Computer Vision
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Lecture – 09
Homography: Properties Part - II

So, we will continue our discussion on the properties of projective transformation and we are discussing about the existence of different subgroups in the projective transformations group of projective transformations.

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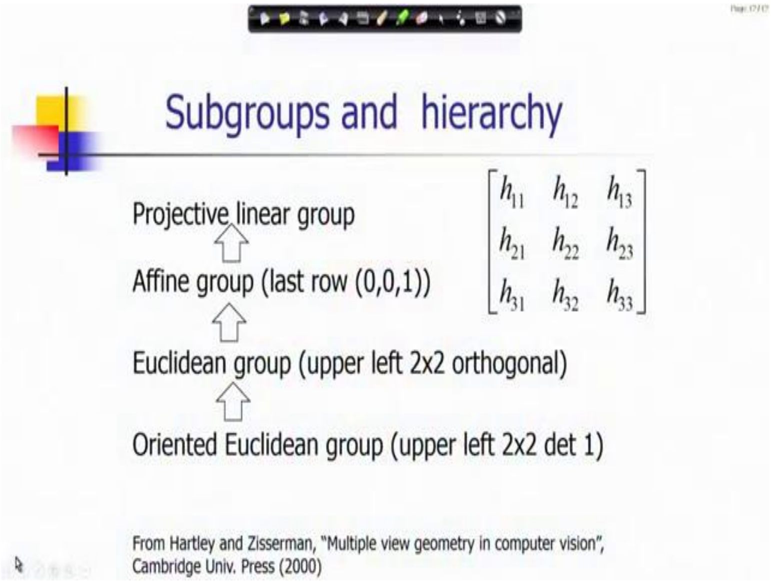

And, we have seen that in general projective transformations they belong to a group call projective linear group and a subgroup of projective linear group is affine group. The characterization is that the last row of the transformation matrix should be (0, 0, 1) or a scalar multiplication of (0, 0, 1) row vector. Then a subgroup of a affine group is Euclidean group, where the upper left transformation sub matrix 2×2 transformation sub matrix it should be orthogonal; which means if you take dot products of rows between 2 different rows it should be 0 and the self dot products should be non-zero.

And then the bottom most subgroup; that means, which is also a special category of Euclidean group is called oriented Euclidean group, when the determinant of these orthogonal sub matrix should be equal to 1. So, these are the different kinds of groups or subgroups those are present in the projective transformations, in the group of projective transformations. Let us discuss that what different kinds of properties are there for each group.

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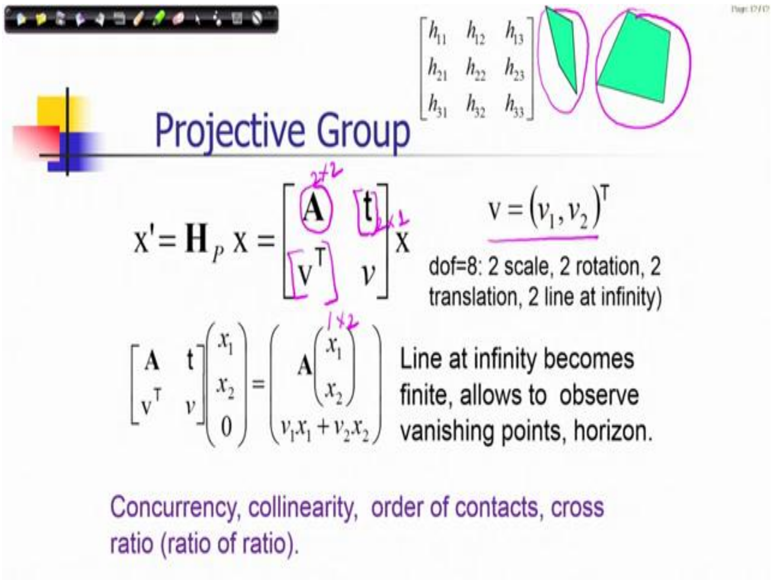

So, the parent group which is a projective group that is represented in this form. You can see that it is a 3×3 matrix, but we are representing it by their sub matrices where A is a sub matrix of 2×2 sub matrix. So, this is a 2×2 sub matrix. And t is 2×1 column vector, V transpose we have already shown how V should be represented its a column vector of $(v_1, v_2)^T$

So, V^T itself it would be 1×2 and this v which is just a scalar quantity. So, this is the representation of any general projective transformation matrix that we have already discussed. Representation we have seen that some typical examples of this transformation of a rectangle or a parallelogram has been shown here, some of the shapes you can see that these shapes they do not preserve the parallelism in this particular transformation in general not necessarily they preserves parallelism.

So, the degree of freedom; that means, a number of independent parameters in the transformation matrix is 8, there are 2 scales, 2 rotations, 2 translations, and 2 parameters are meant for line at infinity. So, when we decompose this matrix we will find out that we can decompose it into this parameters, it will be clear later on when I will be discussing the matrix decomposition of this projective transformation. So, let us take it for the time being that its degree of freedom is 8, 8 which means there are 8 independent parameters by which you can describe this transformation matrix.

$$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

And this is another interesting property which has been shown here; that means, if you transform any ideal point you can note that the third coordinate of the ideal point is 0 here. So, the transformation of ideal point is not going to give you always an ideal point because the third coordinate is $v_1 x_1 + v_2 x_2$ which may be non-zero which means this point would be a finite point in the transformed projective space.

And that is why line at infinity will be finite in the transformation space, it will be a vanishing line in the transformation space and it allows you to observe vanishing points or horizon after the transformation. So, what are the properties which are conserved even after transformation? Their concurrency; that means, if you find 3 straight lines which are concurrent; after transformation also you will find transformation of all those straight lines will be also concurrent.

It preserves the co-linearity that is the best property of any projective transformation from definition itself co-linearity has to be preserved; it preserves order of contacts it preserves cross ratio which is ratio of ratio. Let me explain this cross ratio further. So how do you define a cross ratio-?

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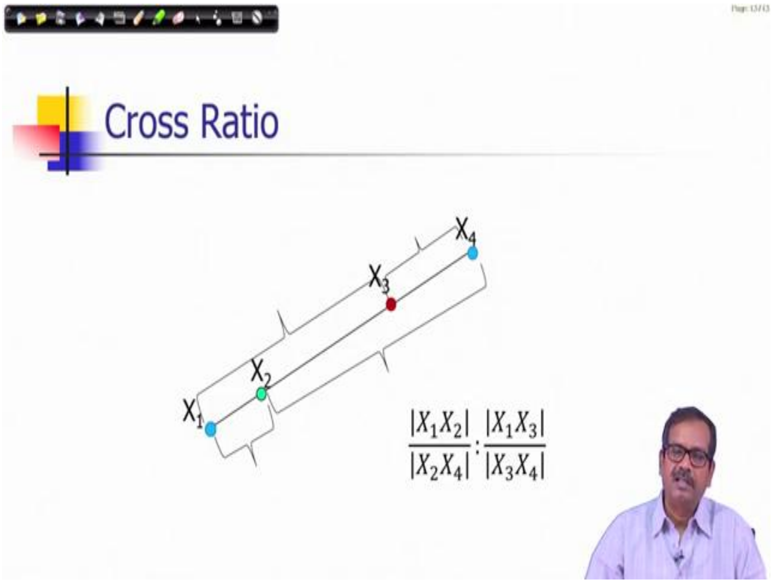

Let us considered four points lying on a straight line; so, we can consider the ratio the cross ratio is defined in this way that is the ratio of ratios. So, you can see that ratio of length between $|X_1X_2|$ and $|X_2X_4|$. So, this is the ratio first ratio this numerator of this ratio of ratio, the numerator ratio is defined in this way ratio of length between $|X_1X_2|$ and $|X_2X_4|$ and the other ratio is $|X_1X_3|$ and $|X_3X_4|$. So, the ratio of these two ratios defines a cross ratio. And, if you even if you transform this points after applying projective transformation, first thing they will all be collinear as this is what is the property of a projective transformation and they all maintain the same order those are also preserved.

And then this cross ratio is computedc there in the same fashion, you will find that cross ratio and this cross ratio they are the same. So, this is one of the invariants of this particular transformation.

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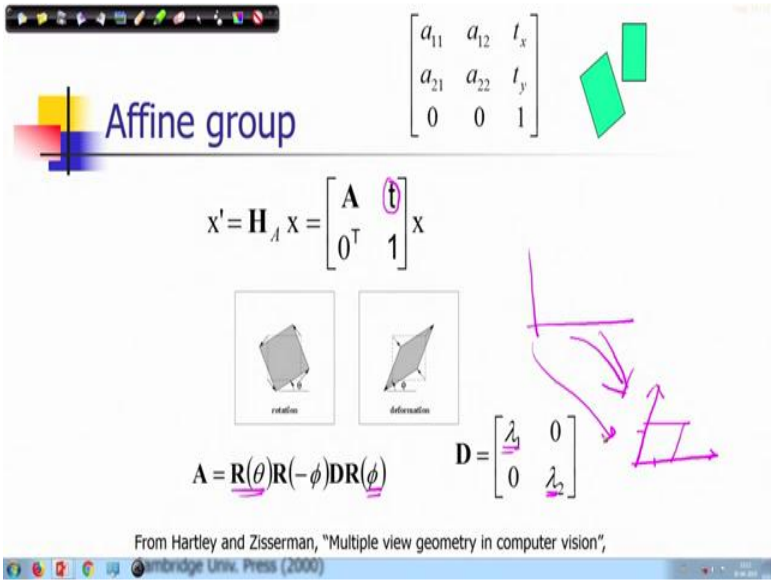

So, let us consider the properties of affine group. As we have already seen or we have already discussed that the third row of the affine transformation should be in the form of a vector (0 0 1) or you can multiply it using any scalar constant. So, let us for the sake of convenience we consider the representation of (0 0 1), one kind of representation. And here I have shown figuratively also how parallelograms when it is transformed in the affine applying affine transformation, still they remain parallelograms which means one of the invariants of this transformation should be the parallel lines.

Parallel lines still remain parallel after affine transformation. And how many degree of freedom we have here? You have only you can count the number of elements there itself there is 6 and in fact, there are six independent parameters and further this matrix A can be decomposed into this operations. So, as if you are applying distortion in the deformation in the corresponding perpendicular to axis directions.

So, you are applying deformation means, you have to apply a rotation first determine align it with respect to the axis where the deformation as to be applied and then you apply the deformation in two perpendicular directions that is given by this diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. So, you are applying you are introducing deformation by using two different scales and your coordinate measurements at two oblique axis. So, as if your coordinate convention instead of considering rectilinear, you know axis representation, we have this oblique axis representation let me draw it once again.

So, you had a rectilinear coordinate representation, but in the transformed space you are following a non-rectilinear representation, but measurements should be all by the parallel lines of those two baselines. Now so, your unit scales they are scaled by different factors there and that is how the co-ordinate transformation is taking place while applying the affine transformation along with this you can rotate the co-ordinate axis that is another parameter θ

$$A = R(\theta)R(-\phi)DR(\phi)$$

So, you can see here that there are four independent parameters in A one is this angle ϕ then this two scale factors and then again you are rotating back and then after that you are applying rotation of the whole transformation points angel θ. So, this is in this way you can account for the coordinate transformations through A and the other thing what you are doing? You are translating the even you apply the translation of the origin so, another two independent parameters. So, there are 2 translation parameters 2 rotation parameters and 2 scale parameters. So, those are the two things those are the things what we discussed in the previous case also those are the parameters in this particular affine transformation.

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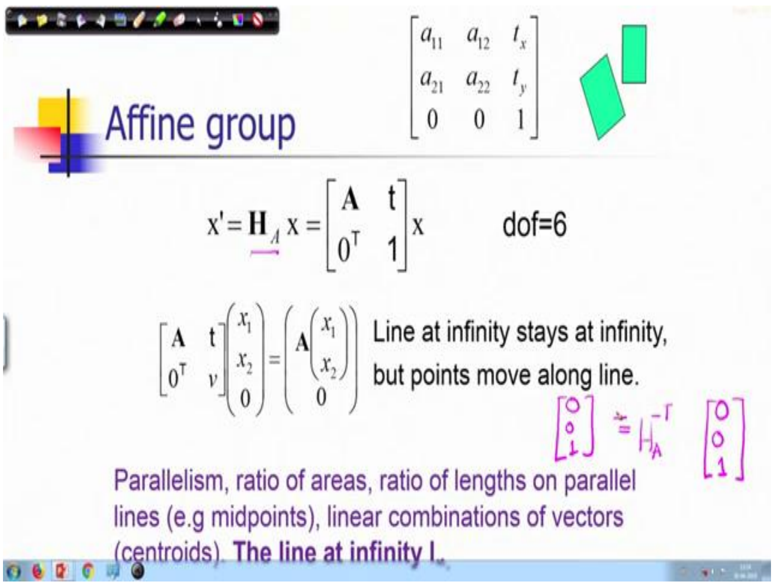

So, just to summarize you have 6 independent parameters and once again if I apply affine transformation on ideal points, you can see that it still remains ideal because your scale factor still remains 0 which means that, your parallel lines they will be still remain parallel because after transformation the intersection points also are ideal points. So, line at infinity stays at infinity. So, line at infinity is the one of the invariants of this particular transformation. So, what are the invariants of affine group? Parallelism, then ratio of areas, ratio of lengths on parallel lines for example, midpoints, then linear combinations of vectors for example, centroids.

So, they all remain invariant and the line at infinity that is one of the major property that line at infinity still remains the same. So, if you transform a line at infinity represented by the vector (0 0 1), then you will still get the corresponding transformation as (0 0 1). So, which means , if I perform this operation say this is the line at infinity and then multiplied by the matrix H_A^{-T} which is transpose of inverse of this particular transformation matrix, then you will get the corresponding representation of line at infinity, it may be multiplied by a scale factor, but still represents the line at infinity as

(0 0 1).

So, the implication is that it means that all vanishing points after transformation they still lie on (0 0 1) which means they are all ideal points and which means after transformation all parallel lines still remain parallel that is one of the very interesting properties in affine group.

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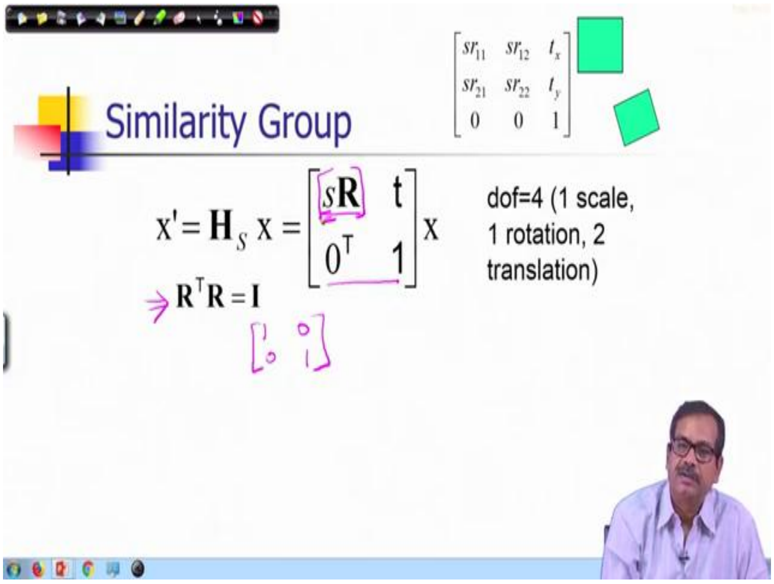

The Euclidean group what we discussed can be also rename that can be called also as a similarity group, because it maintains a similarity property of triangles say when in triangles when all the edges are parallel, then their ratios are preserved ratio of edges they are preserved. So, in this case ratios of distances are preserved. So, that is why it is called similarity group and its particular structure is given in this form you can see that the corresponding 2×2 sub matrix can be simply represented by this decomposition that is means it is a scalar multiplication of a orthonormal matrix R where $R^T R$ equal to identity matrix.

This is identity matrix which means it should be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which means that R is an orthonormal matrix and T is once again it is a 2×1 column vector and so, this transformation matrix is a special case of affine transformation matrix, on top of that it has this constraints that it has to satisfy this particular property which makes the 2×2 sub matrix as orthogonal. So, in this case you have four independent parameters one is for scale. So, this is what is the scale, the other one is rotation, you can keep the axis rectilinear, but you can rotate the axis orientations of this axis and 2 parameters are for translation you can translate the origin.

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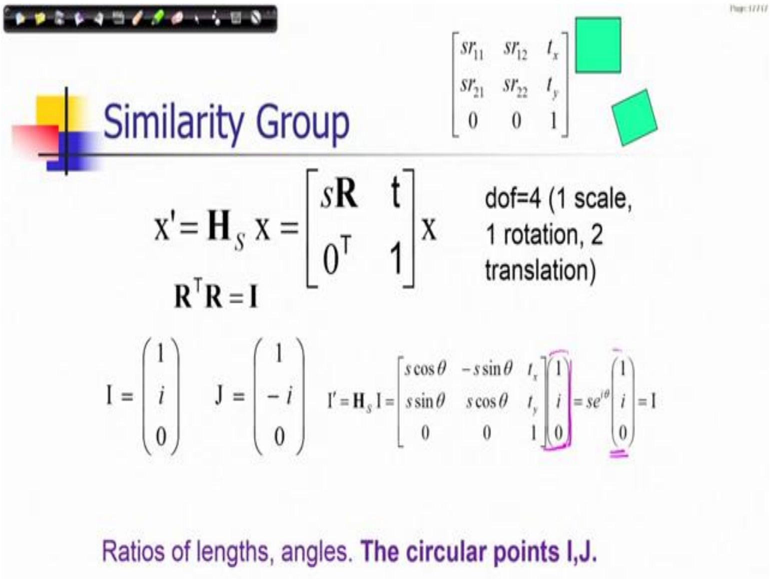

So, there is another interesting property. there are 2 special points which are given in this form that $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ So, you consider instead of a real space you consider a space it is a projective space with complex coordinate systems. So, we have $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$ that is the representation. So, one of the axis is considered as the imaginary axis and that is the representation that is how a two dimensional projective space can be also represented by simple complex numbers.

And then this points are interesting because if you apply transformation on this points any similarity transformation, you will find that it still remains the same point; that means, if I apply transformation similarity transformation of this particular point $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ Then you will find the transformation point is still a scaled factor of $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ which means in the projective space in consideration, it is the same point, it is just multiplied by scale factor this is true for the other point J also. So, these are the invariants of this transformation we will see later that it has useful properties this is very useful for certain computations.

So, these are the things which are invariant under similarity transformation, they are ratios of lengths angles and the circular points I, J. Of course, you have to consider all other invariants which are there earlier for affine group and also projective transformation groups. Since it is a hierarchical relationships all those properties they are also true for a similarity transformations. So, which means the parallel lines they remain parallel even after this transformation, line at infinity after applying this transformation still remains at line at infinity at (0 0 1). So, those are invariants even after this transformation and finally, the oriented Euclidean group which is also called as isometry.

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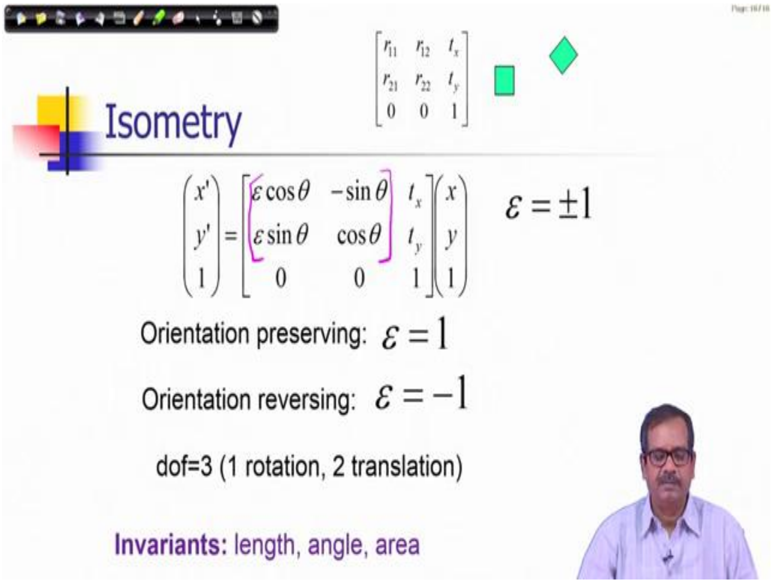

Because it preserves even the distances even the lengths between 2 points and you can see its particular structure, here we can see that the representation could be $\begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$, where ε is ± 1 which means that if I take the determinant of this sub matrix then that should be equal to 1 that is one interesting property. So, the number of parameters as you can see it is θ and t_x and t_y. So, number of parameters in this case is only 3 and it is orientation preserving when ε value is 1. otherwise it is a orientation reversing directions like reflection.

And so, there are 3 independent parameters- 1 rotation and 2 translation as I mentioned; and in this case you have special invariants they preserve the lengths. So, it is not just ratios of lengths, ratios of lengths are preserved in the case of similarity transformation, it is also a lengths those are also preserved angles, preserved areas so, that is why this transformation is called isometry that is another type of transformation. And this transformation is a subgroup of similarity transformation and all the properties of projective transformation, affine transformation, similarity transformation all those properties are also true for a isometry transformation in addition to them you have this three other properties invariants length angle and area.

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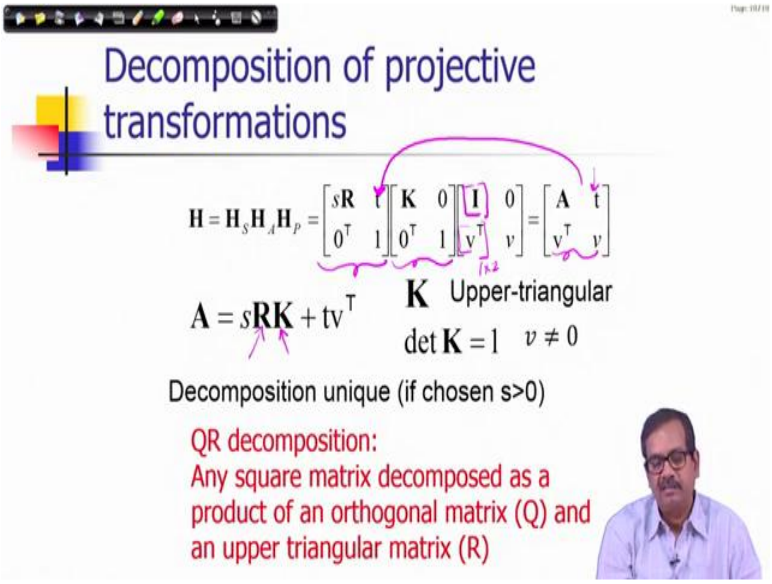

So, let us take an example that of decomposition of a projective transformations. So, any projective transformation can be decomposed as a cascade of all this special transformations like it is a cascade of similarity transformation affine transformation and general projective transformation. And, its forms are also given here you can find out that in this case, similarity transformation has this particular structure and then affine transformation this is affine transformation, the K is an upper triangular matrix and there is a particular form the $\det K = 1$. You should note that K does not form orthogonal matrix, but its determinant is one. So, that is why its an affine transformation and because you are its last row is once again (0 0 1).

$$H = H_S H_A H_P = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix} = \begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$

And this is a projective transformation, I is a identity matrix it is a 2×2 identity matrix and its other parameters are v^T which is a 1×2 row vector and this is a scalar amount. So, the relationship between the original transformation matrix which has been decomposed; you can see this third row it defines the component of projective transformation where as the translation matrix t it is used in the similarity transformation. And, A has to be decomposed into this form $A = sRK + tv^T$ and which is expressed as in this particular matrix multiplication form. So, you know t, you know v^T

So, you have to perform decomposition of this matrix after subtracting tv^T from A and then perform the matrix decomposition. one of them as you can see, R is orthonormal and K is upper triangular matrix. So, this matrix decomposition can be performed to any QR decomposition method where Q is orthogonal matrix R is a upper triangular matrix and that is a very standard method by in linear algebra that you can do and this decomposition is unique if s is positive, if s is kept positive as you know that projective transformation it is a scale factor which matters.

So, we have put the restriction that s has to be positive. So, as I mentioned that through QR decomposition you can derive the corresponding components of R and K. And, this is how you can get all the transformation matrix we can decompose any projective transformation matrix as a cascade of similarity affine and a general projective transformation matrix in a special form.

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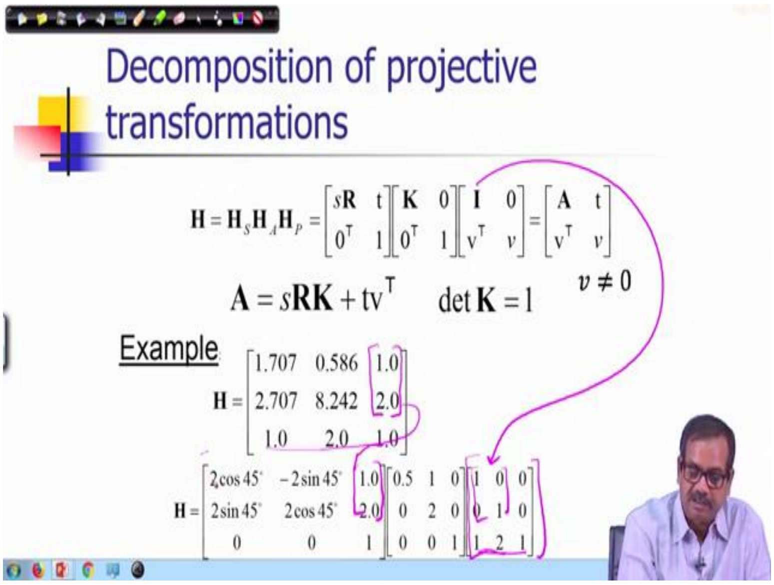

So, let me give you an example consider a transformation matrix H this is given in this form and if I perform this decomposition it would look like this. So, here you can see that this is the projective transformation part, where this is the corresponding identity matrix that is 2×2 identity matrix, this [1 2 1] this is totally defined by the last row of this transformation matrix. Then the corresponding translation parameters they are used directly in the similarity transformation matrix.

And using this after that you can perform the matrix decomposition of A and you can get the corresponding similarity matrix and affine matrix. So, this is one good example of how matrix decomposition can be carried out to make the transformation to show the transformation as a series of different transformations of certain structure.

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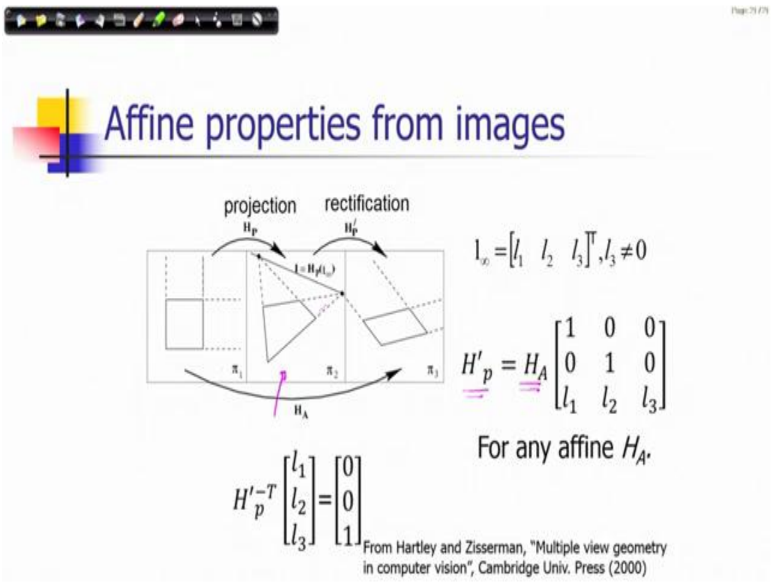

So, this particular this properties could be used for rectifying the images, it is called rectification process. let me explain what is meant by rectification process you consider here you have the image which is observed in this form which has been given in the plane π_2 . So, this is what you have observed you can see the edges . they do not appear as parallel because of transformation; that means, suppose actually in the original space you have a rectangle, you have a square in this particular example.

And, there after applying the two dimensional projective transformation, they take the shape of quadrilateral where the parallel edges in the original space. appear like meeting at a vanishing point and you can get a vanishing line after this transformation. So, one of the task could be that how to make those seemingly parallel lines from our experience we can identify which are parallels and how to make them parallel again, by applying another transformation say H_p'

Which means after this transformation I should at least get this 2 lines parallel it is not really giving me the original square shape where the rectangular property of the straight lines of the corners they are not preserved, but still at least we can get the parallelisms of those edges in this particular transformation. So, this process is called rectification process. So, we are trying to compute a transformation from here to here so, that it gives corresponding edges.

Now you should note that the transformation from the original space π_1 to π_3 that is an affine transformation. However, this transformation is a general projective transformation H_p . Again H_p' is also a general projective transformation. So, the cascade of these 2 transformations can make it an affine transformation. So, you would like to see that how we can compute this H_p , H_p' . suppose you get the vanishing line of this particular vanishing line in the transformed space and which is represented in this form by vector $[l_1 \quad l_2 \quad l_3]^T$ and then you can define a transformation in this form say

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$ So, you can see that this is a general projective transformation because last row is not [0 0 1], but H_A is a affine transformation. So, its any affine transformation. So, if I multiply this matrix with any affine transformation, still we will get a general projective transformation. The property of this transformation is that if I transform this vanishing line then I will get line at infinity. that is very interesting.

That means, whatever vanishing points you had here, now, they will all become ideal point after this transformation H_p' which means this lines now in the transformed space they will appear like a parallel line. So, this is one good method by which at least you can make the edges parallel and this is how we can carry out rectification. So, let me take some example of this process.

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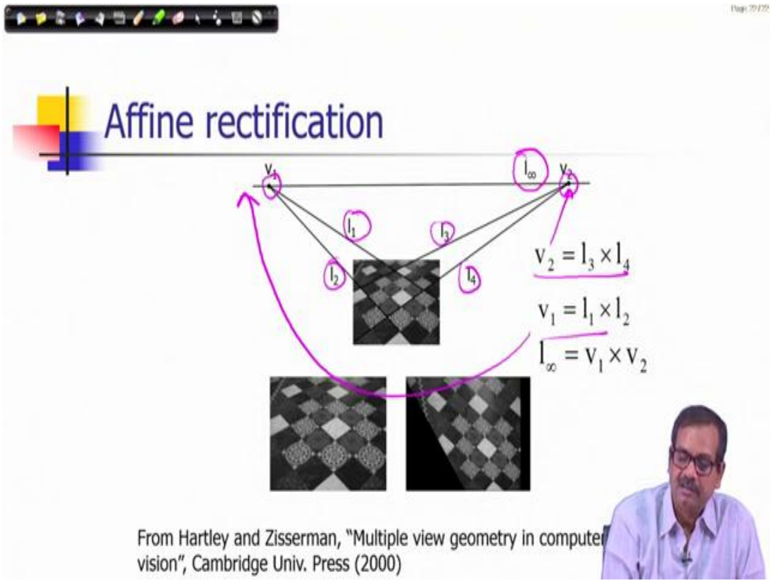

So, first let me give you a figurative examples that how this computation can be proceeded.. See you consider a an input image where you can see here that edges they do not look like parallel in fact, seemingly parallel edges they are meeting in this particular image at some finite point. So, you choose two such parallel lines direct two such set of parallel lines, each one will be giving you two vanishing points. So, with these two vanishing points you can compute the vanishing line.

So, you get the corresponding representation of vanishing line which is the transformation of the line at infinity and from there you can compute the corresponding projective transformation. So, the computation goes like this. So, you take the cross product of this lines l_3 and l_4. So, that would give you

v_2 similarly cross product of line l_1 and l_2 that would give you v_1, and then $v_1 \times v_2$ will give you the corresponding line at infinity which is a finite line which is the vanishing line.

And, from there if I apply the transformation then this is one example that you can find out edges have become parallel in the corresponding transformation. They have not become square or rectangle, but at least they have become parallelograms. So, we will continue with another example.

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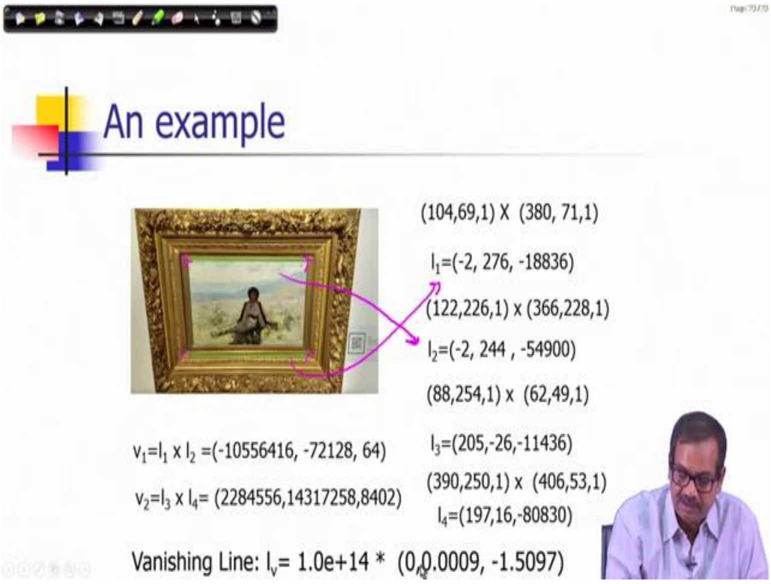

So, take this particular example, you have an image here and you can see that the edges of the image, they are not really parallel though you know that they are not only parallel they form a rectangle. Now, you consider two parallel lines and then you compute their corresponding vanishing point or you compute this lines. So, you take in this case suppose you have taken this parallel line say this line you have considered this two points which defines this lines. So, l_1 is defined by this line, similarly you are computing l_2 which means you are taking this two points.

So, these are the two points and from there you are getting l_2 and you continuing this process you are taking another two lines by defining this two end points. So, in this case lines are define by two end points and from there you get l_3 and l_4 and then as we have computed we take the cross product of l_1 and l_2 which give you the vanishing line vanishing point of $l_1 \times l_2$. That means, you get the vanishing point of $l_1 \times l_2 =$ [-10556416,-72128,64] big numbers are coming which means you see that if I consider multiplication with respect to with respect to 64 these are big numbers.

So, we will see later on. So, this is the vanishing point and vanishing point of l_3 and l_4 is this number your vanishing line is given by this. So, there are big numbers do not bother about that, they are coming due to the computations, but finally, as there are scale factors associated you can see the vanishing line is given by this.

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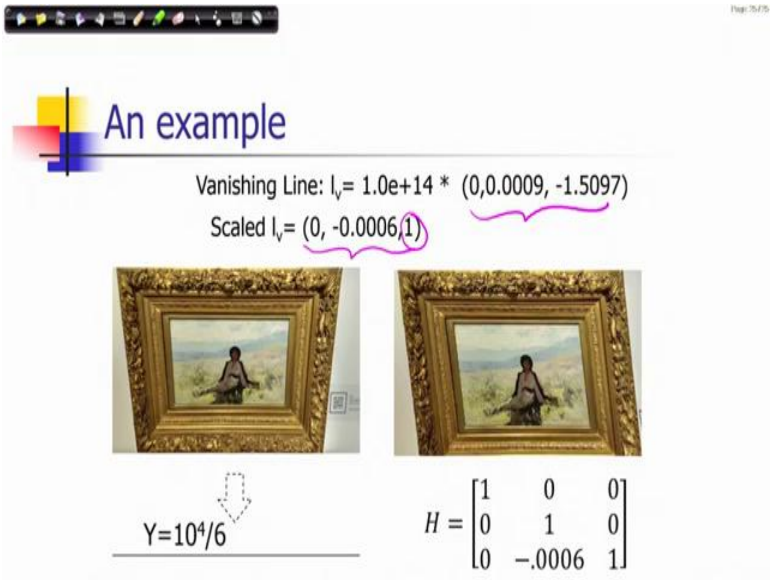

So, the vanishing line as I was mentioning the essential component if I take out the scales can be considered only this vector and then I can represent equivalently by this form also. So, this is equivalent representation by making the scale of this third dimension as equal to 1. And so, actually it will be a vanishing line at a very large distance it is almost horizontal as you can see that this component is 0. So, equation becomes $Y = \frac{10^4}{6}$.

So, that is why these horizontal edges they look almost like horizontal because they are meeting at a point there are almost like parallel to the vanishing line also. And, the transformation matrix will be given $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -.0006 & 1 \end{bmatrix}$. So, you can identify that $[l_1, l_2, l_3]$ those are the 2-3 components they look like this and you can see the affect. If I apply this transformation matrix on these points then you will get this images and where you can find out the edges they look parallel. So, with this let me stop here.

Thank you very much for your listening.