Algorithmic Game Theory
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Lecture - 15
Support Enumeration Algorithm

Welcome. So, till now we discussed some preliminaries of game theory. What is game, what are some basic concepts some basic equilibrium notions and metric schemes. From today we will start computation part. So, that is where really the algorithm parts coming into picture.

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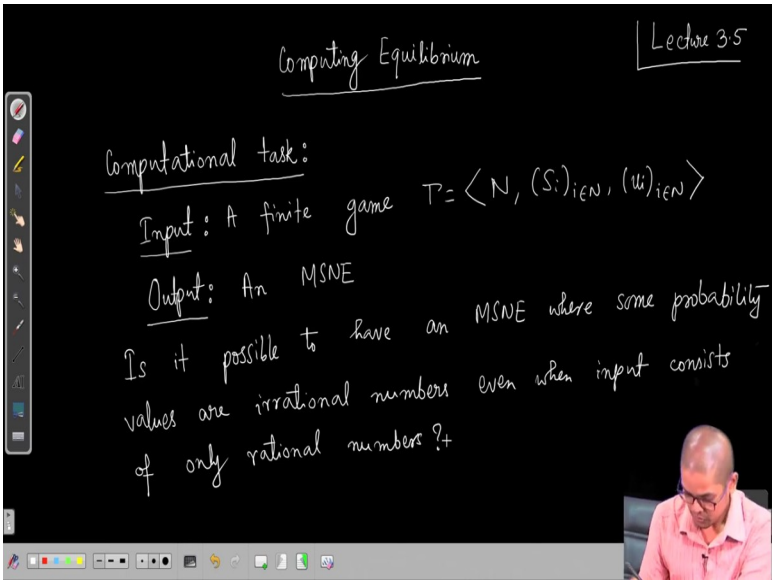

So, our topic is topic for today is computing equilibrium. We will discuss this for next couple of lectures. So, what is the problem? The, ok; so, the computational task: So what is the input? Input is a finite game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. So, how we can give this input? We can list down the set of strategies for all the players and we can give the utility function of all the players as inputs. What is output? Output is an MSNE.

So, that is the most fundamental question that we can think of given finite game in normal form I need to find a MSNE. I would request you to pause this video and think. So, there is some fundamental problem of framing our computational task in this way. So, please pause the video and think is there any fundamental problem. So, the one

fundamental problem with this approach is that. Is it possible to have an MSNE where some probability values are irrational numbers even when input consists of only rational numbers?

So, if all the utility values are rational numbers, then is it possible that still means all MSNEs in a given game involve some irrational numbers? Because if it is true, then we have a problem, because we do not know how to represent irrational numbers accurately on a computer.

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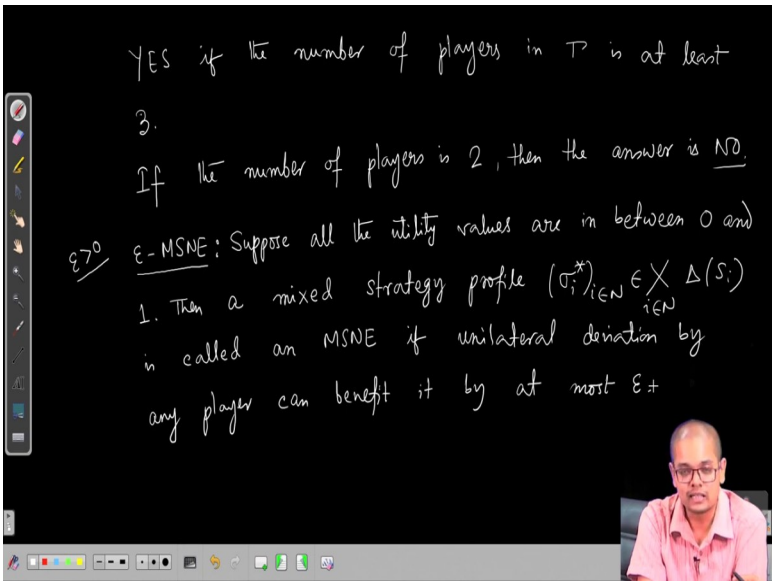

So, the answer to this question is yes, it is possible. So, if the number of players in gamma is at least 3, so what do you mean by that? There exist a game a strategic form game gamma where all the utility values are rational numbers, but all the MSNEs involve some irrational number. MSNEs what is MSNEs? It is a probability distribution over the strategies for each of the player. It is a couple of mixed strategies and it may happen that you know those probability values involve irrational numbers.

So, that is the problem. But of course if the number of players is 2, if the 2 then the answer is no; that means, if I have 2 players and input consists of only rational numbers all the utility values are rational numbers. Then it must be the case that if I take, it must be the case that if I take any MSNEs it will involve only rational numbers ok.

So, for 2 players it is not a problem, but from 3 players onward this problem formulation is bit problematic and so we relax our requirement and we define what is called an epsilon MSNE. So, what is epsilon MSNE? So, where epsilon is a constant greater than 0.

So, suppose all the utility values are in between 0 and 1. Then mixed strategy profile $(\sigma_i^*)_{i\in N} \in \times_{i\in N} \Delta(S_i)$. Then a mixed strategy profile is called an MSNE if unilateral deviation by any player can benefit it by at most epsilon. So, take a mixed strategy profile σ_i^* and any player say player i if it deviates from σ_i^* to any other mixed strategy σ_i, then the change of utility it can either decrease the utility can either decrease that is fine.

But, it is also allowed to increase, but the increase in utility should be at most epsilon. If such a property holds then the mixed strategy profile is called an epsilon MSNE.

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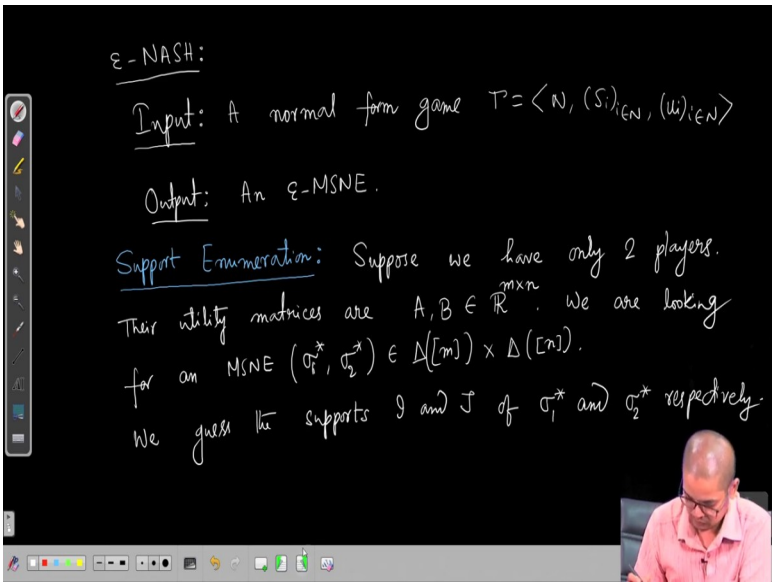

So, this problem, so what is our what is our new problem is called ϵ NASH. So, what is input? Input is again normal form game and we need to output and ϵ MSNE. So, we will see after a couple of lectures that you know this problem is hard and we do not know how to solve this efficiently.

But, nevertheless let us have a look at various algorithms at some of the algorithms. The first algorithm is what is called support enumeration. What is it? It is based on the

indifference principle we had the equivalent characterization of mixed strategy NASH equilibrium and it exploits that. So, suppose so it works for 2 players. So, suppose the number of players is 2.

Suppose we have only 2 players and their utility matrices are A and B in $\mathbb{R}^{m \times n}$. The player 1 has m strategies and player 2 has n strategies. So, we are looking for a mix strategy NASH equilibrium and epsilon MSNE. We are looking for an MSNE because we have only 2 players you can also go with finding exact MSNE.

So, (σ_1^*, σ_2^*). What we do is we guess the support of σ_1^* and σ_2^*? What do you mean by support? Support of a probability distribution is the set of outputs which receive non-zero probability that is the support of a discrete probability distribution. So, we guess the support supports I and J of σ_1^* and σ_2^* respectively.

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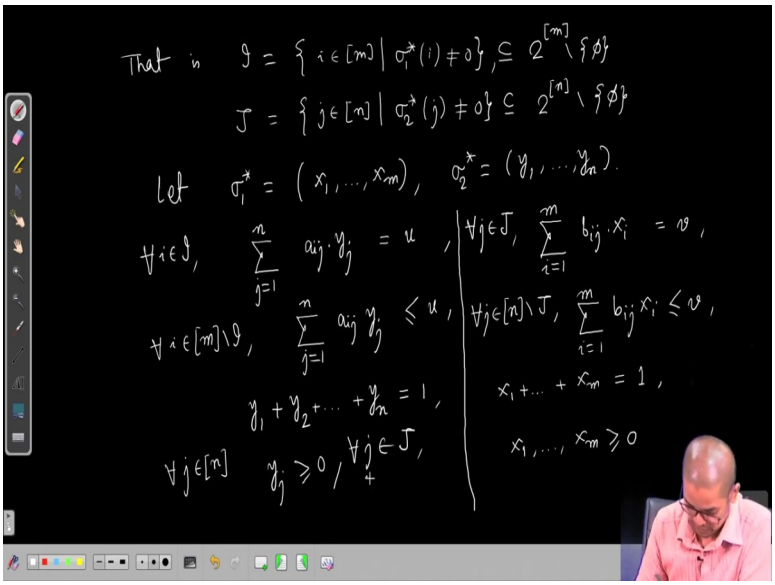

That is we guess that is I equal to small i in m such that $\sigma_1^*(i) \neq 0$ and similarly capital J equals small j in n such that $\sigma_2^*(j) \neq 0$. Remember that we do not know σ_1^* and σ_2^*. So, we do not know i and j. So, we are guessing. So, we so i we all we know about i is a subset of a non-empty subset of power set of m 1 to m non-empty.

So, 2^X for some set X is a notation to denote the power set of X ok. So, because we do not know i and j what effectively, in implementation what we will do is that we will try

all possible 2^m-1 values of i and all possible 2^n-1 values of j and we implement this. What will we implement this?

So, what we do is that; so let σ_1^* be $(x_1,\ldots,x_m)$ and $\sigma_2^*=(y_1,..,y_n)$ ok. So, what does the indifference principle say? Is that all those strategies which where σ_1^* assign non-zero probability. They are best responses with respect to σ_2^*. So, all strategies which gets non-zero probability under σ_1^* and that is exactly the support of σ_1^*, which we have guessed to be I.

So, for all $i\in I$ these are the strategies, which are getting non-zero probability. What is the utility of playing i when the other player is playing σ_2^*? This is the utility, j equal to 1 to n. So, player 1 is always playing the i-th strategy and if player 2 plays the j-th strategy, the utilities a_{ij} and this happens with probability y_j, y_j is the probability with which the column the other player is playing the j-th strategy.

So, this is this should be same is equal to u for some fixed u and what else we have? For other for $i\in[m]\setminus I$ the utility of playing those strategies can only be less or less than or equal. So, j equal to 1 to n $a_{ij}y_j$. This should be less than equal to u ok. And of course, we have this y_j's must be probability distribution. So, $y_1+y_2+..+y_n$, this must be equal to 1 and these variables y j should be greater than equal to 0. This should happen for all $j\in[n]$ and u can take any value, it can be either positive or negative. So, this is a set of constraints and not only this, this is for player 1.

Simultaneously, the equivalent conditions should hold for player 2 also. So, let us write. So, for player 2 all the strategies $j\in J$ the utility of playing small j should be the highest possible when the when player 1 is playing σ_1^*. So, when player 1 is playing σ_1^* when player 1 plays i-th strategy b_i then the utility for player 2 is b_{ij} and player 1 plays i-th strategy with probability x_i and this summation is over i equal to 1 to m. This is equal to say v.

And for the remaining strategies the utility should be at most v and of course, like y_i's x_i 's must be a probability distribution and this should be nonzero non negative $x_1,\ldots,x_m$ is greater than equal to 0.

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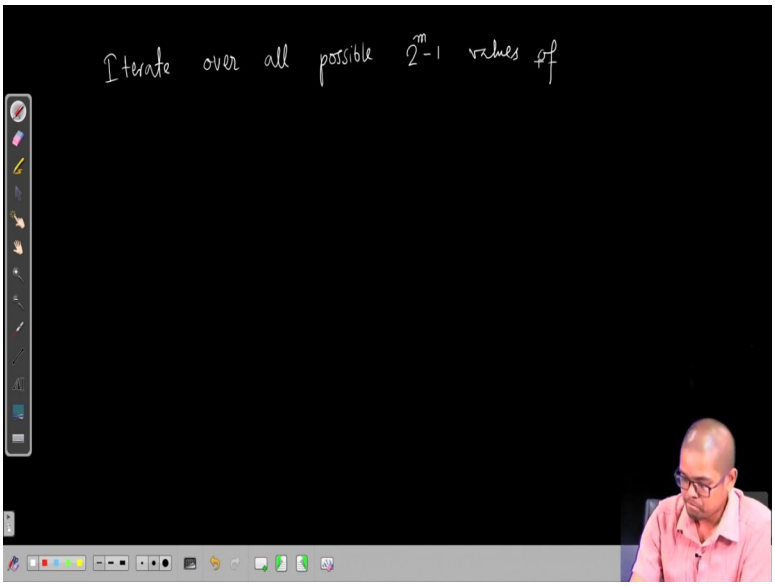

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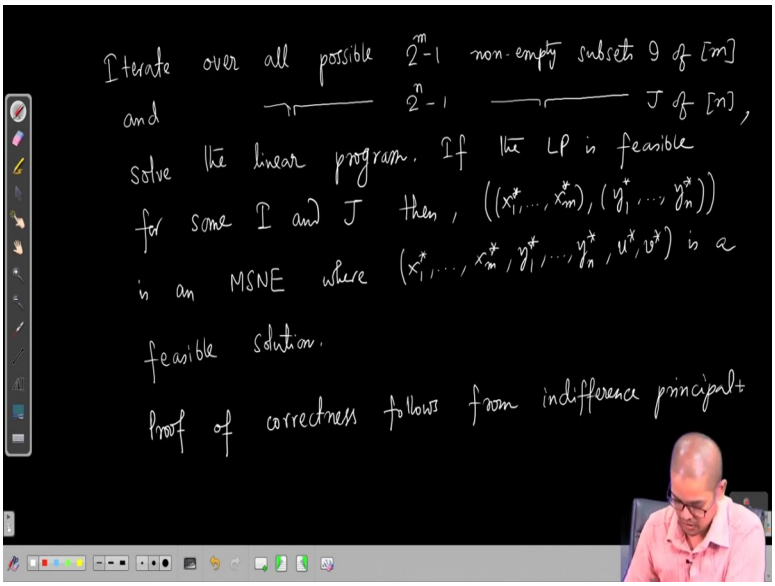

So, what is the algorithm? Iterate over all possible $2^m - 1$ values of values means possible subsets, not values let me write subsets non-empty subsets I of [m] and all possible $2^n - 1$ non-empty subsets J of [n].

Solve the LP, solve the linear problem. There is no optimization function there. So, it is a feasibility checking. Is it possible to satisfy all these linear constraints by setting up values to this variables that is what do we mean by solving this linear problem. If there

exist a solution if there exist a set of values x_i's and y_j's, which satisfy all these conditions all these properties, then that itself is a mixed strategy NASH equilibrium.

If the LP is feasible for some I and J then, $x_1, \ldots, x_m$ take a solution let us call it $x_1^*, \ldots, x_m^*$ and $y_1^*, \ldots, y_m^*$ is an MSNE ok, where $x_1^*, \ldots, x_m^*$ $y_1^*, \ldots, y_m^*$ of course, we have u^* and v^*, but we do not need them is a feasible solution or simply solution.

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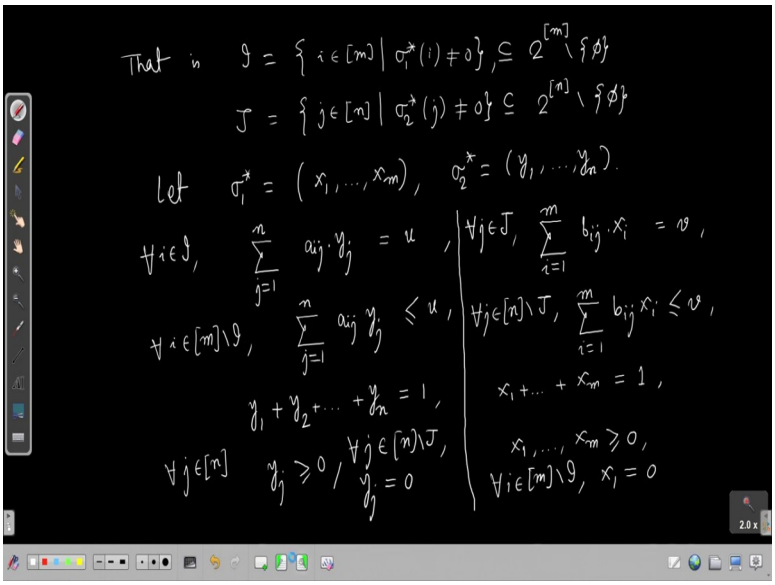

There was one more constraint is that you know for all j small j and capital J y_j must for all $j \in [n] \setminus J$ all those strategies outside j they should get 0 probability. This just ensures that our guess is just our guess is correct. This linear program is consistent with our guess of the support of σ_2^*. Similarly here also for all $i \in [m] \setminus I$ $x_i = 0$ ok. So, very good the proof follows from indifference principle. Proof of correctness follows from indifference principle.

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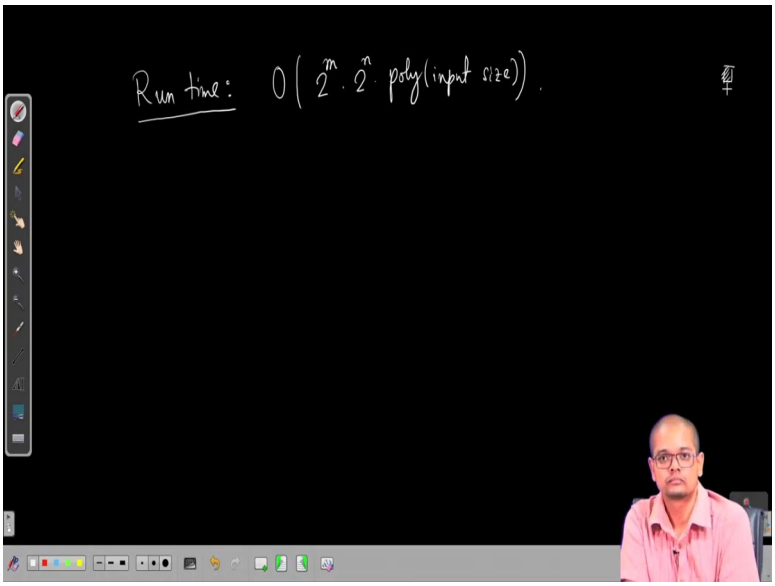

And what is the running time runtime of the algorithm? We are this b go off, we are making $2^m - 1$ guesses for i and $2^n - 1$ guesses for j and then we are solving a linear program we are checking feasibility of a linear program that takes polynomial in the number of variables and the input; that means, all the coefficients..

And what are the coefficients? The coefficients are the coefficients of this linear program are the matrix entries the utility values. So, polynomial in input size ok, so as you see that this is an exponential algorithm and we will see later in this after few lectures that we do not hope to have a polynomial time algorithm for solving this problem, ok.