Algorithmic Game Theory
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Lecture - 11
Iterative Eliminations of Dominated Strategies

Welcome to the 3rd week of the course. In the first 2 weeks we have introduced you basic definitions and basic concepts of game theory. Various equilibrium concepts like strongly dominant strategy equilibrium, weakly dominant strategy equilibrium, very weakly dominant strategy equilibrium, pure strategy Nash equilibrium and mixed strategy Nash equilibrium. There exist some more solution concepts which are further weakening of mixed strategy Nash equilibrium we will see in future lectures.

And that was the content of the 1st week. In the 2nd week we have covered matrix games mostly matrix games and zero sum games and this 3rd week we will begin with problem solving. We will see lots of examples to get used to this sort of concepts.

So, we begin with studying given a game how we can compute a mixed strategy Nash equilibrium and towards that there is a very powerful technique, which is known as Iterative Elimination of Dominated Strategies.

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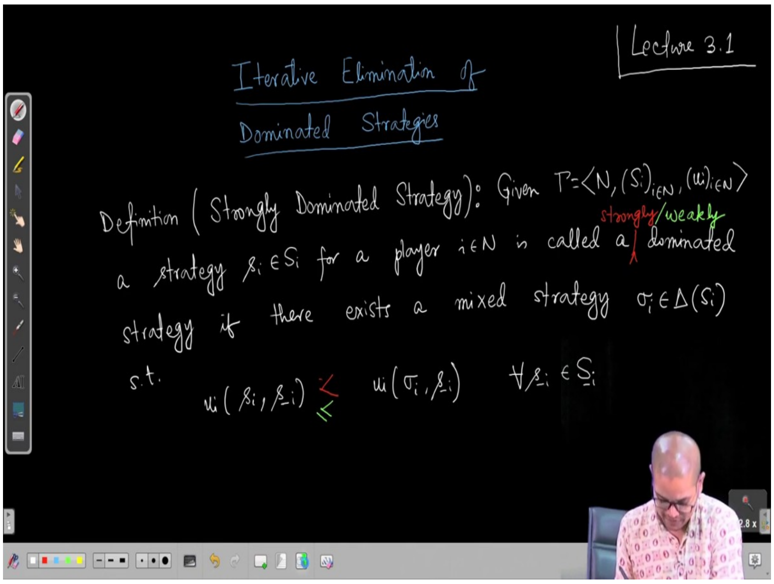

So, which is the topic of our discussion today let me write Iterative elimination of dominated strategies. So, let us define dominated strategies, we have only defined various kind of dominant strategies like strongly dominant strategy, weakly dominant strategy and so on, but what is dominated strategy? So, definition - of say strongly dominated strategy so, let us define it. So, as usual given game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

Given a game in normal form strategy $s_i \in S_i$ for a player $i \in N$ is called a dominated strategy if there exists a mixed strategy $\sigma_i \in \Delta(S_i)$. Let us recall $\Delta(S_i)$ is simply the set of all mixed strategies available to player i which is nothing but the set of all probability distributions over S_i. Mixed strategy σ_i such that the utility of player i by playing s_i is when other players are playing any strategy s_{-i} any pure strategy profile s_{-i} .

This is dominated by $u_i(\sigma_i, s_{-i})$. Now what kind of dominated so if we talk about say if we say that it is strongly dominated then this we should have a strict less than that the utility of player i by playing s_i when other players are playing s_{-i} is strictly less than $u_i(\sigma_i, s_{-i})$. And we can also talk about other kind of domination like strongly dominated or say weakly dominated. For weakly dominated the equality will be less than equal to the utility could be same, but it cannot it should not be more.

Player i cannot get more utility by playing s_i compared to σ_i when other players are playing a strategy profile s_{-i} and this should hold for all strategy profile $s_{-i} \in S_{-i}$. So, if such a strategy exists then we call that strategy a that strategy s_i a strongly dominated strategy or a weakly dominated strategy and so on.

Now, this is very useful to reduce or eliminate useless strategies and focus our attention on the only the useful strategies of the game and which significantly reduces the computation of finding MSNE or PSNE whatever you want.

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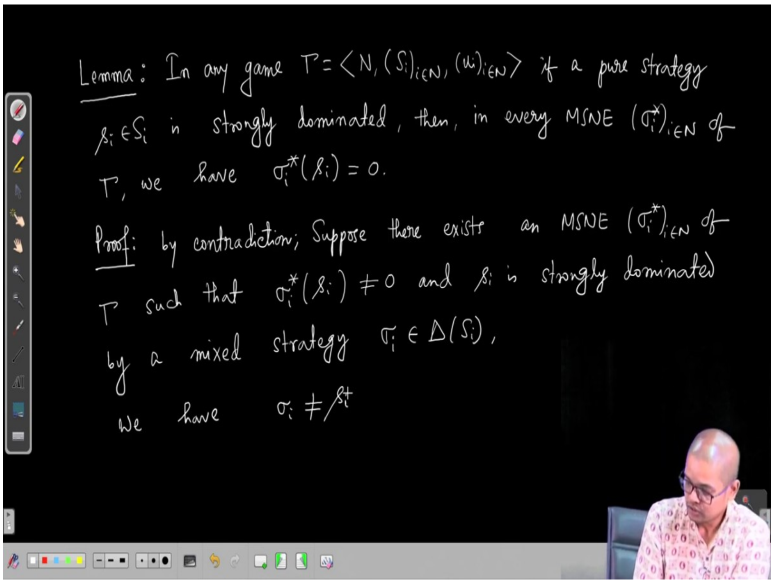

So, towards that this result is needed let me prove it as a lemma. So, it says that in any normal form game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ in any game if pure strategy small $s_i \in S_i$ is strongly dominated. Then in every MSNE say $(\sigma_i^*)_{i \in N}$. Then in every MSNE of Γ the strategy s_i gets zero probability. In every MSNE $(\sigma_i^*)_{i \in N}$ of Γ we have sigma star of s_i is equal to 0.

So, we will prove this result, but the implication is that if given a game if we are looking for an MSNE then we can eliminate all strongly dominated strategies because those strongly dominated strategies will not play any role in the mixed strategy Nash equilibrium of the game. So, that player i in particular will never play a strongly dominated strategy.

Proof: Very easy, so, proof by contradiction. So, it is a proof by contradiction. So, suppose there exists an msne $(\sigma_i^*)_{i \in N}$ of Γ such that $\sigma_i^*(s_i) \neq 0$ and s_i is strongly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$ ok. So, because it is strongly dominated then we have σ_i is of course, not equal to s_i otherwise the inequality cannot hold strictly obviously.

For weakly dominated strategies we assume this condition explicitly otherwise it does not make much sense ok. Now with this assumption I will contradict that this $(\sigma_i^*)_{i \in N}$ is a

MSNE, by how? By exhibiting another mixed strategy which yields strictly more in utility for player i when all other players are maintaining their strategies ok. So, towards that let us see.

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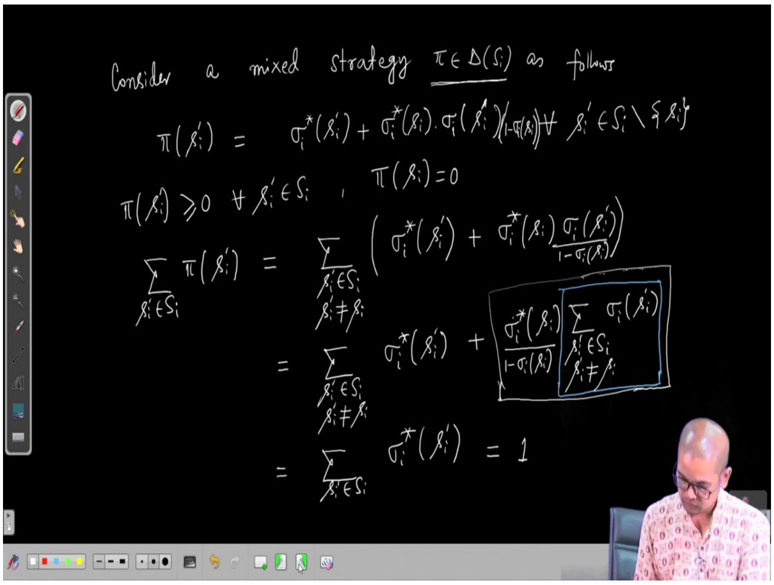

So, consider a mixed strategy $\pi \in \Delta(S_i)$ as follows. So, what is the definition of π? $\pi(s_i)$ is $\sigma_i^*(s_i^{'})+\sigma_i^*(s_i)\sigma_i(s_i^{'})$. For all $s_i^{'} \in S_i$. So, telling that you just consider this particular mixed strategy. First of all it is not clear that it is a valid mixed strategy it is a valid probability distribution. So, let us prove that that pi is a valid probability distribution pi is indeed an element of $\Delta(S_i)$.

So, let us see. So, $\pi(s_i^{'}) \geq 0$ for all $s_i^{'} \in S_i$, why? Because $\sigma_i^*(s_i^{'}) \geq 0$ because it is a probability distribution $\sigma_i^*(s_i) \geq 0$ actually it is strictly greater than 0 and $\sigma_i^*(s_i) \geq 0$. So, we have this inequality. Now what is summation $\pi(s_i^{'})$, what is this?

So, let us apply the definition for all $s_i^{'} \in S_i$ ok. So, this should hold for all $s_i^{'} \in S_i \setminus \{s_i\}$ and we have that $\pi(s_i)$ defining it to be 0. So, this $s_i^{'}$ this summation $s_i^{'} \in S_i$. $s_i^{'} \neq s_i$ add these terms. $\sigma_i^*(s_i^{'})$ plus. So, let us not let us not make this 0 let us make this so, we will see. So, this calculation will dictate what should be this value.

So, push this summation inside. So, summation $s_i^{'} \in S_i \; s_i^{'} \neq s_i \; \sigma_i^*(s_i^{'})$ plus when I push the sum this is $\sigma_i^*(s_i)$ comes out comes outside and we have this sum. $\sum \sigma_i(s_i^{'})$. This is $s_i^{'} \in S_i \; s_i^{'} \neq s_i$.

Now, what I want? I want to ensure that this summation this box is $\sigma_i^*(s_i)$. So, if this box is $\sigma_i^*(s_i)$ then you see that because σ_i^* is a probability distribution I can write it this to be equal to 1, but what is this box, this box is not $\sigma_i^*(s_i)$. This is like $1-\sigma_i(s_i)$ ok. So, to do that to make this 1 I want to make this blue box 1.

Now, to make the blue box 1 I all I need to scale the coefficients. So, what I do is that I keep $\pi(s_i)$ and what I do here is this term I divide by $1-\sigma_i(s_i)$. I divide by $1-\sigma_i(s_i)$. So, then here also I will have $1-\sigma_i(s_i)$ and here also $1-\sigma_i(s_i)$. So, now, you see that this inner sum within the blue box this value is $1-\sigma_i(s_i)$ because σ_i is a probability distribution and so, the numerator and denominator cancels.

And so, we have $s_i^{'} \in S_i \; \sigma_i^*(s_i^{'})$ and I merge this $\sigma_i^*(s_i)$ inside and so I skip this condition that small $s_i^{'}$ should not be equal to $s_i^{'} \neq s_i$. And because σ_i^* is a probability distribution this is 1. So, this shows that π is a valid probability distribution. Now we need to show that π is a better strategy by playing π instead of σ_i^* player i s utility is more. So, what is the utility of?

So, let us see where we are and σ_i the we have σ_i gives more utility to player i than s i strictly more inequality.

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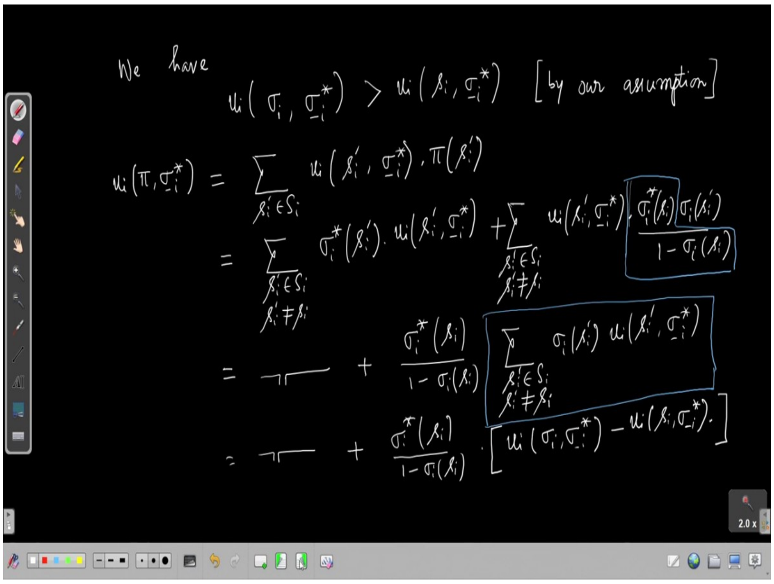

So, we have $u_i(\sigma_i, \sigma_{-i}^*)$ this is strictly more than $u_i(s_i, \sigma_{-i}^*)$ why? Simply because s_i is dominated by σ_i. This is again by our assumption. Now let us compute what is the utility of player i by playing π. This is summation over all strategies $s_i^{'} \in S_i$ $u_i(s_i^{'}, \sigma_{-i}^*)$.

Now what is this? This times $\pi(s_i^{'})$ right. So, what is $\pi(s_i^{'})$? $s_i^{'} \in S_i$ let us recall what was $\pi(s_i^{'})$ by our definition it was $\sigma_i^*(s_i^{'})$ plus this. So, we have $\sigma_i^*(s_i^{'})$ times $u(s_i^{'}, \sigma_{-i}^*)$. Let us keep $s_i^{'} \neq s_i$.

I am just plugging in the value of π of $\pi(s_i^{'})$ plus. Again $s_i^{'} \in S_i$ $s_i^{'} \neq s_i$ then $u(s_i^{'}, \sigma_{-i}^*)$ times the weightage, what was the weightage? Let us see it was $\sigma_i^*(s_i)$ times $\sigma_i(s_i^{'})$ times $1-\sigma_i(s_i)$ of let us write see $\sigma_i^*(s_i)$ $\sigma_i(s_i^{'})$ by $1-\sigma_i(s_i)$ ok. And then of course, you have another term of s_i and which is 0 ok.

Now, you see, what is this? The first term remains as it is and look at the second term again you can take this outside let me use some other colour, the terms independent of $s_i^{'}$ can be taken outside. So, $\sigma_i^*(s_i)$ $1-\sigma_i(s_i)$ summation $\sigma_i(s_i^{'})$ $u_i(s_i^{'}, \sigma_{-i}^*)$ this is $s_i^{'} \in S_i$. $s_i^{'} \neq s_i$ ok. Now compare these term what is this, this is does all the terms of $u_i(\sigma_i, \sigma_{-i}^*)$ of so, let me write.

This is the first term remains as it is plus $\sigma_i^*(s_i)$ by $1-\sigma_i(s_i)$ and then we have this is $u_i(\sigma_i, \sigma_{-i}^*)$. So, that is it. Now let us we need to compare this with $u_i(\sigma_i^*, \sigma_{-i}^*)$. So, let us write $u_i(\sigma_i^*, \sigma_{-i}^*)$ and then we will adjust the terms.

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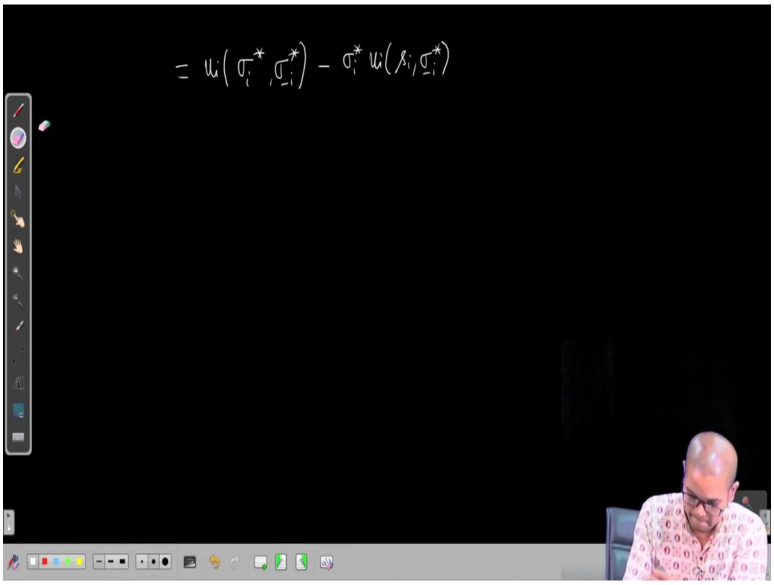

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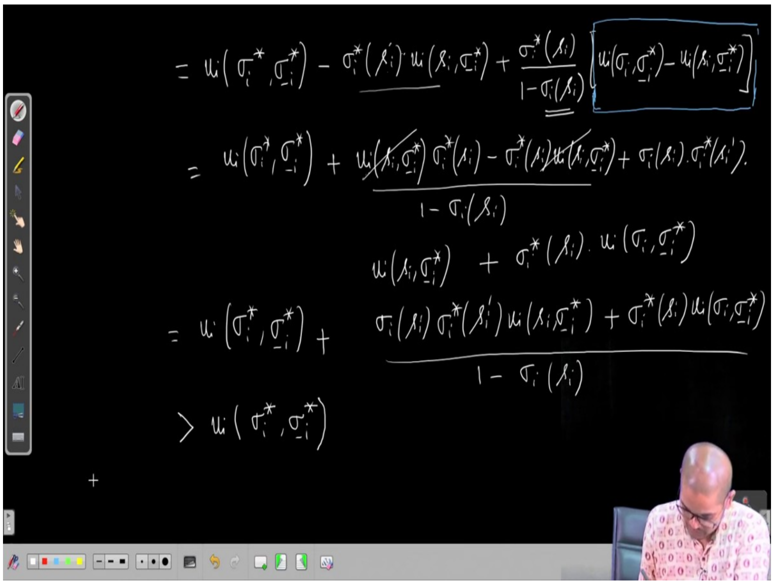

So, this is $u_i(\sigma_i^*, \sigma_{-i}^*)$ and let us see what are the terms we have. So, the extra term that we need to adjust here is $u_i(\sigma_i^*, \sigma_{-i}^*)$. No not this $u_i(\sigma_i^*, \sigma_{-i}^*)$. And what else we had?

We had this term σ_i^* by $(1-\sigma_i^*(s_i))$ by $(1-\sigma_i(s_i))$ and what else we have? We have this term.

So, $u_i(\sigma_i, \sigma_{-i}^*)-u_i(s_i, \sigma_{-i}^*)$. Now you see term by term this term is positive or let me further simplify it this is $u_i(\sigma_i^*, \sigma_{-i}^*)$ plus what is $u_i(s_i^*, \sigma_{-i}^*)$. What is this term? So, let us see this is this by $(1-\sigma_i(s_i))$ and minus sorry. So, this times $\sigma_i^*(s_i)$ minus $u_i(s_i, \sigma_{-i}^*)$.

So, this is one times this plus $\sigma_i(s_i)$ times $\sigma_i^*(s_i^{'})$ times $u_i(s_i, \sigma_{-i}^*)$. So, I have multiplied the denominator term $\sigma_i(s_i)$ with this term and did I miss any term yeah. So, one more term is remaining plus $\sigma_i^*(s_i)u_i(\sigma_i, \sigma_{-i}^*)$ ok. So, it looks complicated, but you know some terms cancels these two terms cancels and what we have is that $u_i(\sigma_i^*, \sigma_{-i}^*)$ plus the remaining terms is all positive that is the thing.

So, let me write $\sigma_i(s_i)$ $\sigma_i^*(s_i)u_i(\sigma_i, \sigma_{-i}^*)$ plus $\sigma_i^*(s_i)u_i(\sigma_i, \sigma_{-i}^*)$ by $1-\sigma_i(s_i)$. Now the numerator is positive denominator is positive. So, the additive term is strictly more than 0.

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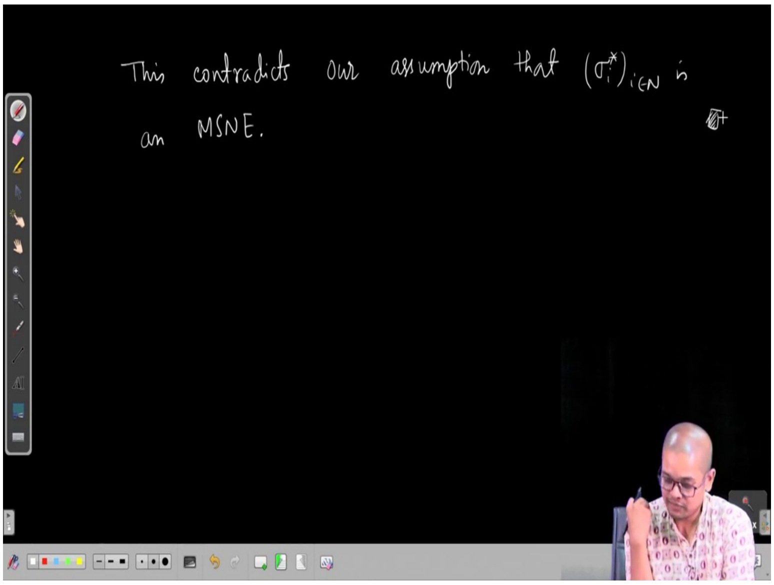

So, what we have is that this is strictly more than $u_i(\sigma_i^*, \sigma_{-i}^*)$. And this, this contradicts let me write contradicts our assumption that this $(\sigma_i^*)_{i\in N}$ is an MSNE, which concludes the proof. Now what we will do is that we will apply this lemma iteratively we will see an example in the next lecture.

And we will see that this reduces the complexity of the game very much and then some most of the times or many times it becomes very easy to compute the MNSE so, that we will see in the next lecture ok.

Thank you.