Algorithmic Game Theory

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Lecture - 13
Braess's paradox

Ok. Welcome, so, in the last two lectures we have studied iterative elimination of dominated strategies and we see this powerful technique how this can significantly reduce the complexity of a game. So, we will continue our example of seeing concrete games and finding its various equilibrium.

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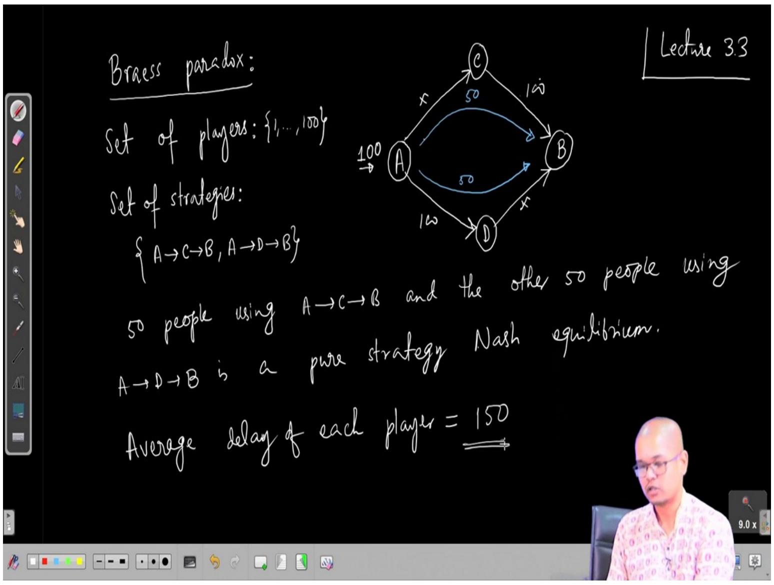

So, today we will take a very interesting example which is called Braess paradox named after the discoverer; so, it is regarding a congestion game a network congestion game. So, think of a network with four nodes A B C D and these are the edges. Now what is the delay function? Suppose one unit of traffic we need to route from A to B and this traffic is suppose it is sort of liquid. So, any fraction of traffic can be routed on any path; so, let me write this way one unit of traffic will need to be routed from A to B.

Now, how much time, what is the delay of each edge delay? So, for example, the delay of edge A C is x, where x is the traffic in that edge. So, if the if we route x amount of traffic on this edge x on this edge A C then it the delay is x. The same is the case for the edge B D, on the other hand the delay on the other two edges are independent of the traffic and suppose this is 50 ok.

So, or instead of 50 let us write it is something 0.5; so, or let me change it slightly; so, that it is the number of players is clear. Suppose 100 people want to go from A to B and the traffic is the number of people using an edge. So, if X is the number of people who is using the edge A C, then the time required is also X, X minutes or second that does not matter. And the time required to traverse this edges C B and A D these are 50.

So, the set of players is let us write the let us write the game clearly set of players 1 to 100 and what are the set of strategies for each player? Set of strategies all the players have two strategies, there are two paths; one is A to C to B, A to C to B and another is A to D to B these are the two paths available. And what is a what is a mixed strategy Nash equilibrium; so, it is not difficult to find that 50 percent of the traffic following the top path; that means, A C B and these 50 percent 50 people.

And the other 50 percent following the bottom path, this is a mixed strategy Nash equilibrium. So, 50 people using A C B and the other 50 people using A D B is a pure strategy Nash equilibrium, pure strategy Nash equilibrium ok very good. So, what is the average delay of each player? Average delay of each player what is it? So, the delay for the edge A C and D B both are 50 so and so the delay for all the players is 100, it is the amount of time needed to reach from A to B good.

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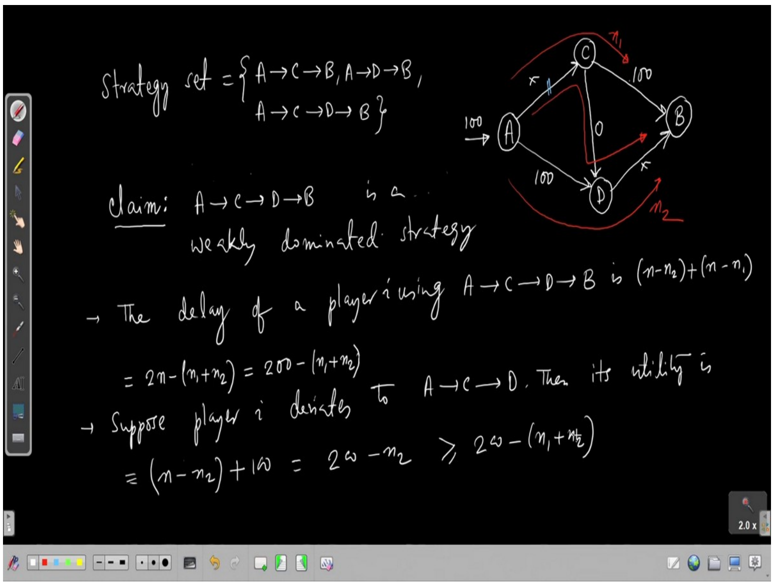

Now, suppose to we want to we want to reduce the average delay and we modify the network and add a super-fast edge A B C D, this is x this is delay here this is x 50 50 this was the network before. Now, you see suppose I add an a very fast edge of delay 0 from C to D, now see what is a mixed strategy Nash equilibrium let us find a mixed strategy Nash equilibrium.

So, what are the strategies that again the again we have 100 players who want to go from A to B, the strategy set has changed. So, strategy set; so, strategy set is a set of all paths from A to B. So, the paths before they exist A to C to B and A to D to B and A to C to D to B this is another path, which was not before C to D to B these are the three strategies.

Now, we claim that A to C to D to B is a weakly dominant strategy. Let us see, so let us to prove that let us consider a situation where there are I do not know, n_1 number of people who are following the top path n_2 number of people who are following the bottom path. And the remaining people which follow this A to C to D to B, we need to show that the utility of anyone who is following A to C to D to B they are better off irrespective of n_1 and n_2 .

So, let us write the utility that the delay they want to minimize, the delay of a player using A to C to D to B is how much? The edge A C is used by $n-n_2$ people and the edge D B is used by $n-n_1$ people. This is $2n-(n_1+n_2)$, suppose the player say i. Now, suppose player i deviates to some other strategy; so, and we will show that you know any strategy deviates it is not better off. So, suppose it deviates to the top path A to C to D ok.

So, what is this? This is 2 n is 100, this is $200-(n_1+n_2)$. player i deviates to A to C to D. Then its utility is how much, now there is how many players are using the top path? It is $n-n_2+50$. So, this is $n=100$ $150-n_2$, and which one is larger? n_2+1 this is plus 1 here.

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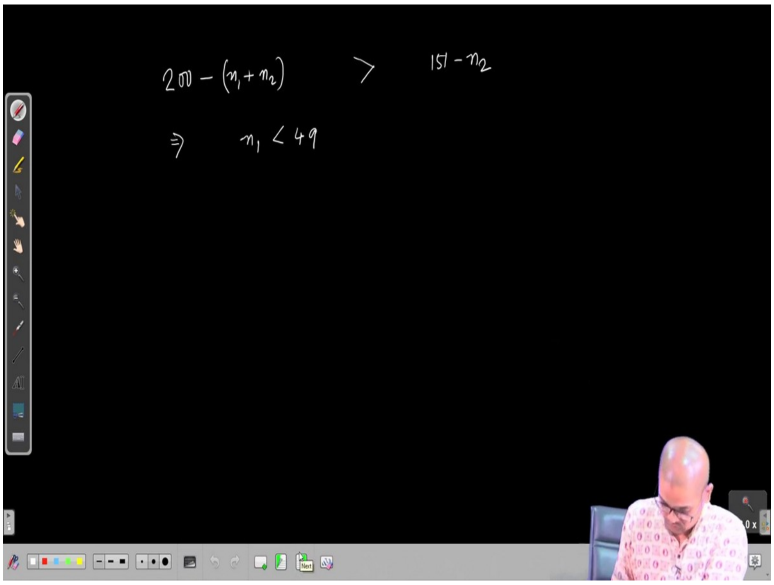

Now, does it delay increases or decreases let us see, on the one hand we have $200-(n_1+n_2)$ and the on the other hand we have $151-n_2$. Now, can this be less, if this is less what will happen? Then n_2 gets cancelled and $n_1<49$.

So, the; so, this is not correct; so, the claim; so, let me modify the claim here, let me modify the claim it is not a weakly dominate strategy. But it will be a pure strategy Nash equilibrium, all players playing this will be a pure strategy Nash equilibrium all players playing this is a pure strategy Nash equilibrium ok.

So, what you have is n_1 and n_2 these are 0 this is 200 ok; so, if one player deviates from this A C D B to the top path then let us compute it from some other angle. So, here in this A C this traffic remains same and instead of playing paying X it is now paying 50 C. So, it 50 it do not work here I think these numbers are correct, make this 100 yes; now, it should be fine yeah, this is a weakly dominated strategy; is a weakly dominated strategy ok.

So, let us see; so, it is 200 on the other hand if it shifts then this will not be 50; so, let me write this inequality again, correct with 100 it should be correct. So, it is for the edge A C this cost is this cost utility is $n-n_2$, this and the $n-n_2$ and this is plus 100, this is $200-n_2$; on the other hand here, it was $200-n_1$.

So, let me change the utilities here also its going to be 50 make it 100 and then you will see the paradox 100 and the average delay is then 150; yes, now everything is fine. So, you see that this is that the delay this is strictly more than or greater than equal to $200-(n_1+n_2)$. Now, because this is a delay; so, utility is minus of delay.

So, player i its utility gets reduced by changing its strategy from A C D B to A C D. So, this is this and this inequality will be strict if n_1 is strictly more than 0. So, it is a it is a weakly dominated strategy; so, this is not needed; so, what we have is this is a weakly dominated strategy for all the players and.

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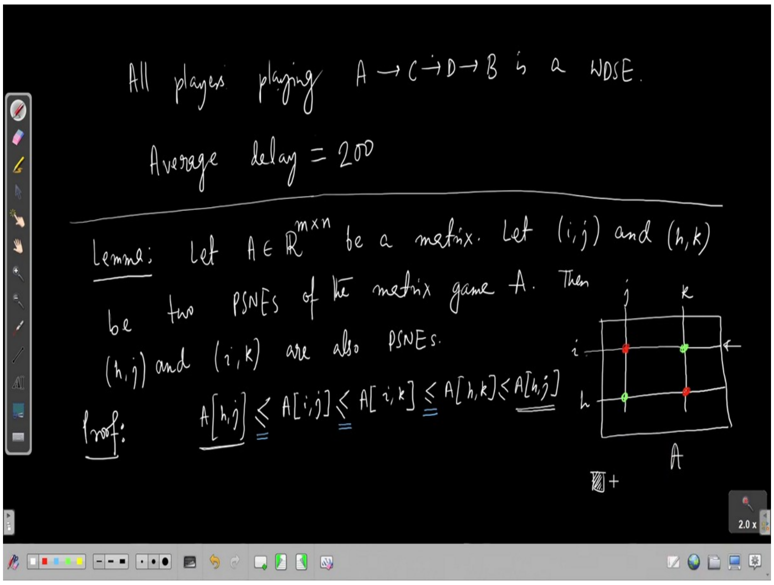

So, all players playing A C D B is an WDSE even stronger than PSNE, it is a weakly dominated strategy dominant strategy equilibrium. Now, what is the paradox? Here the paradox here is that, now you see what is the average delay? All players are following this path A C D B. So, the delay for A C is 100 delay for C D is 0 and delay for D B is 200.

So, the average delay is 200, what was the average delay before adding this high-speed edge it was 150. So, this is practically also very significant that you know just adding a high-speed link anywhere in a network be it a road network, be it a internet, internet network, routing network it can happen that this instead of reducing congestion from the network it reduce it increases the congestion.

And it affects it adversely affects the system performance and it affects the performance of the all the players and so on ok. Now, let me go over with another example of matrix games it also as I was telling in the last weeks lecture that matrix game has so many excellent properties. So, let us see one property of matrix game kind of an example or exercise.

So, here is a sort of a lemma for matrix game its kind, it is a kind of cute kind of an exercise. So, let A be a matrix B are matrix um, let i comma j and h comma k be two PSNES of the matrix game A. Then, so suppose the picture looks like this; suppose these are this matrix A, this is the i th row, this is the j th row this is the h th row, and this is the j th column, this is the k th column. And suppose this i comma j is a PSNE and h comma k is also a PSNE, then our claim is that the other two points are also PSNES.

Then means h comma j, and h comma j, and i comma k are also PSNES ok proof. So, in a matrix game for a matrix entries i comma j to be a PSNE, it must it must satisfy two conditions simultaneously it should be a rho it should be a rho minimum. So, for i comma j this entry should be the minimum of A i j should be the minimum entry of this i th row, otherwise if this is not the case then player i is better off deviating to some other column from j.

On the other hand, it is also a i j is a maximum of the j th column ok; so, what we will show and that is a characterization. So, if there exist an entry in a matrix which is simultaneously row minimum and column maximum then it is a PSNE of the matrix game. What we will show is that, both the other two entries like h comma j and i comma k they also satisfy this property and hence they are also PSNES.

So, one by one; so, let us prove that A say h comma j is a A h comma j is a row is a row minimum, let us see how will you prove it. So, we have A h comma j is less than equal to I can write A h comma j is less than equal to A i comma j is this notation. Why? Because A i comma j is this column max, max of this j th column, but A i comma j is also a row minimum; so, A i comma j is less than equal to A i comma k.

But A i comma k is also less than equal to A h comma k, why? Because, A h comma k is again a column maximum, it is maximum of the k th column and it is a it is a row min it is a row minimum; so, A h comma k is less than equal to A h comma j. So, A h comma k you know you take these two things so; that means, this, this all. So, at the beginning we had A h comma j and at the end we had A h comma j; that means, all inequalities are basically equalities ok.

And what we have then because A h these green entries are equal to red entries all four entries are same. So, because say take any red entry say A i comma j because it is a maximum of j th column A h comma j is also maximum of the j th column. And because A h comma k is the minimum of this h th row, A h comma j is also minimum of h th row and hence this follows; so, this concludes the proof ok.

Thank you.