number of paths that are passing through it ok.

So, this is and each resource has a cost function as usual. Each resource has a non-decreasing cost function and the utility of a player in a strategy profile is the some of the cost of the resources it is using ok. So, as you can see it is a network congestion game is a special kind of congestion games.

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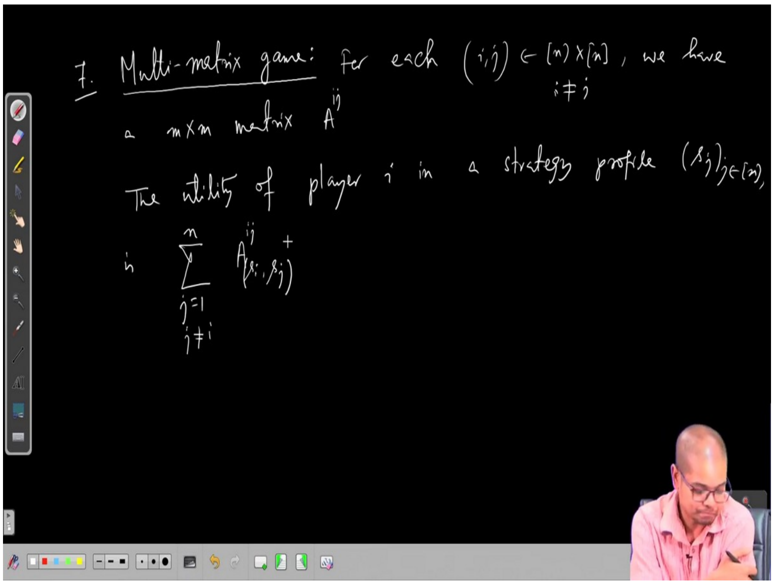

Our last, but not the least example of succinct games is what is called multi-matrix games. So, here what is it? So, for each pair of players, for each i, j pair of players we have a matrix say $m \times m$ matrix say A_{ij} . Suppose, m is the number of strategies for each player. So, for each pair of players i, j we have a matrix. So, we have n square many n choose too many actually, n choose too many.

So, $i \neq j$ right. So, you have this n choose too many matrices and the utility of player i, the utility of player i in a strategy profile $(s_j)_{j \in [n]}$, there are n players is $\sum_{j=1, j \neq i}^{n} A_{ij}$; A_{ij} is the name of the matrix and look at (s_i, s_j) entry and you sum them up. This sort of games are very useful in social networks and this sort of friendship network.

Think of each strategy each player is playing a strategy and each link is giving some utility, some amount of happiness and its total utility or total happiness is the sum of the utilities that it receives from all the friends. So, you see that you know for this sort of games finding a PSNE is not a trivial task. And, the problem becomes the computational problem becomes interesting, because the naive algorithm takes exponential time ok.

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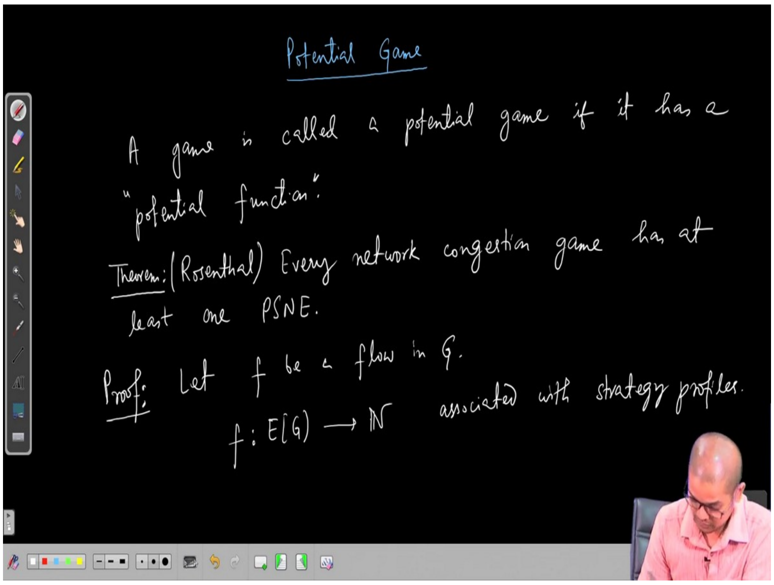

So, next we study a very important class of games which is called Potential Games. The beauty of this class of game is that these games always have a pure strategy Nash equilibrium. Any finite potential game always is a pure strategy Nash equilibrium. So, why potential game? You know intuitively speaking game is potential a game is called a potential game, if it has a potential function.

Now, what is the potential function? So, let us define the potential, we will define the potential function. But, before that let us see and get a feel of the potential function, then we will formally define it. So, towards that let us prove a result and its proof will gives rise to the definition of potential functions. And once we define potential function, the definition of potential game is also complete. Games which has a potential function.

So, what is the theorem? So, we will so, this due to Rosenthal it says that every network congestion games with arbitrary real valued cost functions, but cost functions has to be non-decreasing has at least one pure strategy Nash equilibrium. Proof, how do we prove it? Let us see. So, let if f be a flow in G. What do you mean by flow? Flow is a function. It simply says that how many players are using each edge.

So, from the edge set to natural numbers. For each edge it tells how many players are using this edge. And so, flow is always associated with the strategy profile. These are associated with strategy profiles. Now, we are defining the potential function is always a function of. So, let me write this way, let me write.

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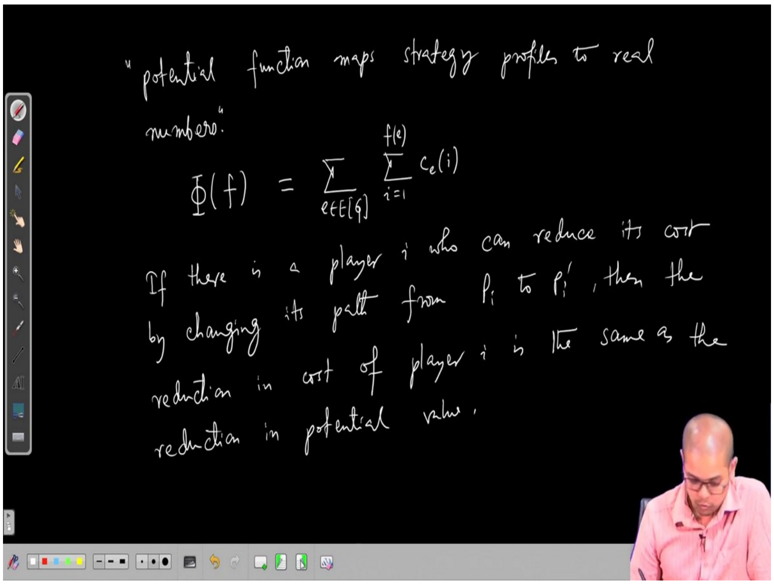

So, potential function, let me write maps strategy profiles to real numbers. And, here you know strategy profiles and flows are in one to one correspondence, it is just another way to represent a strategy profile. So, we are defining a potential function of a strategy profile whose corresponding flow is f as sum of our edges in G, you sum of our edges in G. And, you suppose the flow in this edge is $f(e)$, you $\sum_{i=1}^{f(e)} c_e(i)$.

Now, why this sort of crazy function? No, justification. It is just that it will help us in the proof and the reason or why this helps us in the proof, that will be the definition of potential function. So, what we claim is that what is the beauty of this function is that, if there is a player i who can reduce its cost by changing its path from P_i to P_i', then the reduction in cost of player i is the same as the reduction in potential value.

Let us call phi a potential function. And, this is exactly the definition of potential function. If there exist a player i who can deviate unilaterally and improve its utility, then in this case decrease the decrease its cost. Then, the reduction in the cost of player i is exactly the reduction in the potential. But why? Let us prove it.

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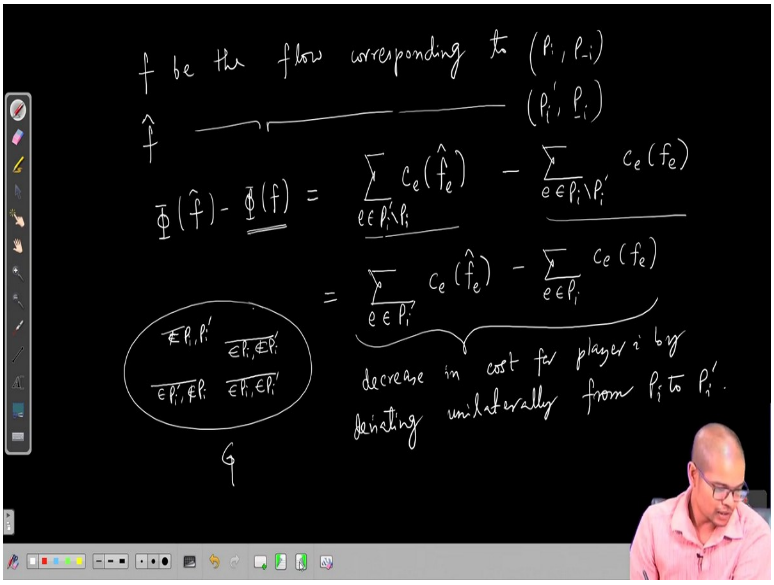

So, suppose f was the initial flow, f be the flow corresponding to (P_i, P_{-i}) and the ith player deviates from P_i to $P_i^{'}$ and let f hat be the flow corresponding to $(P_i^{'}, P_{-i})$. So, what is the change of potential? $\phi(\hat{f}) - \phi(f)$. So, what is $\phi(f)$? Let us recall. It was sum over all edges and for each edge we were summing, we are looking at its flow $f(e)$ and we are $\sum_{i=1}^{f(e)} c_e(i)$.

So, suppose this is the graph G and if some edge is not part of P_i to $P_i^{'}$ neither, then this then that edge contributes equally to $\phi(\hat{f})$ and $\phi(f)$ and thus gets cancelled out. So, suppose this edge does not belong to P_i and $P_i^{'}$, then it gets cancelled out. Suppose, there is an edge which belongs to $P_i^{'}$, but not in P_i, then you see that, then the contribution of that edge is $c_e(\hat{f}_e)$. And, this will hold for all edges which in $P_i^{'} \setminus P_i$. Think of paths as set of edges ok.

And, there are edges, suppose there is an edge which belongs to P_i, but does not belong to $P_i^{'}$. Then, it is contribute it will contribute to $\phi(f)$ only, it will be minus edge $e \in P_i \setminus P_i^{'} c_e(f(e))$. Now, there are edges which belong to both P_i and $P_i^{'}$. They again contribute to both $\phi(\hat{f})$ and $\phi(f)$ and thus gets cancelled out. So, let us add those edges in both the sums, both these two summations. Then, that will be a clear writing. Just for sake of simplicity, let us write that away because for those edges $\hat{f}(e)$ and $f(e)$ is same.

So, in the first sum let me write $\hat{f}(e)$ and for the second sum let me write $f(e)$. This is nothing, but the decrease in cost for player i by deviating unilaterally from P_i to $P_i^{'}$. This is exactly what we need to show; that means, the drop of potential is the is the decrease in cost, the benefit that the player i gets by unilaterally deviating from its current strategy to the new strategy.

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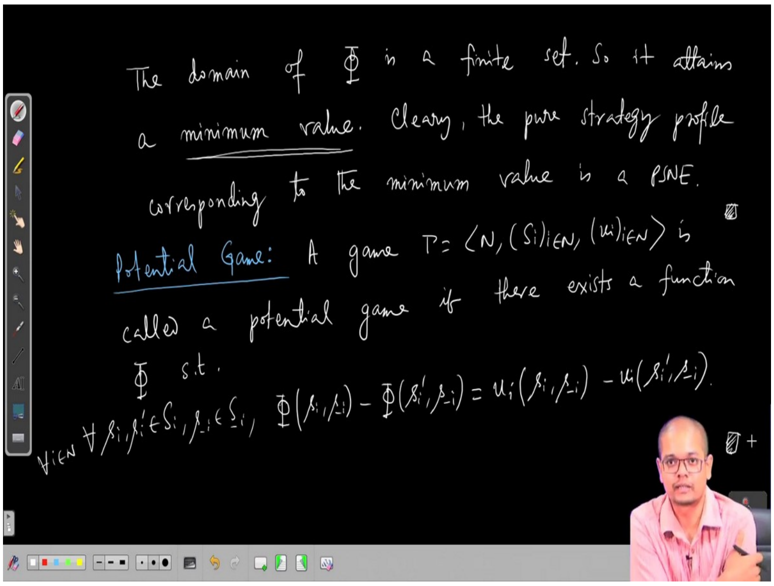

And, I claim that this we can write that hence from this we can conclude that it has a PSNE. Why? Because, see that the domain of ϕ is a finite set, it is a set of all strategy, strategy profiles, pure strategy profiles; domain of ϕ is a finite set. So, it attains minimum value. So, we claim so, clearly the pure strategy profile corresponding to the minimum value is a PSNE.

Why? Prove by contradiction. Suppose not, then there exist a player for which there exists a unilateral deviation which reduces the cost of the player by some non-negative amount. Then, look that reduction, then go to that new strategy profile and look at the corresponding potential. Because, it is a potential function that the decrease in the cost of the deviating player will be exactly the drop of potential. But, because we have started with the minimum value of the potential, it cannot decrease anymore. So, it must be a

PSNE. So, what is the formal definition of a potential function or potential game? Let me write.

Potential game; so, a game is called a potential game if there exists a function ϕ from strategy profiles to real number such that for all $s_i, s_i^{'} \in S$, for all player i in n. The drop of potential when player i deviates unilaterally from strategies s_i to $s_i^{'}$ is exactly the drop of drop in utility or drop in cost $u_i(s_i, s_{-i}) - u_i(s_i^{'}, s_{-i})$. And, from the Rosenthal's theorem it immediately follows that a potential game has always has a PSNE ok.

Thank you.