Algorithmic Game Theory
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Lecture - 12
Iterative Eliminations of Dominated Strategies (Contd.)

Welcome to the next lecture. So, in the last class we have explained the Iterative Elimination of Dominant Strategies, as a strategy for computing or help us compute the mix strategy Nash equilibrium or various other kinds of equilibrium in a game. So, let us see some concrete examples.

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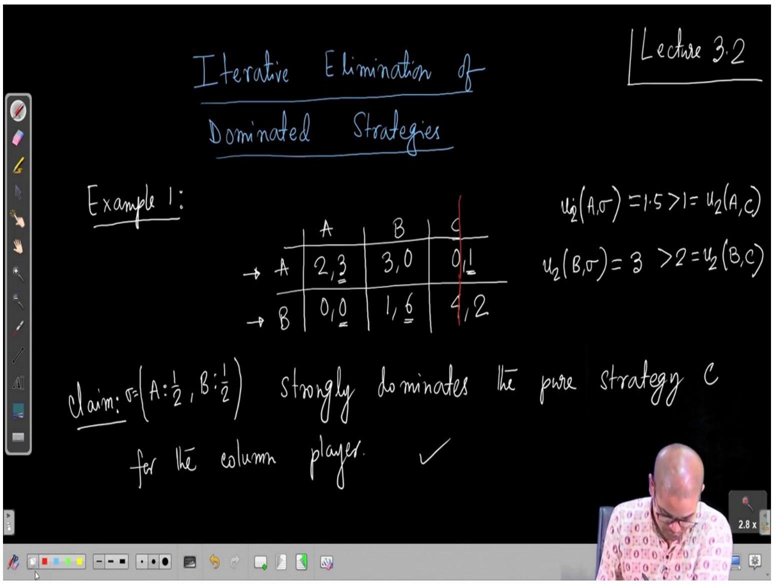

So, let us see, let me write the topic Iterative Elimination of Dominated Strategies. So, let us see an example. So, example 1: So, I have a game, A B C. The column player has three strategies A B C and the row player has two strategies A B and these are the payoffs 2 3, 0 comma 0, 3 comma 0, 1 comma 6, 0 comma 1, 4 comma 2 and suppose we want to find a mix strategy NASH equilibrium in this 2 player game.

So, can we find a dominated strategy? So, just by inspection means no strategy dominates no pure strategy dominates any other pure strategies so, but there exists a mix strategy which dominates the pure strategy and here is a mix strategy. So, consider the mix strategy of playing A with probability half and C with sorry with C, A with probability half and B with probability half not C it is B. This particular mix strategy claim this particular mix strategy strongly dominates the pure strategy C for the column player.

So, let us check. Suppose the row player plays A. So, by playing C the utility of column player is 1 which is this cell, on the other hand by playing A with probability half and B with probability half the utility of player of the column player is half times 3 plus half time 0 which is which is 1.5.

So, utility of player say player 2 column player utility of player 2 by playing let us call this sigma, this mix strategies playing sigma when the row player plays 1 plays A this is strict this is 1.5 which is strictly more than 1 which is the utility of the column player or second player by playing C when the row player plays A. How about when the column row player plays B?

When the row player plays B what is the utility if the column player plays sigma? The utility is if column row player plays B column player plays A with probability half so, half time 0 and B with probability half. So, half time 6 so, which is 3 this is greater than 2, which is the utility that column player would enjoy by playing C when row player plays B, this will be B comma C we will write properly.

So, I have changed mixed up messed up the order. So, let us see, this should be A comma sigma, this should be A comma C, we are writing the strategy of row player first and then the column player. So, not A, A comma C and same here, not sigma comma B, B comma sigma and here B comma C so, this claim is just is just verified.

So, what we can do is that we can get rid of the strategy C for the column player, we can assume we can safely assume that the column player will never play strategy C, because it is always better off by playing the mixture playing A with probability half and B with probability half whenever or whatever the other player plays.

So, column player will never play C. So, we can erase this column, we can assume that this column does not exist. Now you see that why because you know and the players have players are rationale and intelligent. So, column the row player can also see that. So, row player also observed that this strategy C is a strongly dominated strategy for the column player.

Recall that you know this entire the utilities entire game is a common knowledge, all the players know this game, they know that all the players know this game and so on up to infinite term. So, we get rid of this column C. Now, we are now we are in a reduced game. So, does there exist anymore? So, let us write the reduced game what is the reduced game in the next page.

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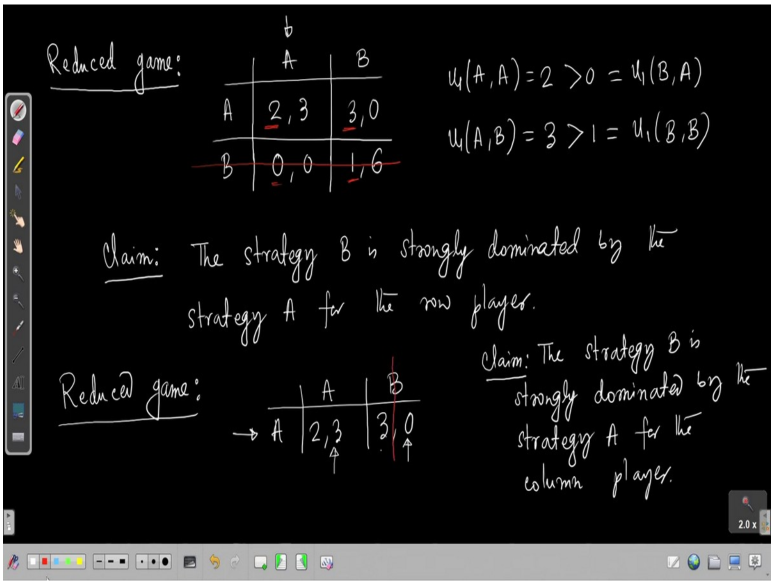

Let us write A B A B, we are writing reduced game ok. 2 comma 3, 0 comma 0, 3 comma 0, 1 comma 6. So, does there exist any other strongly dominated strategy? Yes is it. So, now, you see that we claim that again we claim that there exist another strongly dominated strategy namely A for the row player. So, let us write the strategy A the strategy B sorry.

The strategy B is strongly dominated by the strategy A for the row player. Let us check suppose the column player plays A then the row player gets an utility of 2 if it plays A and it will get a utility of 0 if it plays B. If the column player plays B then row player gets a utility of 3 by playing A and utility of 1 by playing B.

So, what we have. So, we can write that utility of player 1 by playing A when the column player plays A this is strict is 2 which is strictly more than 0 which is utility of player 1 when it plays say B when column player plays it, same with when column player plays B its utility is 3 which is greater than 1 which is the utility of playing B when the column player plays B. So, what we can do is that we can get rid of the rho for B.

So, what is the reduced game now? A B, A 2 comma 3, 3 comma 0. Now, you see that why we need to iteratively remove strongly dominated strategies because you know. So, for example, this strategy B for the column for the row player was not a strongly dominated strategy initially, but only when we recognize that the strategy C is a strongly dominated strategy for the column player and we eliminated that after that only the strategy B becomes the strongly dominated strategy for the row player.

Now, you see again we have another strongly dominated strategy which is namely B for the column player again. So, claim we write here the strategy B is strongly dominated by the strategy A for the row player for the column player. Indeed row player has only one option playing A and if the column player plays B then it gets a utility of 0, on the other hand if it plays A its utility is 3.

So, strategy B is strongly dominated for the column player and we can again eliminate this strategy. So, what is our reduced game now?

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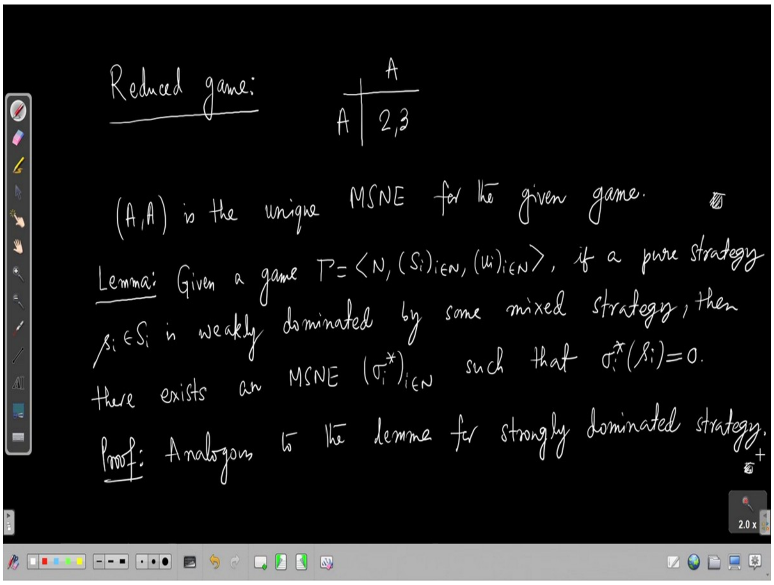

Now, reduced game: both the players have only one strategies A A and its utility is 2 comma 3 and this is the only a MSNE and also this is the this is an MSNE and this is a unique MSNE. So, A comma A is the unique mix strategy NASH equilibrium for the given game ok.

So, you see how powerful is this strategy so, it is not always useful, it is not that every game must have a strongly dominated strategy, but if it has if some game has a strongly dominated strategy then you can eliminate it and we can keep applying this process and get our game simpler and simpler.

So, this there is an our lemma yesterday was for strongly dominated strategy it says that if there is a strongly dominated strategy it cannot participate in any mix strategy NASH equilibrium, but what about weakly dominated strategies? So, let us write the lemma, I leave the proof to you the lemma the proof is along the similar lines it is almost similar you have to you have to follow the arguments analogously.

So, let me write given game given a game in normal form $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, if a pure strategy $s_i \in S_i$ is weakly dominated by some mixed strategy, then there exists there exist a there exists an MSNE mixed strategy NASH equilibrium $(\sigma_i^*)_{i \in N}$ such that $\sigma_i^*(s_i) = 0$. So, if there exist a weakly dominated strategy s_i there it then we can only conclude that there exist an MSNE where this particular weakly dominated strategy s_i gets 0 probability.

It can very well be the case that there exists some other mixed strategy NASH equilibrium where this strongly dominated strategy s_i has non zero weightage. So, if we are interested in finding any or one mixed strategy NASH equilibrium and we do not want to we are not interested in finding say all mixed maximum strategy NASH equilibrium then this is useful, we can still use this and because every strongly dominated strategy is a weakly dominated strategy, but not the not vice versa.

Then, the set of games which has more the set of games which has weakly dominant strategy is a strict superset of the set of game which has only strongly dominated strategies ok. I leave the let me write the proof, proof is analogous to the corresponding lemma for strongly dominated strategies. So, just follow that ok.

So, for example, let us let me give an example of a game which has which has a weakly dominated strategy and there exists a MSNEs where it gets non - zero probability.

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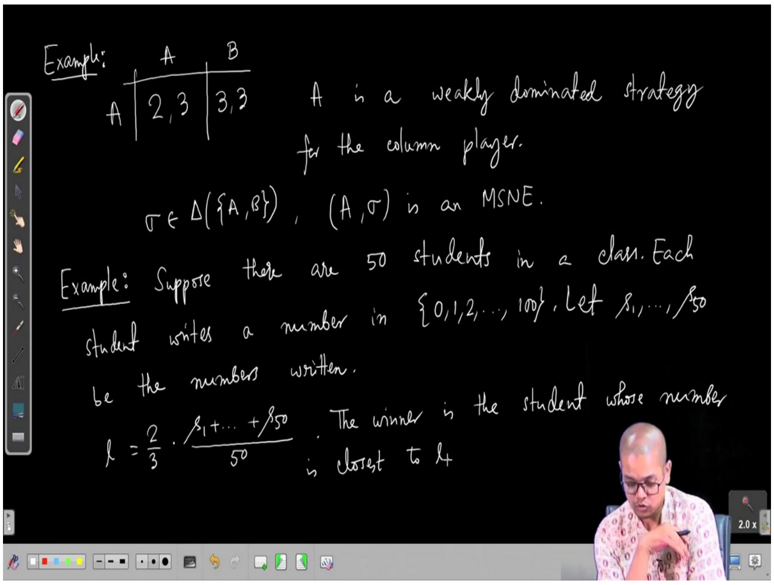

So, let us take an easy example, say a 2 player game column player has two strategies A B and the row player has only one strategy say A, suppose the utilities are 2 comma 3 and 3 comma 3. So, you see that you know both the strategies A and B both are same for the column player. So, this is an example ok.

So, you can say that A is a weakly dominated strategy for the column player weakly dominated by B and also B is also weakly dominated strategy for the column player, but the point is that because column player is indifferent between two strategies A and B it can play any mixed strategy. So, if you take any probability distribution sigma in delta A B. So, e comma sigma is a mixed strategy is an MSNE mixed strategy NASH equilibrium ok.

Let us check a the for the row player there is no other strategy to deviate to and for the column player both A and B gives the same utility. So, this shows that you know even if you have a weakly dominated strategy there can exist an MSNE where that weakly dominated strategy gets non - zero probability ok good.

So, now, let us give another example where this iterative elimination of strongly dominated strategy this becomes very powerful. So, here is an example. Suppose there are 50 students in the class in a class ok so, in a class who are the players, ok. So, we have 50 students in the class and we are playing a game. So, each student writes simultaneously of course, writes number in the set 0, 1, 2 up to 100. So, on a piece of paper they write it independently and simultaneously on a any number between 0 to 100 any integer number. And what is the rule of the game?

So, let s 1 to s 50 be the numbers written by the players, what we will do is that we will take? We will take the sum and take the we will take the maximum and take the average, let us see what is the two - third of the average here. So, let us take the average.

So, what is the average, $\frac{s_1 + \ldots + s_{50}}{50}$ this is the average and I take two- third of it let us call this l. The suppose it is a win loss game, the winner is the student whose number is closest to l. If there exist more than one student whose numbers are closest to l then suppose we pick winner uniformly random from them or suppose the utility of 1 is distributed among them and so on. So, what is the question?

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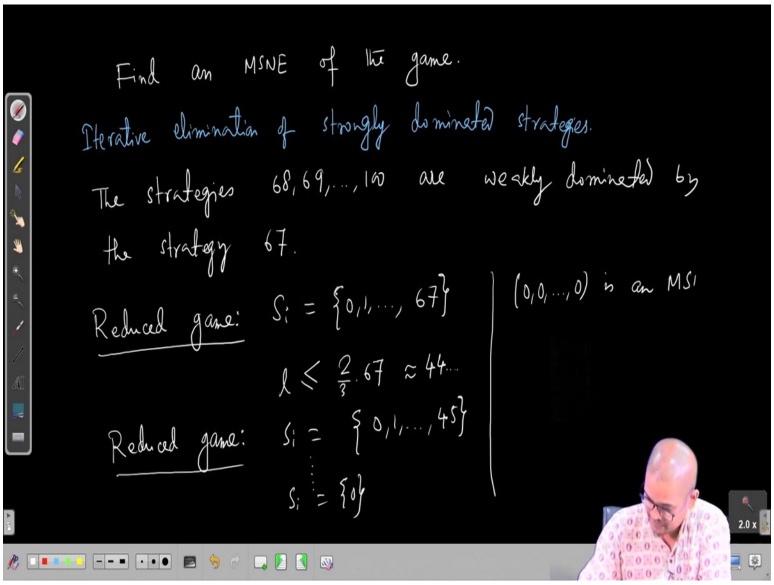

The question is, find an MSNE of the game find the mixed strategy in NASH equilibrium. Now, how will you proceed? Is it a finite game? Yes, there are only 50 students and the strategy sets are all finite. So, there must exist a mixed strategy NASH equilibrium by NASH theorem, but how will you find it ok. And the clue is iterative elimination of strongly dominated strategies iterative elimination of strongly dominated strategies.

Is there any strongly dominated strategies? Ok. So, let us see. What could each s_i be? What could be the maximum value for each s_i ? Each s i could be 100. So, the average also could be at max 100. So, the two-third of the 100 which is l would be at max like 67 66.67. So, what is the so, let me write this way.

The strategies 68 onward 68 69 till 100 are strongly dominated not strongly dominated it is dominated, but weakly dominated. So, this domination you can also clarify; strongly dominated, weakly dominated and very weakly dominated. So, what we have defined weakly dominated greater than equal to you take it as very weakly dominated and weakly dominated if like it is greater than equal to, but there exist something for which it is which is this inequality strict, but that way also the lemma does not change.

The strategy is these are weakly dominated by the strategy 67 because l is always less than equal to 67 and by playing 67 it is always you are always closer to the to l compared to 68, 69 and 100 and so on. Now, you so, eliminate all the strategies 68 to 100. So, the reduced game, the reduced game the strategy profile the strategy set S i is now from 0 to 67.

And again so, l what could be l, l is again the average could be at max 67. So, l is less than equal to two-third of 67 and now again you use apply this principle and what is this two-third, this is 22 like roughly 44 point something. So, now, you have again reduced game S i is all numbers greater than 45 is dominated by 45.

So, you keep till 45 and so on you keep doing this. So, at the end it will be left with only 0 and so, you conclude that all 0 is an MSNE and you can verify also unilateral deviation does not benefit any player ok.

Thank you.