Computer Vision
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Lecture – 06
Projective Geometry Part - II

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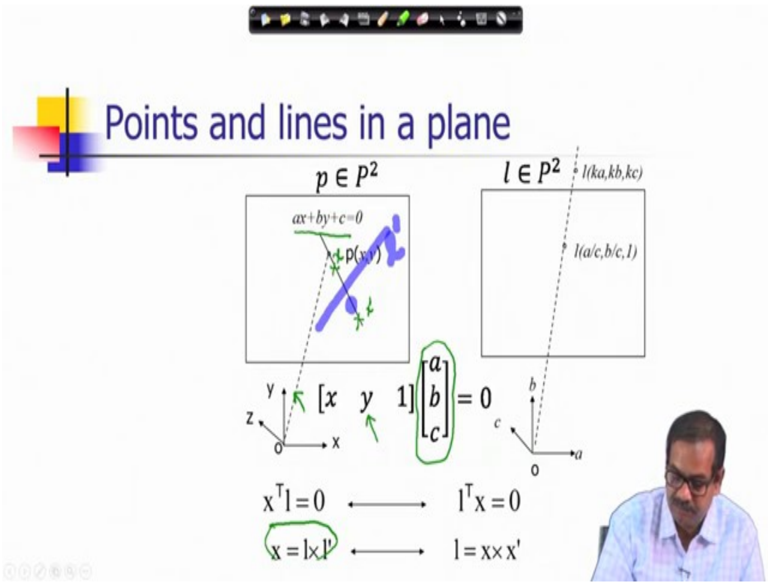

We will continue our discussion on Projective Geometry. We have seen how points and lines are represented in a two-dimensional projective space. We have seen that a point in the projective space can be represented by an additional dimension, because there is a implicit three-dimensional representation involving the two-dimensional projective space.

So, in this case, any point that represents a particular element, i.e, a ray passing through the origin connecting to that point, and every point in this space is being represented by this element. Similarly, a line in the plane of projection which is given by the corresponding equation, say, $ax+by+c=0$ which is also represented as an element of a projective space, which is a different projective space representing a line. There also a point in that projective space is representing a line and which is also representing an element that is a ray passing through the origin connecting to these point and extending it to towards infinity.

And as you can see that the parameter of this equations are now used to represent this line in the two-dimensional projective space. Also we have learned the relationships between points and lines, there is a duality in expressing these relationships. For example, if you have point contentment relationship, then this can be expressed in this form,

this is a point is represented that say transposition of the column vector of point representation in a canonical form, and this is the line representation of the line what we have shown here. So, if I take this matrix product that should be equal to 0, as if this is a dot product of these two vectors.

So, this could be expressed as in this form that I can write it as x transpose l equals 0. So, this transportation is represented here and symmetrics multiplication. And we can see in this relationship that if we interchange the position of point and line same relationship holds, so that is the dual principal. Similarly, there is another example of this kind of dual representation that is if you would like to compute a line, given two points in this space, suppose you have another point in this line and you would like to represent it.

And then ah how do you get that relationship? That is a point is a intersection of two line. So, we consider there is another line which is l prime. And this representation this l cross l prime will be the operation that would give you the corresponding intersection intersecting point. Similarly, if we consider a line is defined by two points say this is x and this is x prime. So, you get x cross x prime as line l. So, this is the duality what I was talking about you interchange the point and line into this relationship, and still that relationship holds. So, now, we will continue this discussion and we will further see what are the properties are there in the projective geometry.

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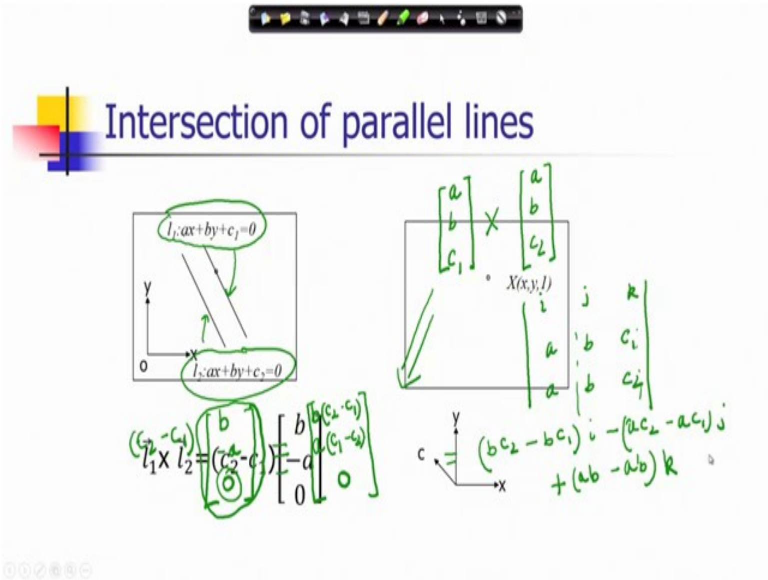

So, one of the interesting property in this space that how do you express the intersection of parallel lines? We know in normal two-dimensional real space which we studied in our school geometry, two parallel lines they intersect at infinity, but there we could not qualify the nature of infinite point, nature of point of intersection at infinity. We will see in the two-dimensional projective space this could be qualified. Let us see how, let us compute this intersection.

So, here we are going to compute, here you can see that there is an there are examples of two parallel lines this is given by this equation. Suppose, this is this parallel line and take another parallel another line which is parallel to it by this equation. You can observe that the coefficients a and b they remain same; they remain the same, so that is why this parallelism is established.

So, to compute the intersection of these two parallel lines, we can apply the cross product operations of three vectorial representation of these lines. So, we will perform that competition say line l 1 represented by this three vectorial form, and l 2 is also represented by another three vectorial form. And we would like to take the cross product of these two to compute the point of intersection. So, as we did this exercise in the previous lecture, we will carry out the same computations in the similar fashion we will be computing it.

So, we are computing the cross product. So, let me consider the components of these vectors arrange them in rows, and then expand the determinant. So, let me do it as you understand that this is the sub determinant which unit to compute as a component of i. So, this would be b c 2 minus b c 1 i. The middle part, so we will write it as minus of these two which is a c 2 minus a c 1 j. And finally, the third component by suppressing the third column, it would be ab minus ab k.

So, if I write it in the vectorial form, I can write this the resultant vector as b into c 2 minus c 1, then a into I can write it as c 1 minus c 2. You note the change of sign because of this negative sign here, and then 0 that is the third component. So, this is the intersection point. In fact, this is equivalently I can write these vector as b minus a 0 by taking the scale factor c 2 minus c 1, no outside.

So, this equivalent representation itself it is sufficient to say that this is the point of intersection of these two parallel lines. So, now, you can see that this point if you notice that the scale value is 0. So, if I divide the scale value, divide the other coordinates by the scale value, those coordinates will become infinite. But the nature of infinity is captured here, because it is qualified by these two values b and minus a. Let us try to understand the significance of this representation.

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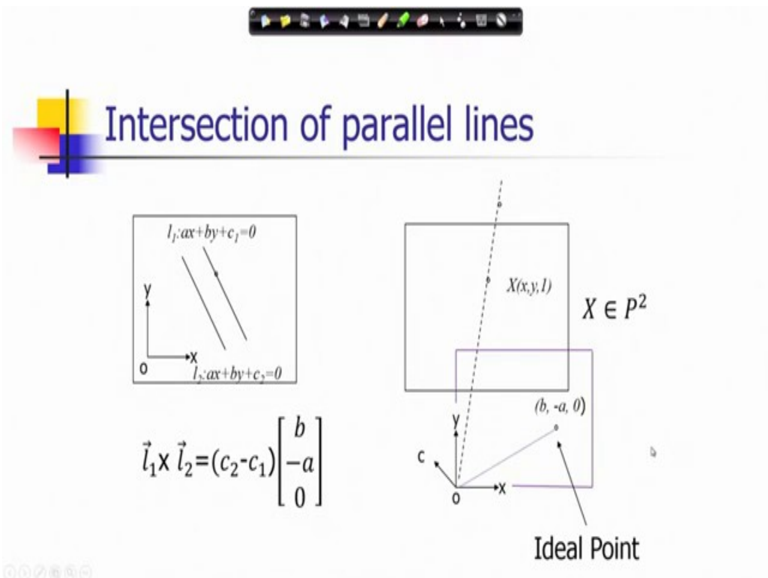

So, if I rub this you know computations, so you can see that we have the point of intersections in this form. And how this line is represented here, this line this particular point of intersection, how it is represented? So, we will say that in the two-dimensional projective space this point b minus a 0 it is represented as a point in a plane which is parallel to the projection plane. And this plane is called principal plane, because it contains the access x and y.

And not only this point b minus a b minus a 0, but also the ray passing through this point connecting to the center O, the whole ray itself is representing this point because that is how the elements in the projective space is represent and this is the point of intersection in this representation. This point is called ideal point that is a technical term will be using it more often. And the plane where all these points are line for all of them the third coordinate is 0 that plane is also called ideal plane which is incidentally is the principal plane of this representation. This form of representation is called canonical form of representation.

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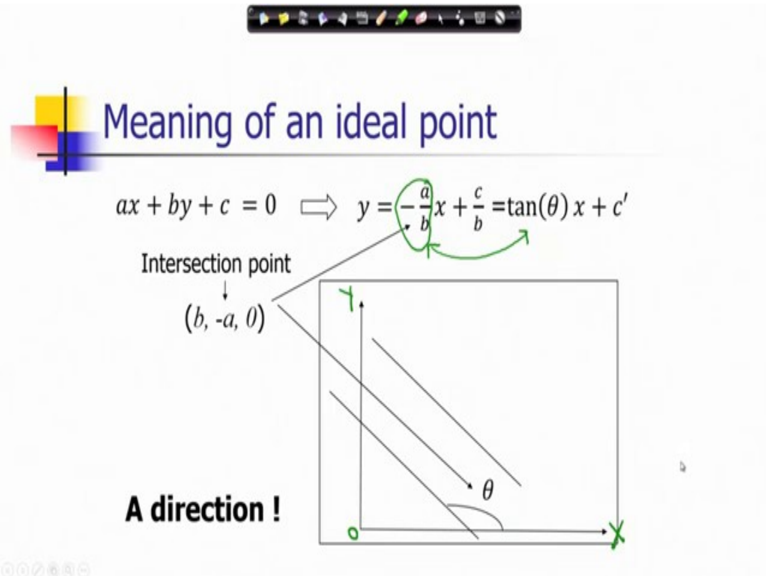

So, let us understand the meaning of an ideal point. So, we consider a two-dimensional plane, where you have these two parallel lines, and these are the x axis, and this is the y axis. So, this is x axis; this is y axis; say this is the origin of this representation. So, a straight line given this equation ax plus by plus c equals 0, one of the straight lines in this representation, and you know the other straight line which is parallel to it should can be represented as ax plus by plus some value c 1 which is not equal to c in this case that should be equal to 0.

So, this straight line particularly if we notice that this can be represented also in another form very well known analytical geometric form. I can represent as y equals minus a by b x plus c by b, where you see that this is a slope of this representation, and the relationship between the slope and the angle of this line which it makes with x axis that is also known to us. So tan of this angle that would give you the slope.

So, we can see how a and b they are related with this representation. So, intersection point is given by this b, minus a, 0 that you have computed and this point is related with this slope. So, what is a point, ideal point? In that case it is simply representing a direction, a direction in this two-dimensional plane. So, a point ordinary point in the two-dimensional perspective projection space or two-dimensional projection space is representing a ordinary point, there is an one to one correspondence with the ordinary point of a two-dimensional real space also. Whereas for the ideal point, it corresponds to a direction in that plane, and that direction is given by this angle theta which makes an angle with respect to x axis that is the implication of an ideal point.

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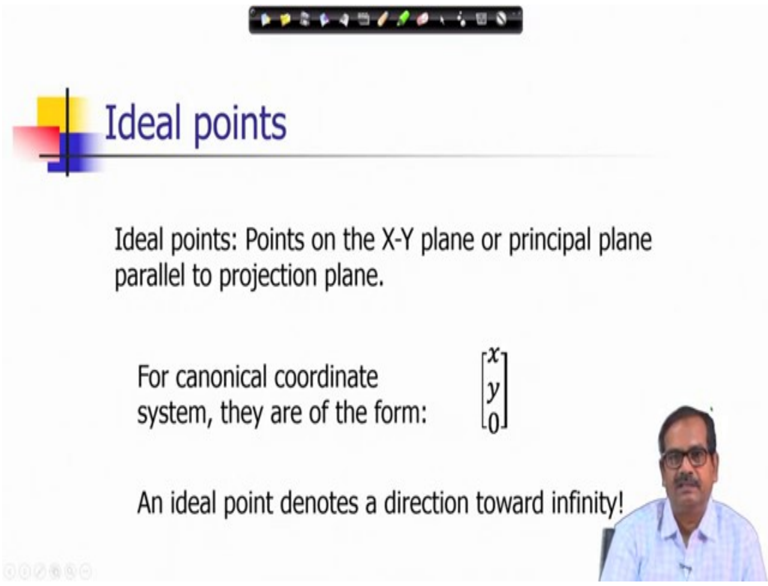

So, just to summarize this fact, ideal points are points on the x y plane or principal plane parallel to projection plane. And for canonical coordinate system, they are of the form x y 0. So, the third dimension which represent the scale that would be 0. An ideal point denotes a direction toward infinity that is the implication of an ideal point.

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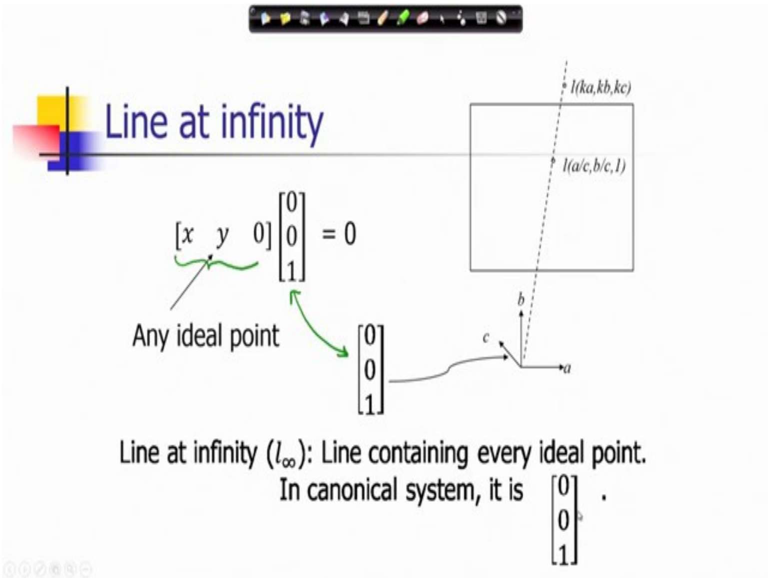

There is another interesting concept in this projective space in this representation, this concept is called line at infinity. So, let us try to understand what is a line at infinity. You notice this particular axis which is extended towards the direction of parameter c that is represented by this particular representation 0 0 1, this is the column vector. This is also representing an element in the two-dimensional projective space which are which is representing all the lines. So, this is a representation of a spatial line.

Let us see what is the property of this spatial line. Let us consider this particular operation. It says multiplication of the transpose of a point incidentally which is an ideal point. So, this is an ideal point and this is the line, what I was referring at. If I perform this multiplication, you can say that this is giving you 0, it is very simple to check this computation.

So, what does it signify? You choose any ideal point and you perform this operation you will get 0. This is the relationship between a point and a line that is a point contentment relationship, which means all the ideal points they lie on this particular line and this line is called line at infinity. So, to summarize their definition of line at infinity, it is a line containing every ideal point and in canonical system it is given by 0 0 1.

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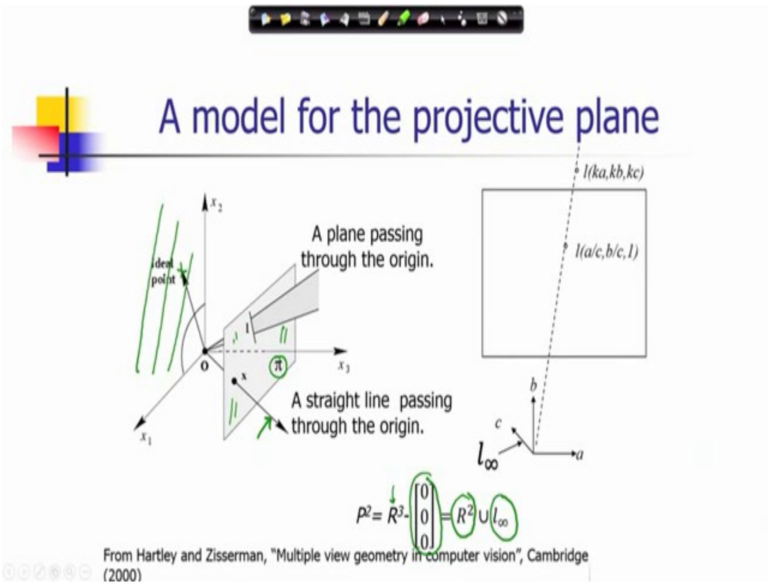

So, what should be a model for the projective plane? In this case, we can represent all the points in the projective plane using this geometric concepts, this is a geometric model. So, you can observe that this is a plane of projection, this is a plane of projection which is represented by the symbol pi. So, all the points which are in the real space and which corresponds to a point in the projective space directly, they lie on this particular plane of projection. And every point corresponds to a ray passing through this point connecting the origin. So, any point is related with a ray connecting to origin passing through that point.

Similarly, if I have considered a point in the principal plane or ideal plane that is also an element of the projective space. So, all this point which are lying in this plane there was a part of the projective space. And they are representing all ideal points and as I mentioned they are representing a direction with respect to this plan of observation. And any straight line on this plane you can see it is geometric interpretation is that it is a intersection of a plane containing the origin and intersection with the plane of projection.

So, this is what is your a geometric model by which we can understand the two-dimensional projective space, so that is what a straight line passing through the origin, that is how a point is represented in a projective space. And a plane passing through the origin intersection of that plane with respect to the projection plane that intersection represents a line in that on that plane, or any line is actually representing a plane passing through the origin.

So, mathematically we can say the set of all points in a projective space is also related or they are equivalent to set of all points in the three-dimensional real space excluding the origin, as I mentioned earlier origin is a singular point of the projective space. Similarly, I can consider also a real space two-dimensional, real space every point in that real space representing some point in the projective space.

In addition to that, there is another plane parallel to the real space that is the canonical in the canonical representation or ideal plane, all points in that ideal plane is also represented. So, instead of writing it as a plane containing all points, simply I can write all those points, they lie on a particular line which is called line at infinity. And this line at infinity is given by this particular you know structure. So, these itself represents all the points in the ideal plane. So, this is a summary of this representation.

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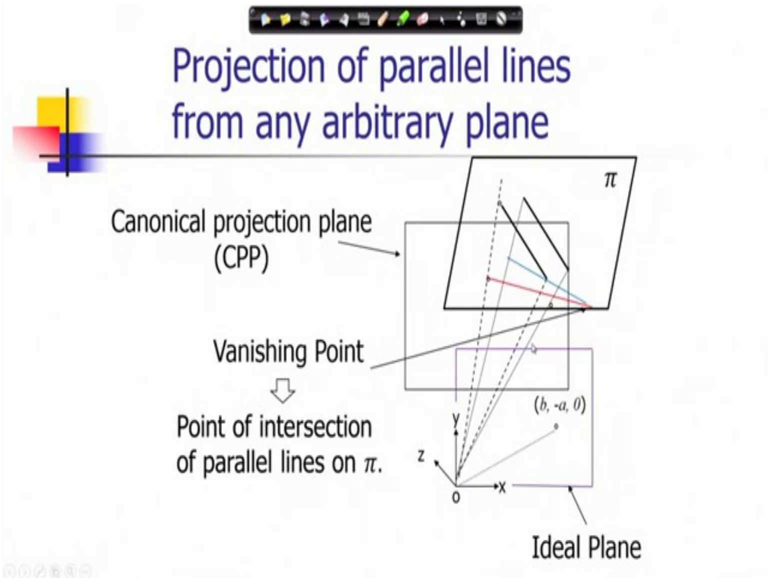

Let us try to understand another particular feature in the two-dimensional projective space that is called projection of parallel lines that feature we would like to check from any arbitrary plane how this projection appears in the projective space. Let us consider a projective space given by this representation, that means, there is an implicit three-dimensional representation. You have this ideal plane, you have those access a plane of projection, and any point in this projection plane is represented through this plane of projection.

And let us consider a plane in a arbitrary plane, and a parallel line two parallel lines in that plane. So, a plane is denoted here by the symbol pi. And if you would like to project this parallel line on the canonical plane, let me draw these two rays passing through any points lying on this plane. So, these rays they intersect the plane of projection, and the intersection would be given by a straight line lying on that plane of projection.

Similarly, consider the other line which is parallel to the parallel to this line. And if I consider the other line and perform the same representation, same projection, and projection of that line on the canonical plane which means I have to get the intersection of rays connecting two points lying on that straight line, and those intersecting points they will form a line. What do you observe that though the lines are parallel in plane pi, but in the canonical projection plane these lines they are meeting to a particular point. And this line this point is called vanishing point. So, vanishing point is a point of intersection of parallel lines which are projected on the canonical plane.

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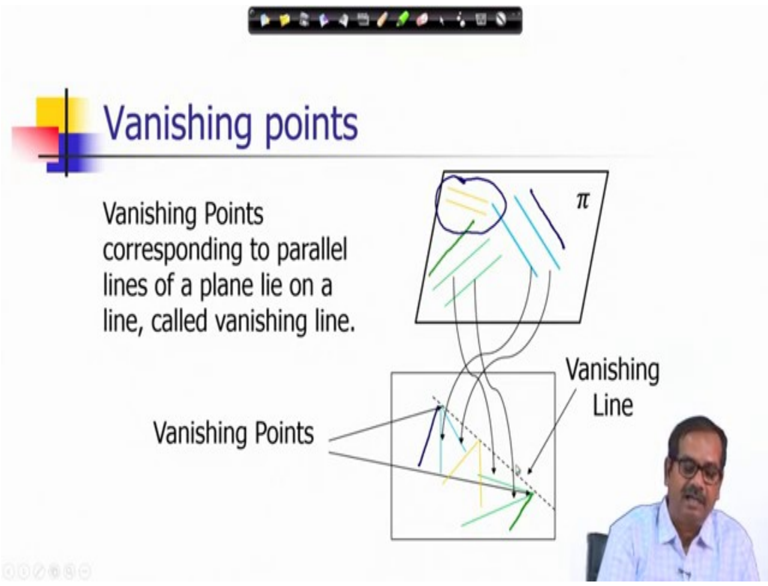

We try to understand a bit more about this vanishing points; the their implications would be more clear here. You consider parallel lines on plane pi in various directions. Suppose, you take two directions and there are two representative lines which parallel lines which are denoting those directions. And if I take the projections of those lines in the canonical projection plane or in the plane of projection, as we can see that these two lines they would appear like meeting at some point which is a vanishing point.

Similarly, say other two lines, it would appear also meeting at some point which is a another different vanishing point. So, all parallel lines in that direction, so if I take another parallel line in this direction, that would also meet at the same vanishing point. If I take another parallel line see in this direction, that would also met in the same vanishing point here for this group of lines. Interestingly if I connect these two vanishing points, then we get a line, and this line is called vanishing line, because any parallel lines set of parallel lines in any directions they are vanishing points in the plane of projection will lie on this particular line which is called vanishing line.

For example, if I consider a parallel lines see in this say if I consider this is another set of parallel lines, and if I take the plane of their projection on the plane of projection. So, what I will get? I will also observe that we will observe that those two lines they are meeting at a point which would be the vanishing point. And that point will also lie on the line on the same straight line connecting to the vanishing points earlier we have seen, that means, that point is lying on the vanishing line. So, this is a summary that vanishing points corresponding to parallel lines of a plane lie on a line and that is called vanishing line.

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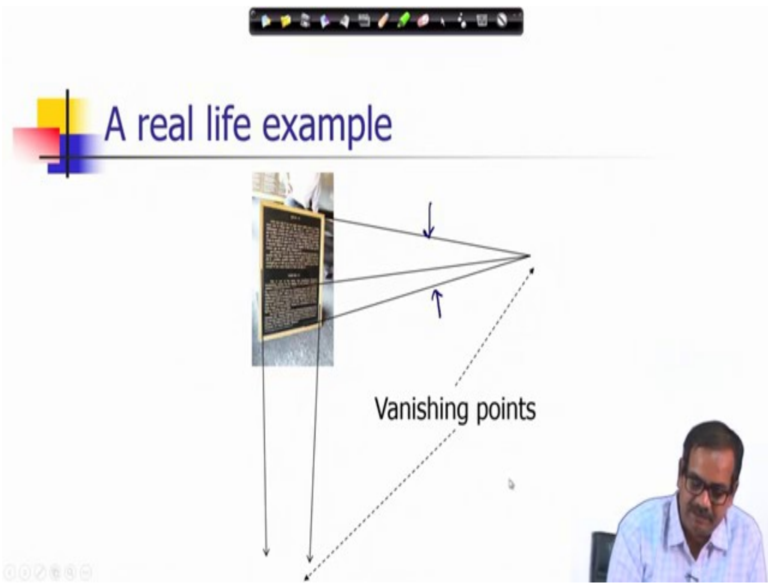

Let me draw a real life example to show that how vanishing points do exist. You take this particular image and you can see that the edges in the horizontal direction as we understand from that notice board and edges in the vertical directions, they are meeting at some point.

For example, in the horizontal direction, if these two edges this particular two edges they are meeting here and in the vertical direction.

So, here what is shown here that even you take another parallel line, another line parallel to same direction like this text; text are also in the horizontal direction. So, this line also will be meeting at the same vanishing point, because as I mentioned all lines parallel to a given direction will meet on a single point that is the vanishing point. And similarly the vertical edges also they will also meet some vanishing point, and connecting these two vanishing point we will give you a vanishing line.

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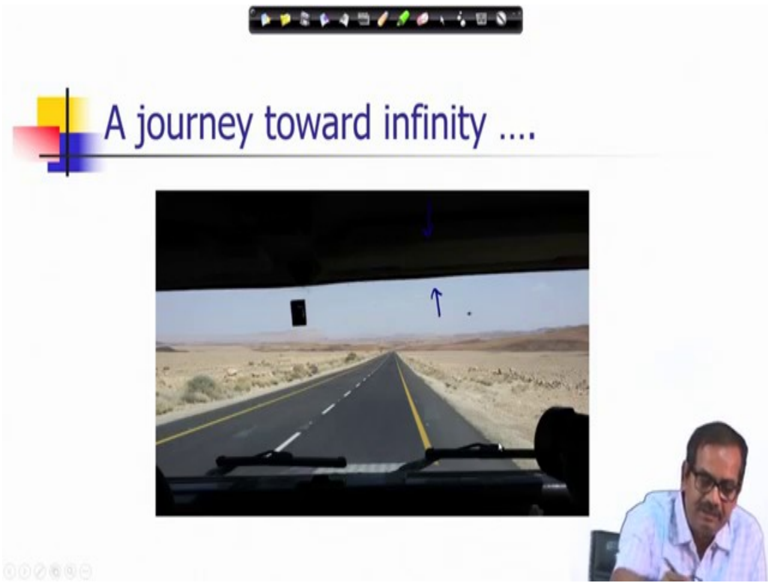

This is a visual demonstration of vanishing point that we can see that this is an image of a road, and which is captured from the front of a car. And you can see that how the edges of that road is meeting at a point at infinity, but this point we can sense, but it remained ever (Refer Time: 29:18) let say. So, our journey is to our infinity, we can say it from our perspective projections point of view, but really cannot touch it that is how a vanishing point could be also interpreted.

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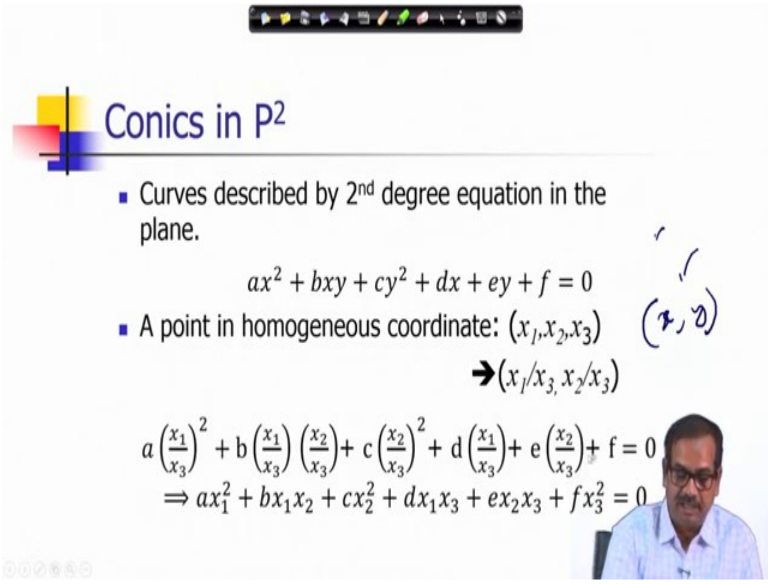

There are there is another element in the two-dimensional projective space which is called conics, which are conics. And we will be considering their representation also in a projective space. So, how conics are represented? They are curves described by secondary equation. And this is the form of the equation which has been shown here.

So, and if we consider this representation translate this representation in the homogeneous coordinate, each point instead of represented by 2D real coordinate of x and y, so in a 2 d real coordinate a point is represented by x and y. So, there in the homogeneous coordinate we know how this coordinates are represented by using this scale factor. So, x is equated with x 1 by x 2; and y is equated with x 2 by x 3. And if I replace this in this equation, then we will get a representation of conics in the homogeneous coordinate representation, and this is how this representation looks like.

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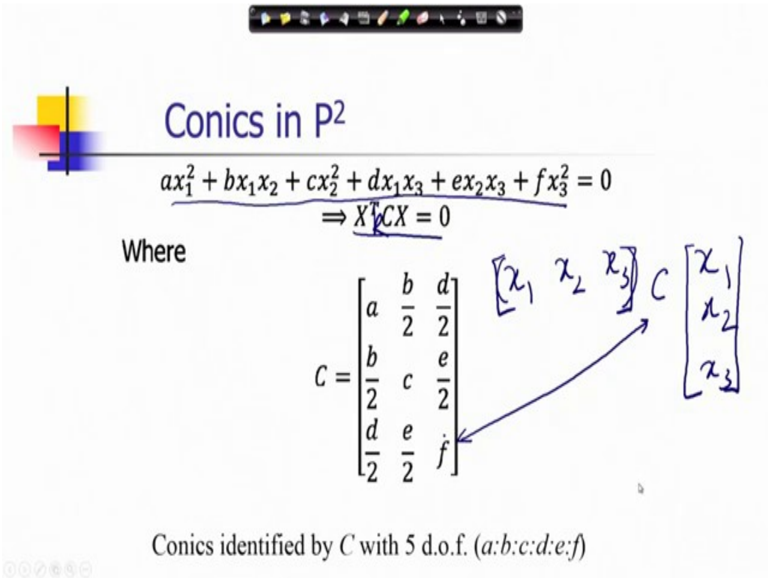

So, to make this representation brief once again we will be using the matrix form of representations using vectors for representing a point. And we can see a two-dimensional matrix represents a conics, this is how it is represented. So, this is the general form of representation of a conics. And we can see that this coefficients a, b, c, d, e, f, they are representing a conics. And these equation can be simply represented by this particular form. So, if you are wondering how I could get it, I can consider the homogeneous coordinate representation say let me write it as x 1, x 2, x 3, that is what x transpose, then C is given by this matrix. And then we have the column vector representation x 1, x 2, and x 3.

So, if I perform this matrix multiplication, you can check you will simply get this expression. So, finally, a conics is represented by this , you can observe that this is a symmetric matrix, its dimension is 3 cross 3. And you have how many parameters are there? there are six parameters a, b, c, d, e, f. But as you know in this equation if I multiply this C with k, still it remains the same conics. So, it is an element of projective space. So, one of them can be treated as a scale factor. So, ultimately the degree of freedom in this representation is 5 though there are 6 parameters. So, I can represent a conic by this 6 elements, but one of them is a scale. So, degree of freedom is 5.

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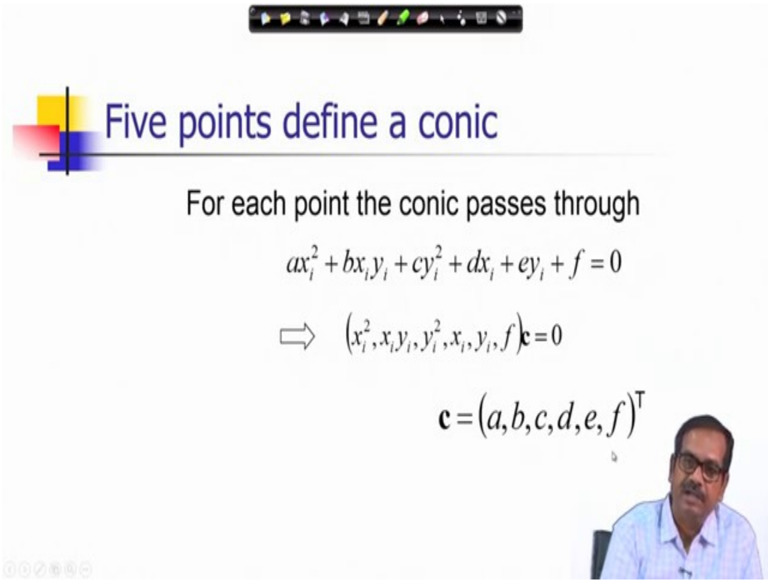

So, naturally to define a conic uniquely I need at least five points in the two-dimensional projective space and I can write those equations using that five points. So, this is the equation, they should satisfy this equation, and this is a representation of a conic also in a vectorial form.

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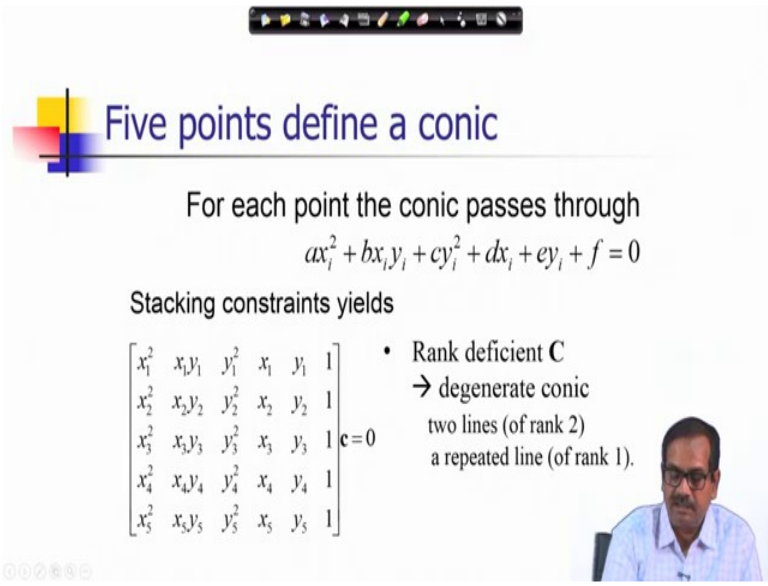

And if I get the five points, I can write five equations and then I can solve this equations because there is a 5 degree of freedom by taking one of the parameter by fixing it at some value, I can solve it. There could be rank deficiency in this representation rank deficiency in

C. And in that case here is degree of freedom is less than 5, and the there are special cases those are called degenerated conic, like there could be two lines of rank 2. And a repeated line of rank 1 those are the rank deficient representation of C. We will also check how this representations could be done, could be expressed analytically.

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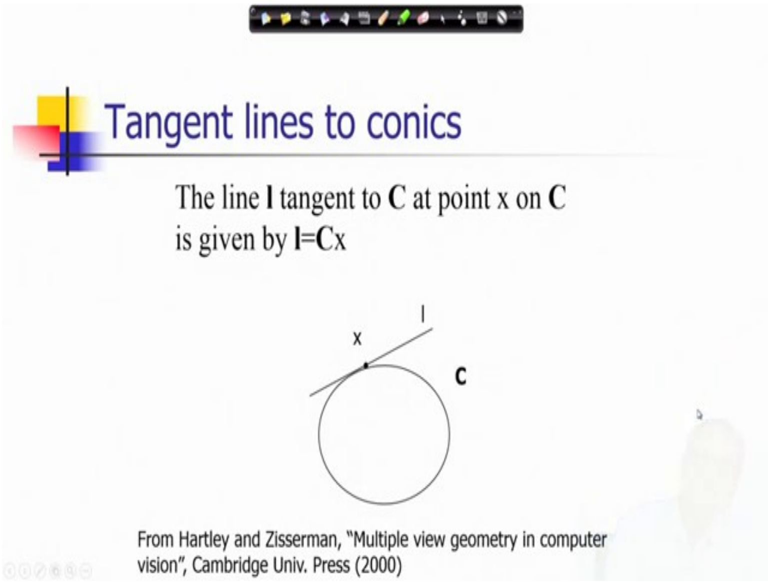

For a conic, its tangent lines are well related very in a convenient from it is related with the point which is lying on that conic, and this is a relationship that is given by simple linear relationships. If I multiply the point with the matrix C, then we will get the corresponding tangent line l.

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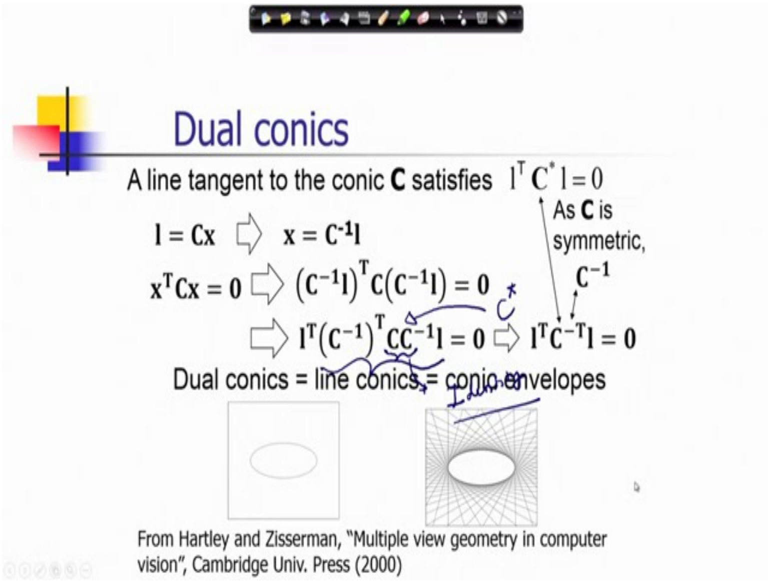

Now, this gives an interesting relationship of a conic representation, we have a dual representation of a conic. In this case a conic can be represented by all tangential lines which is enveloping which forms an envelope of the conic. And you can see that the expression is also in the similar form in the previous case it was x transpose C l, now it is l transpose another representation of conic C star it is a different matrix 3 cross 3 matrix, and l transpose C star l equals 0.

So, the relationship between the original conic representation with the dual conic representations can be found from this particular case. See if I have l equals C x, then I can get x equals C inverse l, then x transpose C x equals 0 that is the original representation. From there I can derive the representations involving only line l, and this is how the algebraic manipulations, every x is represent is replaced by C inverse l. And if I take the transpose operations of those matrices that property, finally we see that we get l transpose, then all the composite matrices involving C involving constants and another l that is equal to 0.

Now, the this whole thing can be considered as a representation of another conical form, another form of conics which is the dual representation. We can simplify these expression further by using matrix algebra, and we can represented as l transpose C minus, that means, transpose of C inverse because you know that that C into C inverse this is this is equal to the identity matrix ok.

So, this is the identity matrix and so we can just simply you know ignore it from this term, and then we gets you know this is what is C star incidentally, since C is a symmetric matrix. So, transpose of its inverse, this transpose is same as the original matrix, that means, this is equal to C inverse itself.

So, finally, the dual conic representation of C is nothing but its inverse there is a interesting and very beautiful relationship involving the conics. So, a nice you know picture from the book from Hartley and Zisserman. You can see this is the original representation of conics in the point space, and these are representation of conics with the lines these are the two dual representation.

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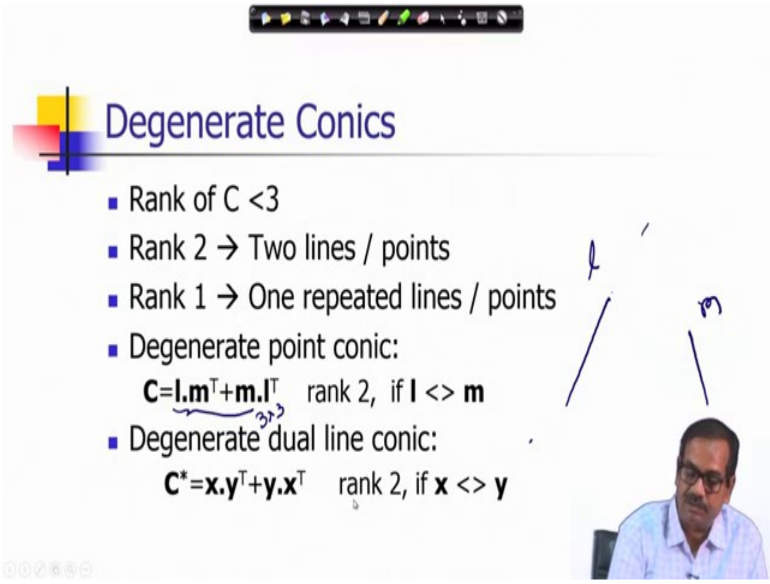

And degenerate conics we mentioned earlier if the rank of matrix C itself is less than 3, then we have some degenerate conditions of representation. Like with rank 2, a conic is defined by only two lines or two points, which are contained in a conic and they are defined only that lines and points. For example, a rank 1 it is the repeated lines and points.

For example, in a degenerate point conic, we have to we have to specify it using two lines, say a line given by parameter l or vector l and m. So, l dot m transpose plus m dot l transpose that itself we will give you the conic representation. You note that the vectorial form of l, it is a 3 vector. So, if I perform this computation, it is 3 cross 1, and this is 1 cross 3. So, the dimension could be 3 cross 3, and which is a conic representations, but its rank is 2. There are only two you know directions, it will involves only two parameters.

So, if I take any two line l and another line m this itself these operation the pair of these lines is representing a conic, because there are there is a point of intersection of this line which lies on both this point, and that is what is representing a very generate degenerate conditions. Similarly, degenerate dual conic is represented by two lines, two points x y transpose plus y x transpose. So, this is what is the degenerate representation of conics.

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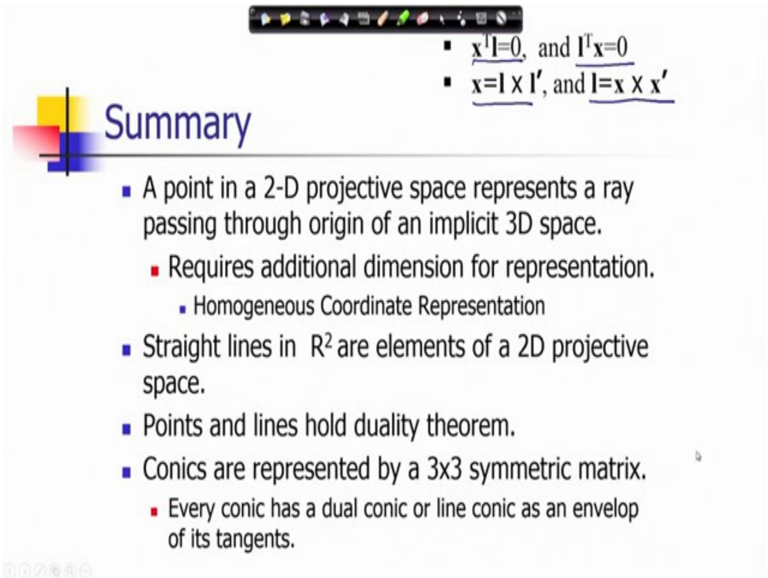

So, we can summarize our discussion on this projective geometry is two-dimensional projective space that a point in a 2-D projective space it is represented by a ray passing through origin of an implicit 3D space. It requires an additional dimension for representation. And we call that representation as the homogenous coordinate representation. Then a straight lines in 2D real space those are also can be represented as an elements of a 2D projective space that is the space representing for lines of 2D real space.

So, points and lines they hold duality theorem. So, these are the duality theorem which we have learned that is x transpose l equals 0 that is a point contentment relationship which can be expressed in the dual form also l transpose x equals 0, x equals l cross l prime that is the intersection of two lines keeps a point. A dual form is intersection of two points gives a line which is also in the same kind of operations. Then there are conics in 2D projective space which are represented by a 3 cross 3 symmetric matrix. And every conic has a dual conic or line conic as an envelope of its tangents. So, here we come to the end of this particular lecture.

Thank you very much for listening.