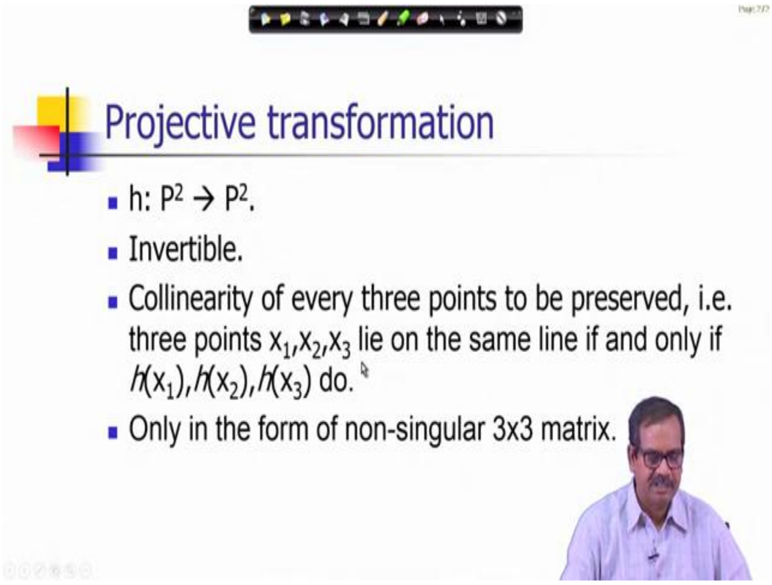
Computer Vision
Prof. Jayanta Mukhopadhyay
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture – 08
Homography: Properties Part – I

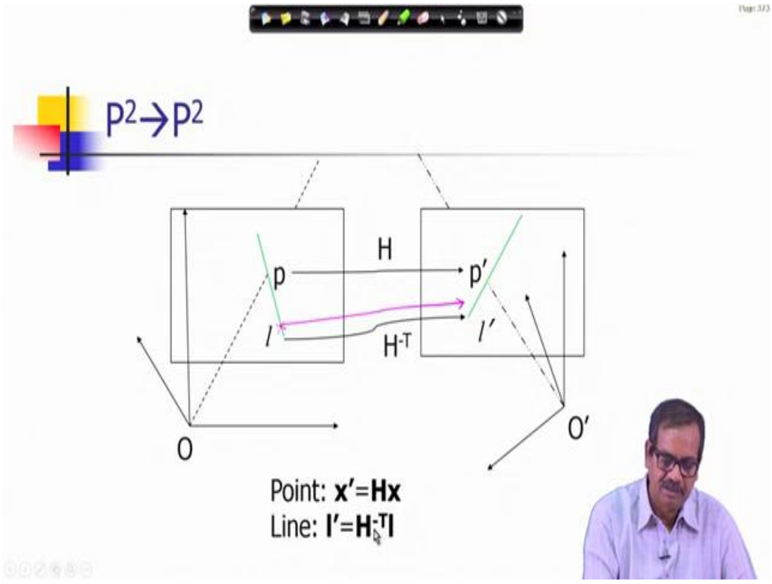
In this lecture, we will talk about different Properties of Homography.

(Refer Slide Time: 00:21)



So, let us summarize what are the features of a projective transformation. We know that a transformation is projective first it has to be a mapping from a two-dimensional projective space to another two-dimensional projective space. Then, it should be invertible and collinearity of every three points to be preserved that is three points for example, which lie on a straight line, then after transformation transform points also should lie on a straight line. And also we have discussed that there is only one form of this kind of transformation and that is in the form of a non-singular 3 cross 3 matrix.

(Refer Slide Time: 01:19)



So, let us consider a typical case of this projective transformation a kind of a schematic diagram by which we can explain these properties. So, you consider a projective space two-dimensional projective space and you can see that O is the center of projection and a projection plane which is in the canonical form is placed at a distance 1 along the z axis of this particular projective space. And let us consider another projective space in the same representation, similar representation, but it has a different coordinates of the center of projection, it has different orientations of the coordinate axis and also accordingly with respect to that its canonical projection plane is also defined.

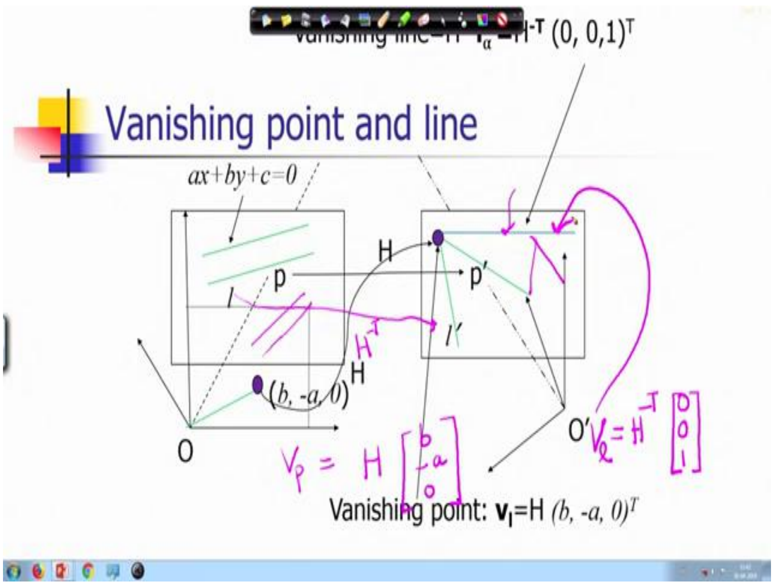
So, a point in this case O' is the center of projection of the other projective space. So, a point in the projective space which has been shown earlier that is represented in this form. We know that any point in the canonical space is represented by a ray connecting to the center of projection and whereas a point in the transform space that is also represented in this form and here we are showing that there is a mapping between the point p and p'. So, there is a mapping, and that mapping is defined by a 3×3 transformation matrix as we have discussed earlier. And this is how we denote a projective transformation. We can visualize the projective transformation, for every point in the space there will be another unique point, every point in the space there will be another point in the transformed space, where it will map to that particular point uniquely.

So, to summarize this property also we can say that , if there is a straight line and if the point lies on that straight line, , so if we map all the points in that straight line , they will also form another straight line and in that straight line all the mapped points will lie. So, we can say there is a mapping from straight line l to l' under this transformation.

And we have also discussed how this transformation is related, how we can derive the transformed points or transformed straight lines from the points or straight lines of the original projective space. So, the relationship is that for straight line you have to apply the transformation matrix which is transpose of the inverse of the transformation matrix for point transformation.

So, the relationship can be summarized in this form that for a point if I multiply the point x in the projective space representation which is an homogenous coordinate representation, if I multiply with a 3×3 transformation matrix H, then we will get the corresponding transformed point also in the homogenous coordinate representation. Similarly, for a line if we multiply it with the H^{-T} matrix which is transpose of inverse of H, then also we will get the corresponding transformed line.

(Refer Slide Time: 05:47)



So, let us discuss again that how vanishing points they are related in the transformation, in the transformed space. So, in this case we have drawn a pair of parallel lines and one of them is denoted by l. Consider l is transformed in the transformed space by a line l' and we know how l and l' is related. So, you multiply l, represent it in a two-dimensional projective space by transpose of inverse of this matrix H and then you can get the corresponding transformation of l'. That means, you have to multiple l with H^{-T}. Similarly, if I transform the other point of that parallel line we will get another straight line and this straight line incidentally we observe though they are parallel in the original projective space, but in the transformed space they will be appearing as a converging straight lines or actually they will intersect at certain points. Now, this point is called the vanishing point of this straight lines.

So, the interpretation of this vanishing point in the transformed space can be very easily given, if we understand that what is a intersection between this parallel lines in the original space. We know that we have already discussed that when two lines are parallel in a projective space they intersect at a point which is at infinity, but there is a finite representation of that infinite point.

And in this case for example, if we represent the straight line by this equation $ax + by + c = 0$ and then the all lines parallel to this straight line will intersect at a point in the projective space which can be represented by the coordinate $(b, -a, 0)$ it is a homogenous coordinate representation. So, the ray connecting this point and passing through the origin that same ray is representing this element. And we know that this point is also called ideal point.

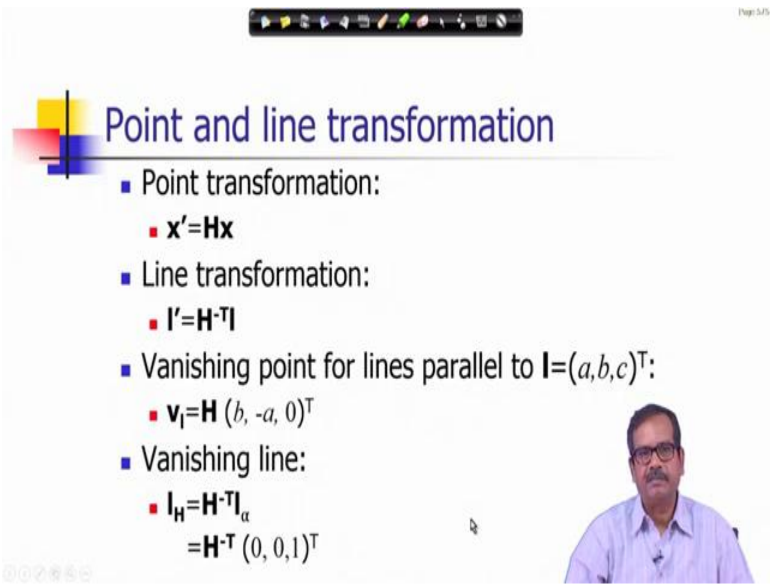
Now, the vanishing point of these parallel lines in the transformed space is nothing, but the transformation of this ideal point into this space, which means if I apply transformation matrix, if I multiply transformation matrix with this point then I will get the vanishing point. So, what we can get is that, we have to represent the intersection point of the parallel lines in the original space which incidentally is $(b, -a, 0)$ this column vector represent represents the intersection point for a line for all parallel lines which are parallel to the line $ax + by + c = 0$. So, if I multiply it with H then I will get the vanishing point, corresponding vanishing point, now that means, the corresponding coordinates of this point in the projective space with this computation. So, this is a simple interpretations of a vanishing point and that is what we have represented here.

Now, another interesting part is that, we know that if I take any other parallel line in the original space any of the set of parallel line in the original space, suppose you take another set of parallel lines in the original space and then if I transform these lines in the projective space they will also meet at some vanishing point and all these vanishing points they lie on a particular line which is called vanishing line. So, how do you get this vanishing line also in the transformed space? What is the representation?

So, there we know that all the vanishing points in the original space they lie on a particular line which is called line at infinity and that is given by the; that is given by this representation $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the let us say that is an element in the projective space representing the all the straight lines. So, these vanishing line is nothing, but the transformation of this line at infinity into this transformation space. So, which means that if I perform transformation of line at infinity by following the same rule of line transformation then I will get the corresponding vanishing line.

So, let me find out this particular equation to which is hidden here. So, let me write it here. So, what we can do we have to multiply the line at infinity represented by this from $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then we will get the corresponding vanishing line which is this line here. So, this is a interpretation of a vanishing point and vanishing line in the transformation space.

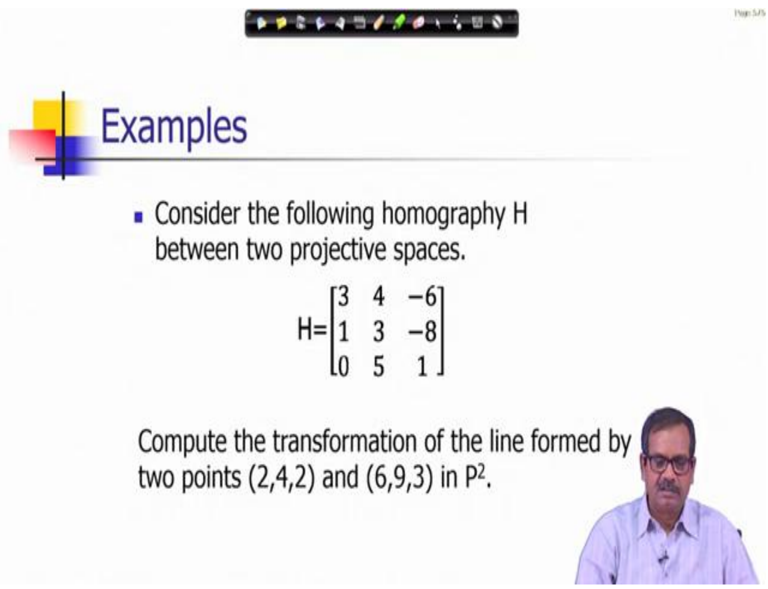
(Refer Slide Time: 11:59)



So, just to summarize all these properties in the transformation space or all the properties of this projective transformation homography. So, we have these are the transformations, those we have studied that is point transformation which means that we have to multiply a point in the projective space by the corresponding transformation matrix, then you get the corresponding transformation point in the transformed space. You should remember that all the representations are in the homogeneous coordinate form. Then line transformation, where a line has to be multiplied by the matrix H^{-T} which is a transpose of inverse of transformation matrix, then you will get the corresponding transformed line in the transform space.

Similarly, for vanishing point, for lines parallel to say represented by this form $(a,b,c)^T$ that is a form of a line then the vanishing points can be derived by multiplying the intersection point of those parallel lines in the original space with the transformation matrix which would $(b,-a,0)$ That is a typical representation of the intersection point of lines which are parallel to the line given by parameters a, b, c. And vanishing line would be the transformation of line at infinity of the original projective space which is given by $(0,0,1)^T$, and that has to be multiplied once again with H^{-T} which is transpose of inverse of transformation matrix and that would give us the vanishing line.

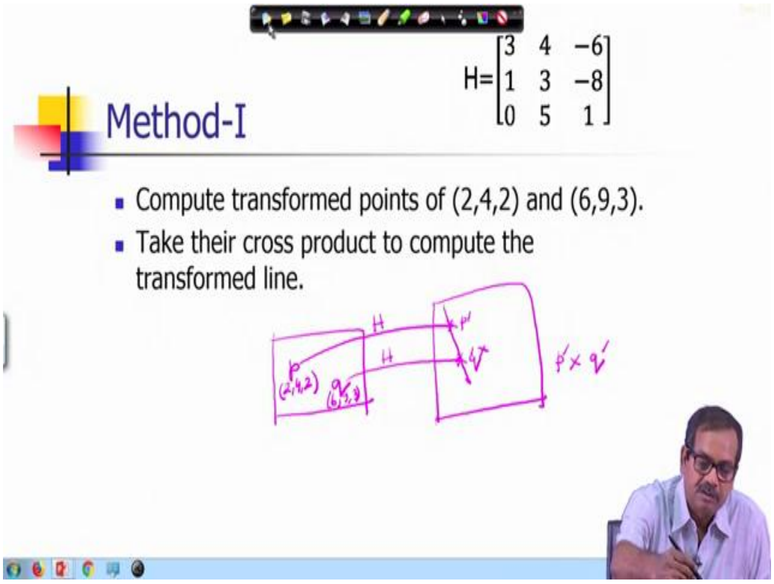
(Refer Slide Time: 14:01)



So, let us consider an example by which we can show all this computations. You consider a homography matrix which means this is a transformation matrix which is given in this form you can say this is a 3×3 matrix and which should be incidentally non-singular you can verify that. So, the computational problem is that you have to compute the transformation of the line formed by two points given by (2, 4, 2) and

(6, 9, 3) both the points in the two-dimensional projective space. So, let us see how we can compute this.

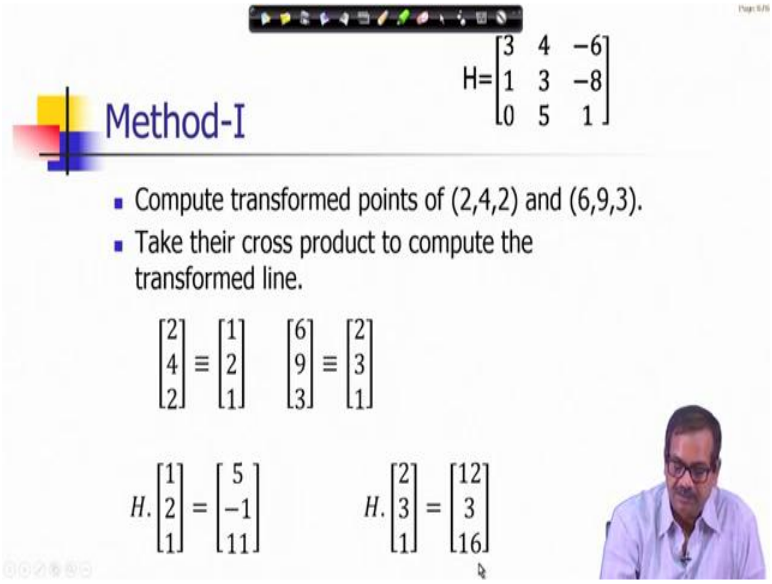
(Refer Slide Time: 14:43)



So, there are few methods by which you can do it, I will discuss two of them. So, the first method, in this method what we can do that first you compute the transformed points of (2, 4, 2) and (6, 9, 3). So, you get the corresponding transformation point in the projective space. So, it is figuratively I can show you. So, you have a projective space where you have this points say p and q, and p is given by the point (2, 4, 2) and q is says (6, 9, 3). So, you find out the corresponding transformation in the transformed space.

Suppose, this point is p' and this point is q'. So, what you need to do? You have to perform the multiplication of transformation matrix H with p and multiplication of transformation matrix H with q, then you get this point p' and q' those are the transformation point. Now, you compute the straight line in the transformed space, which means if I perform $p' \times q'$ I will get the corresponding transformation matrix. So, let me see this computations once again.

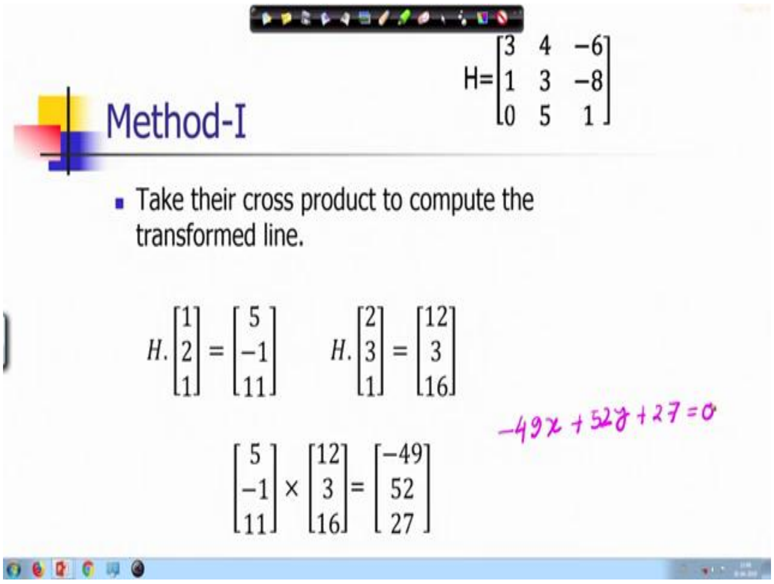
(Refer Slide Time: 16:17)



So, first let me reduce the representation of this points. Since, there is a scalar quantity 2, I can divide all the coordinates to make it more simple representations of (1, 2, 1) and similarly (6, 9, 3) I can simpler representation of (2, 3, 1), you could have performed using (2, 4, 2) and (6, 9, 3) also. So, it is just for the convenience of the computations I have taken these two. Now, you transform (1, 2, 1) and (2, 3, 1). So, if I transform

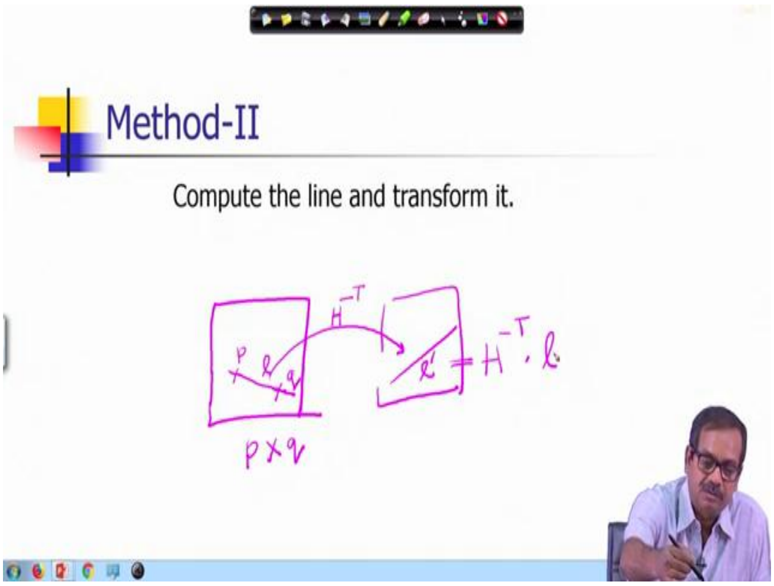
(1, 2, 1) by multiplying this matrix then you will get this point (5, -1, 11). You need to check when you multiply it. And if I multiply (2, 3, 1), then you will get the coordinate (12, 3, 16).

(Refer Slide Time: 17:17)



Now, what you should do that you have to take the their cross product to compute the transformed line. So, you perform the cross product of this two and you can find out that the result would be $\begin{bmatrix} -49 \\ 52 \\ 27 \end{bmatrix}$. So, that is the representation of the straight line which means the straight line in our conventional coordinate form should be represented as $-49x + 52y + 27 = 0$. So, this one of the method by which we have carried out this computations.

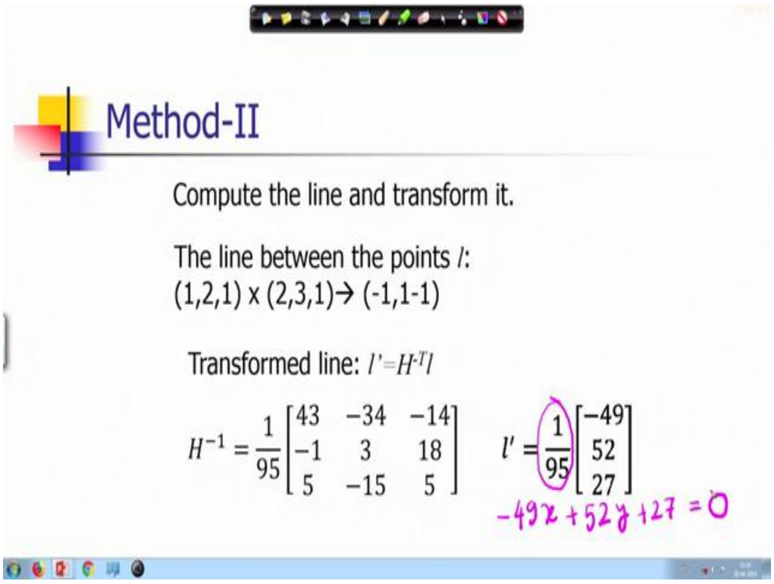
(Refer Slide Time: 18:07)



Let me discuss the other method. So, in this case what we will do. Rather, first we will perform the computation of line in the transformed space. So, let us consider your original projective space and once again two points, p and q. So, you have to compute this transformation or you have to compute line between p and q by performing the cross product of this two points in the homogeneous coordinate representation.

Now, you perform the transformation of this line. And in this case what you should do? You should multiply with transpose of inverse of this transformation matrix H which is represented as H^{-T} . So, if I perform this operation, then I will get the corresponding transformed line in the transformed space. So, l' should be equal to $H^{-T}l$. So, this is the computation we will be carrying out. So, let me detailed out this computation.

(Refer Slide Time: 19:27)



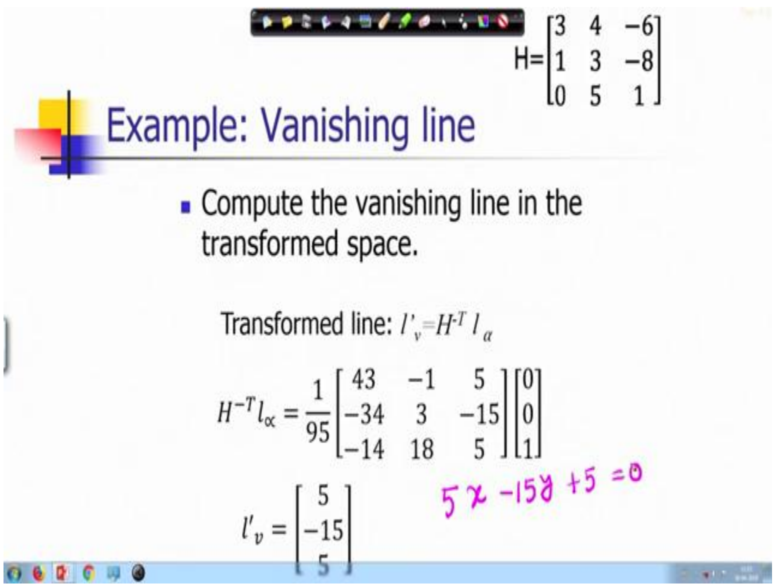
So, first we will be computing the cross product of this two points and the line would be (-1,1,-1), you can verify this computations by computing cross product of this two lines. And then, we will be transforming this line by multiplying this line with transpose of H inverse (H^{-T}) and you get the corresponding transformed line l'. And H^{-1} in this case is given by this particular matrix.

So, you know how to compute the inverse of a homography matrix or inverse of a 3×3 matrix and there you need to compute the determinant and you need to compute the co-factors, take the transpose of the co-factors and then you have to divide the corresponding matrix by the determinant you will get the corresponding inverse matrix.

So, please go through a standard text book of matrix operations, you will find out the steps of doing inverse and you should be familiarized to this computations. Particularly, since it is a 3×3 matrix, it should not take that much of time to get an inversion.

So, after performing the inversion of a matrix, transformation matrix, then you have to take the transpose of this matrix and perform the computations. So, if you do this computations your line will come in this form. So, you can ignore the scale factor here. So, this scale factor you can ignore you know that. This, it is same as representation of (-49, 52, 27) column matrix itself, and it will give you the similar equation of straight line what you have derived earlier also by the method one which we discussed in the previous slides. So, this is the equation of the straight line that you will get. We will continue this example by computing the vanishing line of the in the transformed space.

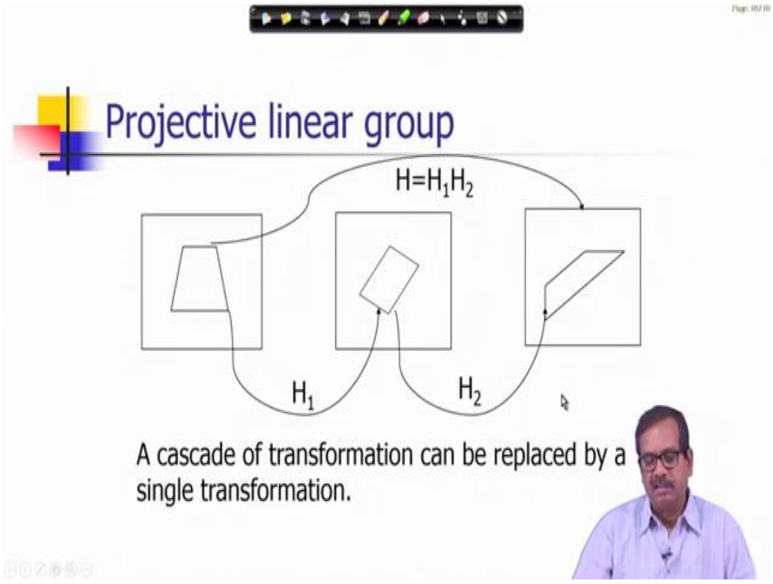
(Refer Slide Time: 21:37)



So, here as we discussed that vanishing line could be computed by transforming the line at infinity of the original projective space which is incidentally $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So, we will be computing, we will be performing this computation and if I multiply the corresponding transpose of this inverse with $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then you will get the corresponding vanishing line as

$\begin{bmatrix} 5 \\ -15 \\ 5 \end{bmatrix}$. And in the conventional coordinate representation this could be equivalent to the straight line $5x - 15y + 5 = 0$

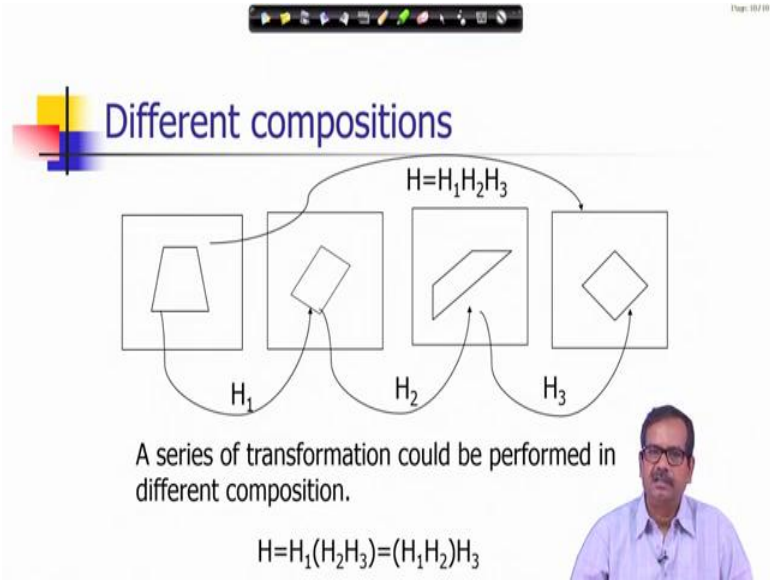
(Refer Slide Time: 22:39)



So, now, we will study another interesting property of projective space. And we will see that projective transformations they form a group, we call it a linear transformation and this is a linear transformation and we call that group as projective linear group.

So, the essence of this property is that if I consider a transformation of a spaceto the points in the another space by the transformation H_1 , subsequently another transformation from this space to the third projective space by H_2 Now, this is equivalent to transformation of the first to the third space directly by a matrix H which can be derived by composing this two matrices H_1 and H_2 , simplify multiplying these two matrices. This property holds because of this; this property of projective linear group, so which means that a cascade of transformation can be replaced by a single transformation that is one of the implications of this property. $H = H_1 H_2$

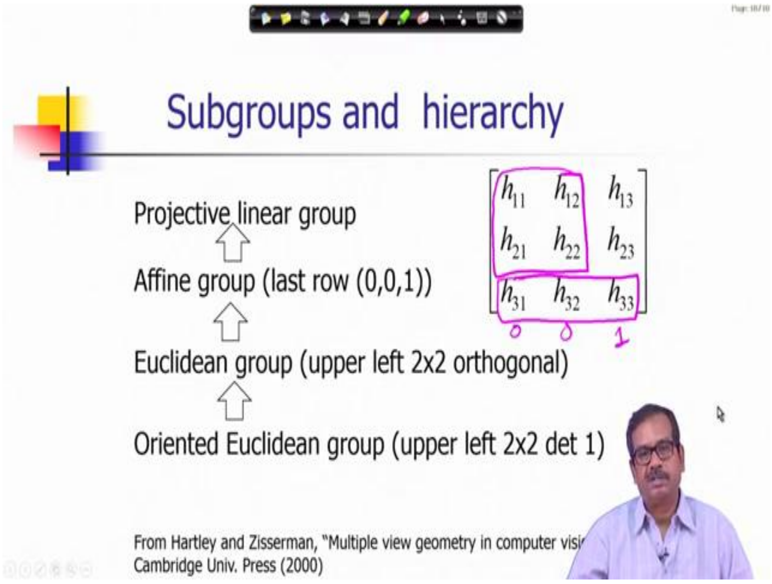
(Refer Slide Time: 24:03)



The other implication is that, if I consider a series of transformation, suppose we are transforming first those points by H_1 , next the transformed points to another transformation space by H_2 , and from the third transformed space to a fourth transformed space by H_3 , then by applying cascade we can simply multiply all these matrices H_1 , H_2 , H_3 and we can get the single transformation we can equivalently represent it by a single transformation from the first to fourth space.

But the interesting part is that as you know the matrix multiplications they are all associative. So, it does not matter in which order you compute this multiplications; that means, we can compute this multi this operations by first taking multiplications of H_2 and H_3 and then multiplying with H_1 or first multiplying with H_1 and H_2 and then multiplying this matrix before H_3. So, this will give me the composite matrix H. So, this is another property we can, it can you can perform this computations with different compositions and this is possible because of that group property.

(Refer Slide Time: 25:27)



So, there are some interesting structure in this particular groups and we will see that there are subgroups and there is a hierarchy among the subgroups. So, our parent group is a projective linear group. So, all projective transformation they fall under this category which is in general which is a projective linear group. We have discussed all those properties. Those properties are hold for this kind of transformations.

Now, there is a special case of projective transformation, and they are called affine projective transformations, and they form affine group. One of the key property of this particular transformation is that their last row should be (0, 0, 1), you can multiply with any scale factor of (0, 0, 1), also. So, it is very easily distinguishable. We can easily determine that whether the transformation is an affine transformation or not.

Then a special class of affine group is called Euclidean group when the upper left 2×2. So, this is the upper left sub matrix this should be orthogonal. As you, as I have told you the bottom most row or the last row it should be (0, 0, 1), it they should have the value of (0, 0, 1), or a scaled multiplication of this vector (0, 0, 1). So, that forms the affine group and a special class of a affine group becomes Euclidean when this sub matrix is an orthogonal sub matrix. And finally, another sub group which is also a special class of Euclidean group which is called oriented Euclidean group, if the determinant of this sub matrix is equal to 1; that means, upper left 2×2 sub matrix is equal to 1. So, we call that group as oriented Euclidean group.

So, as I mentioned as that oriented Euclidean group is a sub class of Euclidean group. Euclidean group is a sub group of affine group, and affine group is a sub group of projective linear group. So, we have this hierarchy in the transformation space. With this let me stop for this lecture, and we will continue this topic in next lecture.

Thank you for listening.