Computer Vision

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Module – 05
Projective Geometry

In this lecture we will discuss about Projective Geometry and its application for the analysis of images in Computer Vision.

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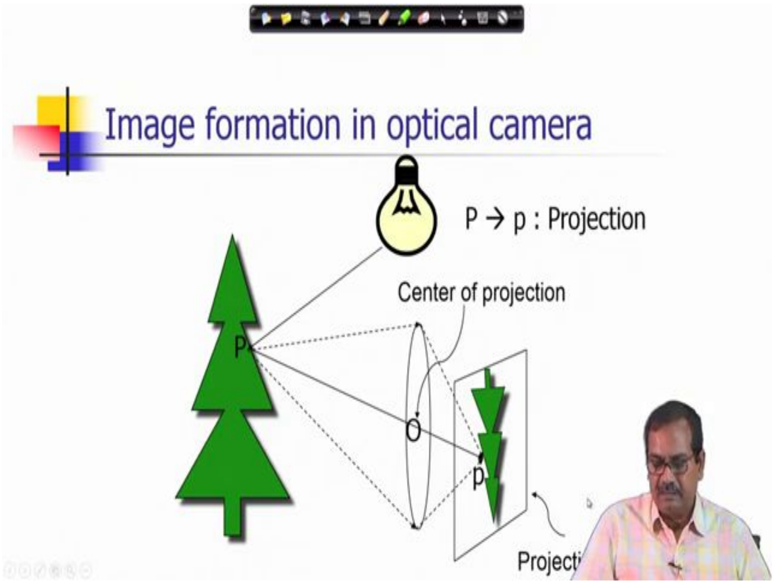

As earlier during the discussion of fundamentals of image processing, we have seen how images are formed in a camera. So, here you can see the rules of projection is that given a same point, then from that point you can draw a ray which passes through the center of the lens and then that ray when it intersects image plane, then that particular point is the image point of the three dimensional scene.

And this perspective projection rule is applied for any scene point for getting the image point of any scene point you take another scene point here and once again let me draw a line passing through this center and when it is intersecting the plane of projection, that makes the image point of the other scene point. So, in this way the projections are formed. So, we can see that this particular geometry can be well understood or well studied in the case of using projective geometry. So, let us understand what is a projective geometry. So, these are the things for same point.

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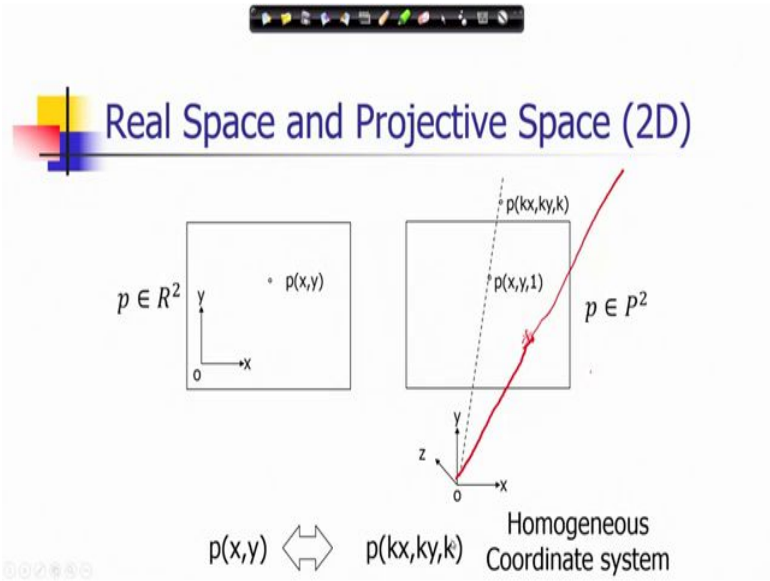

So, there are usually in the mathematics, we study our geometry in the real space you know school book geometry or later on in the analytical geometry. Consider it two dimensional space where any point in the two dimensional space is denoted by a particular pair of coordinates, x and y pair of coordinates and since this is the Cartesian product of real axis, we denote these two dimensional space also as R^2 and this point belongs to the two dimensional coordinate space. And following these coordinate convention, there is a origin and there are x axis and y axis based on which these coordinates are correspondingly defined.

Now, when we are considering a projective space once again a projective space in two dimension, but implicitly there is a three dimensional space behind that definition. it is like an image. When you look at an image; all the points in an image, they are in a two dimensional plane, but finally those points are related to three dimensional points in the scene.

So, if we apply the rule of perspective projections, we can consider that in this three dimensional space, there is a point which is the origin and if there is a ray passing through this particular point p, then that ray itself is representative as if this point is representing this ray itself. For any point on this ray is equivalently represented by particularly this representative point of two dimensional point.

So, this is what we say a point in a projective space and we call this space as a two dimensional projective space. So, you should note that if I take any other point in this space; then once again if I draw a line passing through the origin and extending this ray. So, the whole line itself is represented by the point in this plane. So, this is what is the two dimensional projected point and every point in this particular plane is represented in this form. So, every point in this plane is actually representing a ray that is the basic representation of a projective space.

So, in this case a point (x, y, 1), it is a canonical representation of the projective space where you have a three dimensional coordinate system. As you can see implicitly this three dimensional coordinate system as if this plane is at a distance of one along the z direction and all the coordinate axis in this plane is also parallel to the corresponding coordinates of the implicit coordinate system of the three dimensional plane. If I consider any point in this plane in this straight ray, so that is also represented by the proportional factor of k. Any point can be represented by (kx,ky,k) where k is a parameter.

So, finally, a point (x, y) in a two dimensional real space is in a equivalently represented in the two dimensional projective space as a set of points (kx, ky, k) where k varies which is actually representing this ray. So, this is the relationship when we consider an image as a two dimensional real space . On the other hand when the image is considered as a two dimensional projective space under the imaging perspective of projection.

So, this coordinate system is called homogeneous coordinate system and you have an additional dimension which is denoting the scale of this coordinate system. So, if all other coordinates are divided by this scale, you will get the non-homogeneous representation which has direct relationship with the two dimensional.

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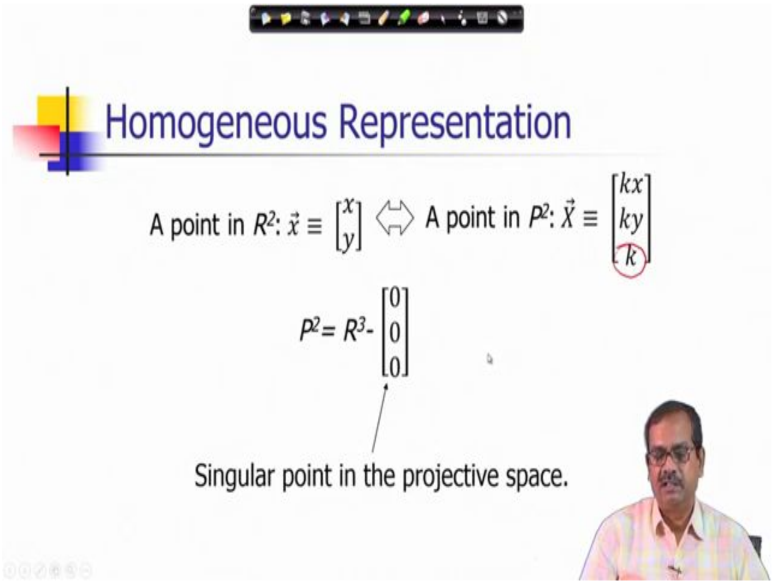

So, let us understand bit more about this homogeneous representation. So, let us consider a point in R^2 which is a real space, the ordinary space which we are used to understand in our geometry. So, let us consider a point x. In this case I am using this notation vector to denote that it is a tuple in this case, it is a two tuple coordinate systems. So, it is a vector. you can also represent this as a column vector of coordinates and this is a representation I will follow throughout in this lecture.

let this point with this. If the corresponding point in the projective space is denoted in this form

$$\vec{X} \equiv \begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$$

and here also you can see that it is a three vectorial representation where you have an additional dimension which is denoting the scale and which is multiplied with all other coordinate system . So, these two are equivalent representations in our perspective. Either I can denote a point as a point in a two dimensional real space or a point in a two dimensional projective space with three coordinate dimensions.

One of the interesting facts in this projective space is that the origin of that space is not included.

$$P^2 = R^3 - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, it is the singular point you cannot form a projection ray between two at that point itself. So, it is a singular point in this space which is not included as a set of points in the projective space.

So, I can consider that any point in their space of three dimensional real space excluding the origin, can be represented as a point in the projective space, however you should understand the elements in the projective space are not points. They are all rays passing through the origin and each ray can be mapped to a point in a particular plane in that space.

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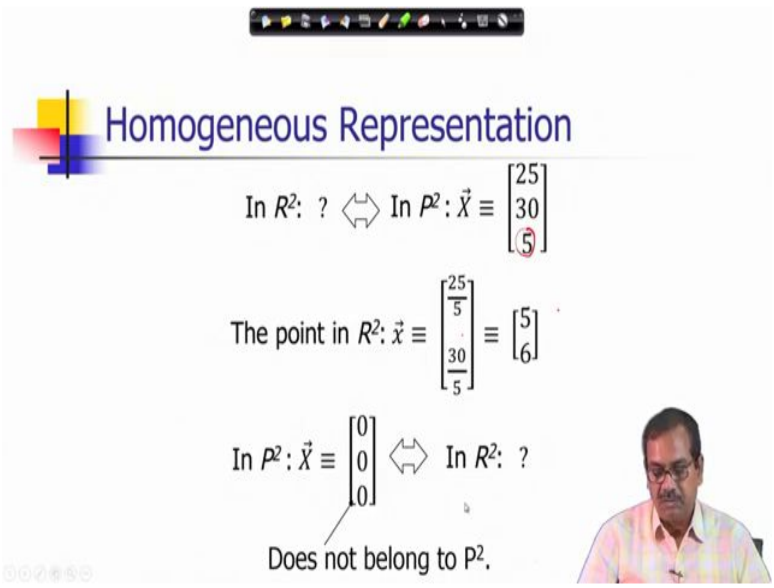

So, this is another example what we would like to understand once again. consider a point in the projective space given in the above form say $\begin{bmatrix} 25 \\ 30 \\ 5 \end{bmatrix}$

So, what should be the corresponding point in the real space? So, in this case how do you convert it? As I mentioned that there is a additional dimension which represents the scale.

So, to convert into the real space what I need to do? I need to simply divide this coordinates by this scale and then we can get the corresponding you know real space point. So, this is how these points will look that know you can divide 25, by 5. So, it should be 5 and 30 divide by 5 which is 6. So, this point is equivalently represented by $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ coordinates in the real space. Let us also consider another tricky situation. Suppose I am considering a point $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in the projective space, what should be the real point in the real space? But as we have discussed that this is not included, so this question is not valid because this is not the point in a projective space. So, I cannot convert these points as you can see that 0 by 0 is also undefined mathematically. So, it is consistent with mathematical notations and operations. So, this is not a point in the projective space. So, I cannot consider any equivalence point of this in the real space. So, it does not belong to the projective space.

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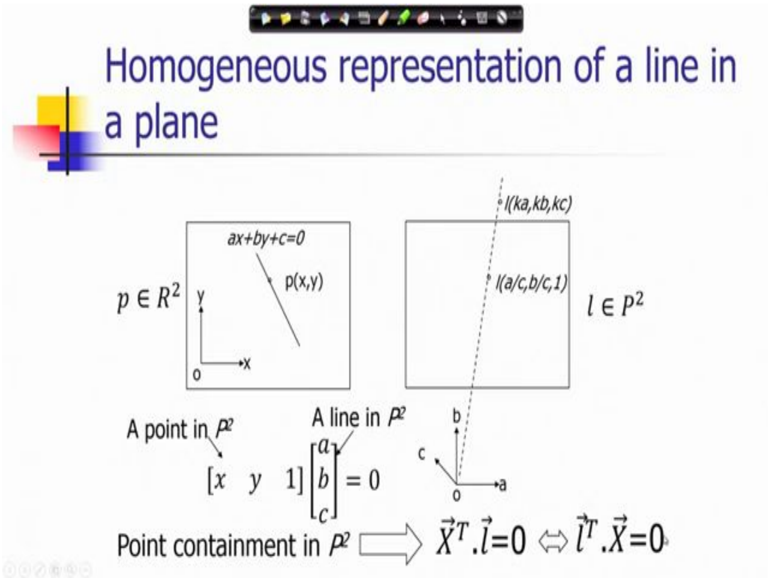

Let us understand another fact that is homogeneous representation of a line in a plane and let us see how the lines are represented here. Say once again you consider a two dimensional real space and p is a point and consider a line between these point which is represented by our familiar equation of straight line equation $ax + by + c = 0$, where a, b, c are the parameters of the straight lines.

Now, this representation itself what I can consider that a, b, c are the parameters that itself is representing the straight line and we know two parameters are sufficient instead of three parameters,because this is the proportional ratios of a, b, c that is uniquely identifying a particular straight line. If there is a projective space, which is representing only lines in this case. So, a line in this two dimensional space is represented by a point in this projective space and follow the same representation.

So, you have a three dimensional implicit, three dimensional coordinate system. It is a two dimensional projective space once again and if I consider any element of this space that should be a ray passing through the origin and incidentally if it passes through this particular point $(a/c, b/c, 1)$ that is a representation of this straight line. So, a straight line is represented like a point in this space. So, it is an element as I mentioned and any point, whether it is (a, b, c) or $(ka,\ kb,\ kc)$, all are straight lines. So, as you can see the third dimension is again representing a scale for representing this element straight line. .

In fact we can denote this relationship of straight line into this form which is a dot product of the two vectors or we can say it is a matrix multiplication, where this is represented as the transpose of the column vectors of the projective point and this is a straight line vector. In a simple form I can represent it represent in this way that this is a point containment relationship and this representation in a simple form we can represented in this way, it is a transpose of the column vector of the point x.

A point in P^2 A line in P^2

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Also, this line multiplied with the column vector of point, represent this line and that is equal to 0 which is essentially representing the straight line equation. I can write this equation because this mathematical operations can be also equivalently represented as :

$$\vec{X}^T.\vec{l}=0$$

So, in this way you can see that whether it is point containment or that equation can be represented in this way.

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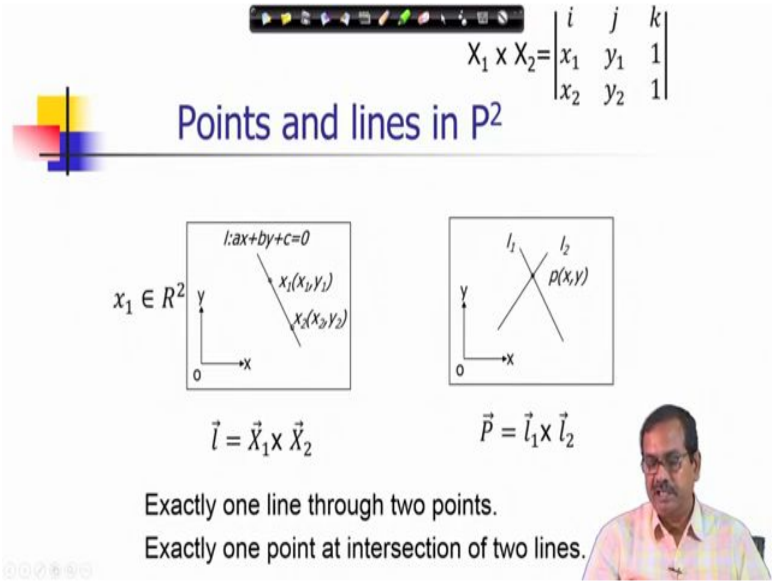
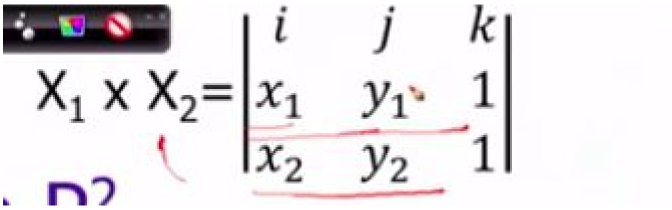


So, points and lines in the projective space have nice complimentary relationships, we call as a kind of duality relationships. So, what we are looking at in this particular case, we will explain the dual relationships. So, any point in the real space or any line in the real space can be defined by two points. we know that two points define uniquely a line. This line can be computed as a cross product of these two-3 vectors. So, $x_1(x_!,y_1)$ is representing the corresponding projective homogeneous representation of point X_1 and $x_2(x_2,y_2)$ is the homogeneous representation of point X_2. So, there are three vectors, so if I take the cross product, you will get this straight line.

In the real space itself any point is also defined as intersection of two straight lines which has a relationship in the projective space is in the following form $P = l_1 \square l_2$

here it is a cross product of these two lines which are represented in the projective form. And then, we are getting the corresponding point of the corresponding intersection point that is also represented in the homogeneous coordinate system or as an element of the projective space. Here, you should note that how the cross product of two vectors are defined this is a usual mathematics that we have learnt in our school book.

So, you should see that X_1 corresponds to its coordinates $[x_1, y_1, 1]$ and $[x_2, y_2, 1]$ are the coordinates corresponding to point X_2 and once we define the cross product of 3-vectors, in the following form



if I get the determinant, then I can always compute to this point if we expand this particular determinant.

So, this is the summary exactly one line that passes through two points and exactly one point at the intersection of two lines and their relationships in projective space, they are quite similar. A straight line can be expressed as cross product of these two points in the projective space and a point can be expressed as cross product of these two straight lines in again all representations are in the projective space.

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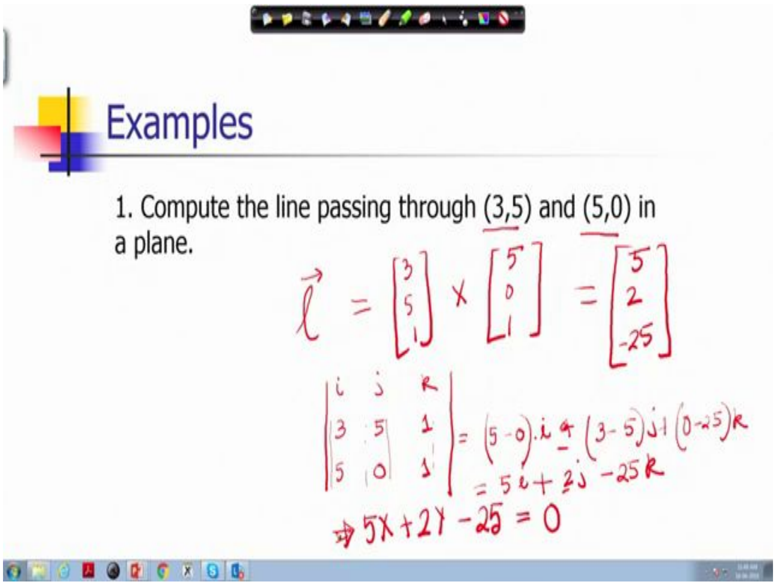

So, one of the interesting applications would be, we would be to find out to the straight line passing through two points by applying this method. We have learnt these method from simple coordinate geometry, but we can see that these computations can be very briefly summarized using this operation of cross product in using the concept of projective geometry.

So, let us consider this particular problem that we need to find out a line passing through the points (3, 5) and (5, 0) in a plane. So, naturally it is understood that (3, 5) and (5, 0) are the points in a two dimensional array space. We will be taking the cross product of the corresponding projective points in the projective space of (3, 5) in the homogeneous representation and the vector which is represented from (5, 0) in the projective space. Which means the vectors say (3, 5, 1) and cross product of (5, 0, 1), ok. So, that is what is the straight line representation.

So, if I expand this ,let me do this sum. So, as I have written earlier, so we have this particular components and you are representing . So, these are the unit vector directions and represent with the corresponding vector components. So, if I expand this determinant what I need to do? I need to consider this sub determinant in to e which means say $(5-0)\square i$ plus, then you suppress this and then you take the determinant and you have to take the negative of that. So, (3-5), so I should write here this is should be minus, then it is $(3-5)j$ and then you suppress you consider this part of the sub determinant for the kth component and you take the determinant which is $(0-25)k$.

So, this should be then $5i+2j-25k$. I hope my computation is correct, let me see with the result and you should also verify it. So, finally this line is represented by a vector as you can see I will take from $5\square 5$ from 2 and this is -25. So, if I write the equation of the straight line, what should be the equation? 5 is that a, 2 is that b and -25 is that c. So, the equation of the straight line can be represented as $5x+2y-25=0$ So, that is what is your equation, ok.

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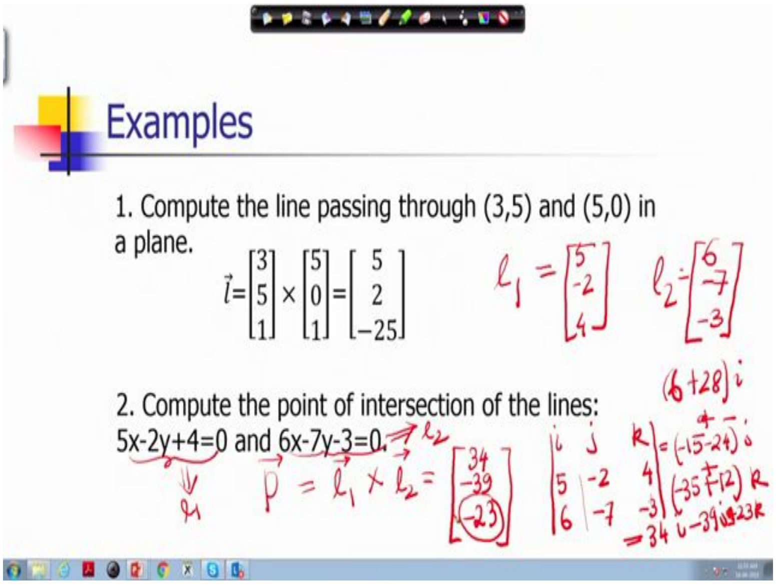
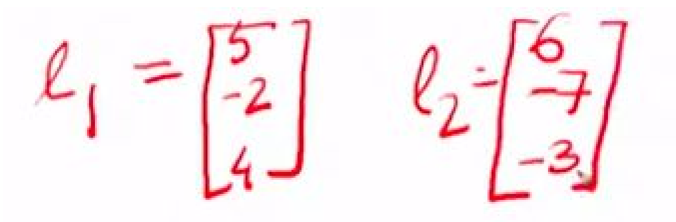
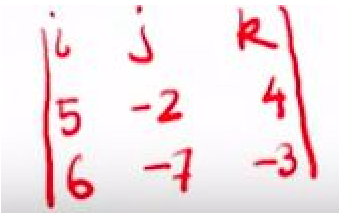


So, let us see that how this result is . So, this is a summary of the result, so as you can see we will get these equation $5x + 2y - 25 = 0$, that is the equation of the straight line in our understanding of coordinate geometry. Let us do another example consider this. So, we have computed the line from two points. Now, let us consider there are two lines and you would like to compute the point of intersection of these two lines and these two lines are given by these two equations.

So, once again let us work out. So, from this line, so now what is the rule or what kind of operations we have to do? If I consider the intersection point is say p. So, that would be a cross product of these two vectors l_1 and l_2.



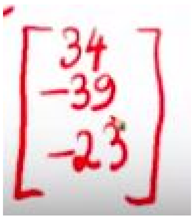
Let us see how we will do it and the above line will represent l_1 and l_2. So, we know that l_1 can be represented by $5x - 2y + 4$ and l_2 is represented by $6x - 7y - 3$ So, we need to take cross product of these two operations. Once again I will consider the same determinant



then if I expand it, what would be the expansion? So, let me rub some areas, ok. So, if I expand it let us consider it I will be expanding in this way say this coordinate this sub determinant. That means, they send this it is plus 6 and then plus 28, ok.

So, I am doing it mentally $(6+28)i$, then $+$ or I should put it as $-$ Then this middle determinant suppressing the middle column it is $-15-24$ and then we consider the third you know component which is $(-35-(-12))$. So, it should be $(-35+12)k$ So, the full you know.

Now, this coordinate can be represented as this expansion can be $34i-39j\ -23k$. So, what is p? P is represented as this vector which is



I hope my calculation is correct and let us see. So, as you can see this is the homogeneous representation of a point, actual point. What you need to do? You need to divide this scale value. You need to divide all the coordinates values with this scale value (-23).

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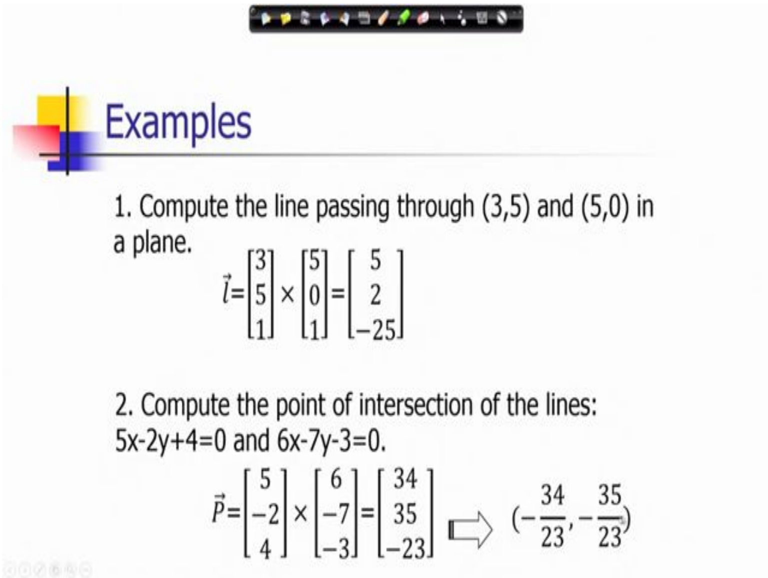

So, let us find out that you know what is the result of this operation. So, you can see that is what we are getting 34, 35 minus 23. I am not sure whether you know I just rubbed it, but as I mentioned this scale value has to be correspondingly divided by -23. So, this is the coordinate that we will get. You have to check my computations once again, there could be some error there.

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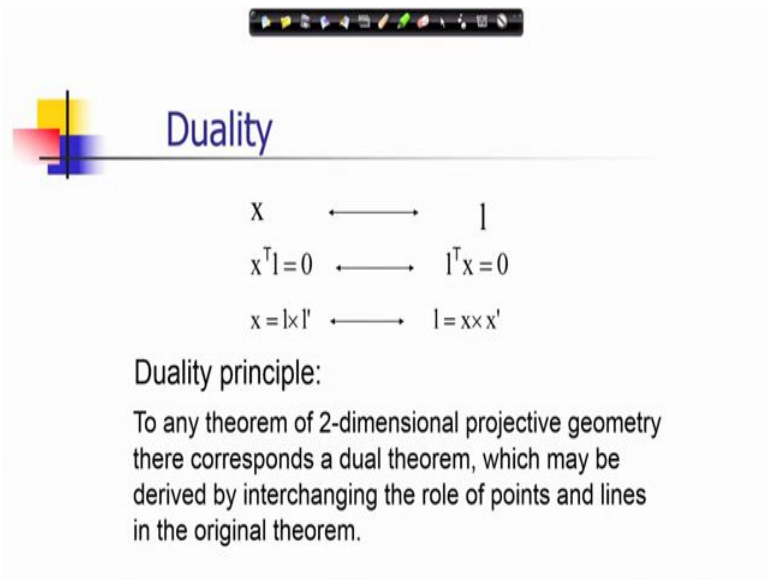

Let us complete this lecture by explaining the duality principle as we can see the point and line, they are represented in the projective space and their relationships. They can be nicely complemented by each other. For example, if I get the containment relationship $x^T l = 0$ and then $l^T x = 0$. so if I interchange l and x, still this relationship holds, that is the duality principle.

Not only this point containment if I consider the cross product if I point in line intersection and point intersections which are expressed by the cross product operations of three vectors. So, in this case the cross product of two lines $l \times l'$ gives you the corresponding point and cross product of two points x and x' gives me the corresponding line.

So, once again the places of lines and points if you interchange, still you get the same principle. So, that is what, is the duality principle. So, to any theorem of two dimensional projective geometry, they corresponds a dual theorem. So, this is any theorem and then there is a dual theorem which may be derived by interchanging the role of points and lines in an original theorem. So, with this understanding of duality principle let me stop here at this point. We will continue this lecture in the next video lecture session.

Thank you very much for listening to my lecture.