Computer Vision
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Lecture – 43
Range Image Processing – Part III

We continue our discussion on differential geometry based analysis of local surface geometry.

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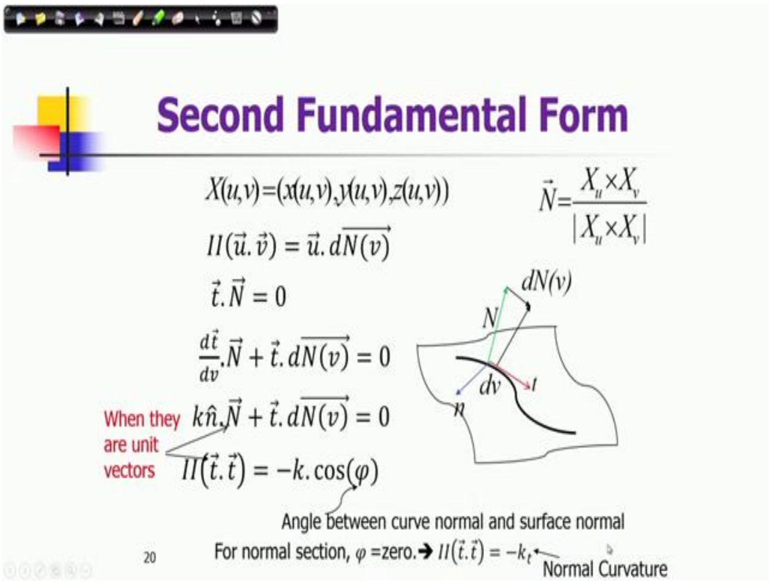

And in the last lectures we discussed that how normal at a surface point could be computed and we have defined two particular entities describing the local surface characteristics. One is first fundamental form, the other one is second fundamental form. In the first fundamental form essentially we are computing the magnitude of the change of the gradient at that point a magnitude along the tangent vector. And, in the second fundamental form it is related with the computation of curvature and we will continue our discussion on this particular aspect.

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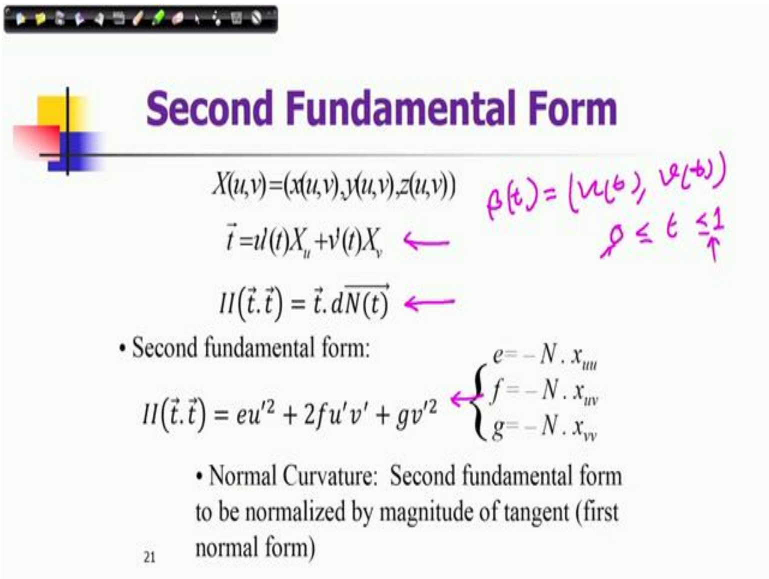

So, we are using the parametric description of a surface point say u and v are the parameters and normal at tangent at a surface point can be computed in this form, where u and v both again can be expressed in terms of a parametric curve in parametric curve descriptions and then if I induce a curve on the surface we have used this parametric representations. So, if $\beta(t)$ which is once again defining the surface point $(u(t), v(t))$ for t within certain range, it could be 0 to 1, it could be any other values of the real interval.

So, with this description we at this particular value of t we can compute the tangent of that curve by taking the derivative epsilon t. So, this is how the derivatives could be computed for the functions as we have seen that the X can be derived with respect to u and then u with respect to t such chain rule is applied here. So, this is how the tangent vector is computed at the point we have already discussed this also earlier.

And then the second fundamental form is defined in this fashion which is the $\vec{t} d\overrightarrow{N(t)}$; that means, as we moved across the along the tangent how the normal is changing its direction. So, that is the vector that changing vector that vector representing that change of normal. So, you are taking the dot product of these two and then you will be you will be getting the second fundamental form which actually measures the local curvatures which represent the which is a measure of local curvatures and which is defined from the functional point of view in this fashion. So, with the same descriptions here you can see e, f, g they are again 3 entities 3 parameters which are related to this functions and this is a definition of e, f, g which are all related to the second fundamental form those parameters.

So, they are the dot product of the normal at that surface point and also the second derivative vector of the variations along constant along constant v curve, this is second derivative vector variations along const both u and v directions and this is first derivative with respect to u, then second derivative with respect to v. So, we understand this kind of interpretations of differential calculus or differential that is the definition. And so, normal curvature is actually has to be normalized with respect to the magnitude of the tangent. So, it is the second fundamental form that should be what normalized.

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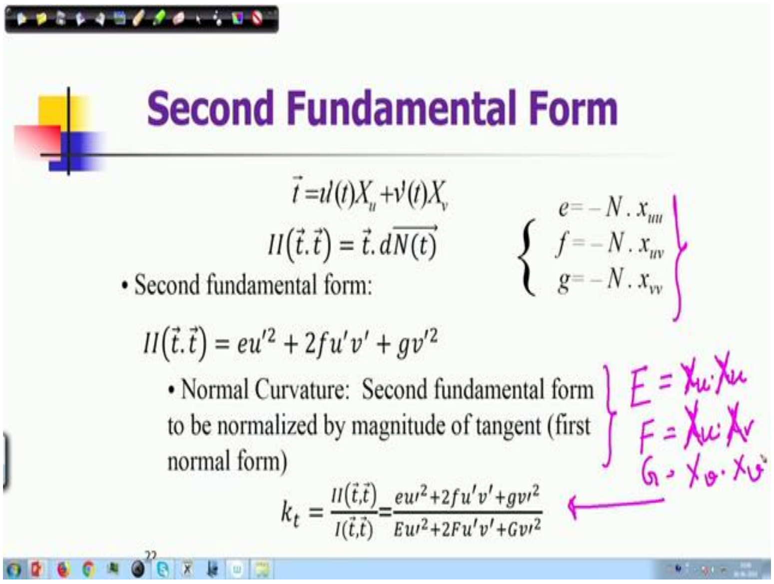

And this is the, these are the expressions. So, this is what is normal curvature which is second form fundamental form to be normalized and as you can see this is the expression for normal curvature at surface point for the curve described for any particular curve with described by the c parameter t and also the corresponding in the tangent directions along that curve. So, for any tangent direction this is the definition and these are the quantities which are defined earlier that e, f and g these are already defined here and E, F, G they are related to the first fundamental form.

And what if we can remember that E should be $X_u.X_u$ then F is $X_u.X_v$. So, these are all X that is a surface point and is G is $X_v.X_v$. So, they are the magnitude of the tangents along the constant u, constant v and the other is basically the dot product of 2 tangents along constant u and constant v curves in this description. So, these are the interpretation from the functional point of view. So, let us continue that how this could be further used for computing curvatures locally.

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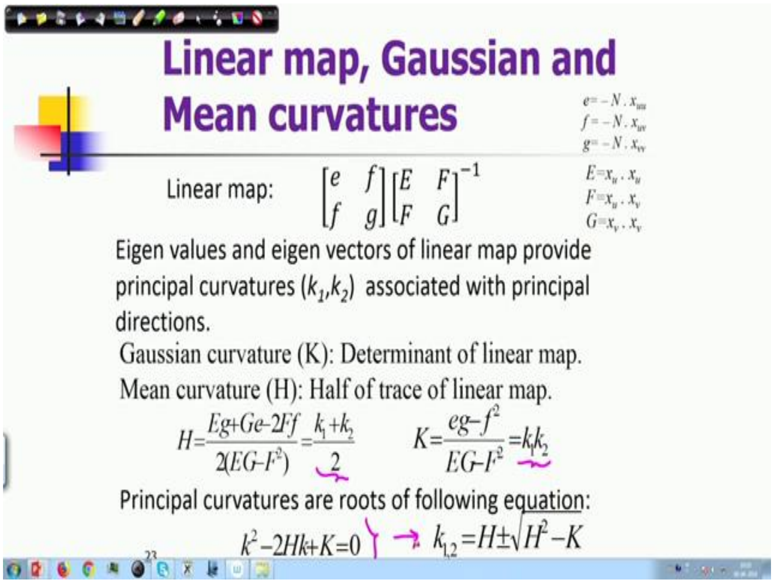

So, there is a concept called linear map using this parameters and this linear map it contains all the necessary informations for computation of curvature. So, this linear map is defined in this way you can note that the inverse of the parameters of first fundamental form is shown by this matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ these are the elements those inverses I show.

$$\text{Linear map}: \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}$$

So, this is related to first fundamental form parameters and these are related to second fundamental form and those definitions can be seen here how we have defined earlier also in this case. So, how linear map is characterizing let us see. So, in the linear map the eigenvalues and eigenvectors they provide the principle curvatures and also the principle directions corresponding to those principle curvatures.

So, these are interesting and we can say that it is basically one of them would be dominant curvature whose value is greater, but note that there is sign involved. So, in the case of computation we call them two principal curvatures, but since their sign involved actually when we are considering with respect to their relative strength then the magnitudes should be considered.

Or earlier we have seen this particular feature when you have detected the (Refer Time: 08:39) and you have considered the magnitudes of curvatures here. For a 2 dimensional function of in that case which is equivalent to a function in the surface also is a 2 dimensional function. So, these analysis are can be quite extended in that under this in that scenario is also except that is not a surface that is a brightness distribution or any other functional distribution.

So, any way coming back to this particular topic of surface geometry; so, curvature is as a very critical information to understand the local surface topology or geometry. And so, there are two entities which are very intrinsic, one entity particularly Gaussian curvature which is very intrinsic property of the surface geometry and that could be computed as a determinant of linear map which is actually product of this two principle curvatures. And, mean curvature is basically half of the trace of linear map or which is the mean of the two principle curvatures.

So, these two also can be an alternative descriptions of local curvatures other than the principal curvatures. So, the expression for mean curvature can be obtained using the parameters of first fundamental form and second fundamental form that is given here and also the expression for Gaussian curvature is also given here.

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)} = \frac{k_1 + k_2}{2} \qquad K = \frac{eg - f^2}{EG - F^2} = k_1 k_2$$

You note that you know since this is a very familiar expression which can be related to the theory of quadratic equation. So, we can define a quadratic equation this is equation and it is solution is also shown here. So, you can also get principal curvatures given mean curvature and Gaussian curvature.

$$k^2 - 2Hk + K = 0 \quad k_{1,2} = H \pm \sqrt{H^2 - K}$$

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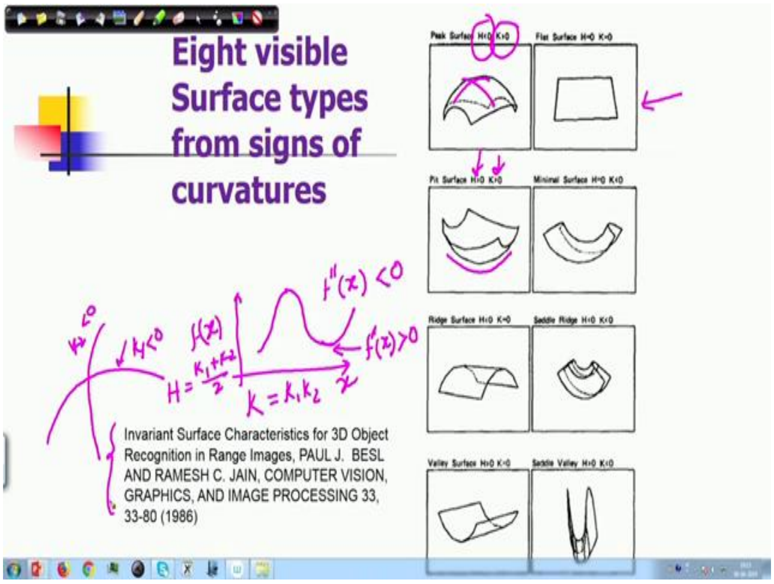

So, the usefulness of this analysis can be understood by this particular technique which is which has found the topology of the local surface curvature. You can see various kinds of local shape that could be attributed by the signs of in this case this is shown with respect to Gaussian curvature and mean curvature.

So, r if the mean curvature is negative so, if it is negative and if the Gaussian curvature is positive then it is like a surface like a peak surface just to draw your attention to the fact in our school geometry also we know that when you have a function as a variable of x f(x) and if we get a peak here. So, at this point the peak maximum could be characterized when we find it is double derivatives; that means, if I take the $f''(x) < 0$

Where as if it is a minimum then $f''(x) > 0$ we have already know this particular feature and it can be explained from differential calculus from there we can explain why the signs should be here negative and positive respectively, and we also know that this mean this maximum and minimum they are characterized by the gradient value that should be equal to 0. However, the fact is that this curvature which is related which is proportional almost to this differential double derivative of the functional value.

So, in a surface in a surface again we can consider any particular curve and which has this kind of shape. So, we expect the curvatures values so, principle curvatures should be negative here and all the principle curvature. So, there are two directions. So, if I consider these are the two principle directions for principal curvatures. So, both should be negative then you can get this kind of peak shape that is what is expected here and you know that the mean curvature it is defined as the mean of this two value.

$$H = \frac{k_1 + k_2}{2}$$

So, since they are negative so, mean also has to be negative and the Gaussian curvature which is given by K it is defined as product of these two values $K = k_1.k_2$ and since w once again they are negative the product has to be positive. So, that is how this characterization of if the mean curvature is less than 0 and or Gaussian curvature is greater than 0 then you can characterize it as a peak surface.

Similarly for the pit surface which is like they like the minimum of a 1 dimensional function. So, this is a scenario this is a pit surface here we expect both of them should be positive both mean curve both curvature should be positive. So, the mean curvature of should be positive as well as a Gaussian curvature should be positive that is what is shown here. So, you can extend this analysis for a flat surface this curvatures are going to be 0 . we know that that radius of curvature is infinite for a flat surface. So, that value of curvature is 0. So, since for a flat surface value of curvature is 0 at any directions so, you get both of them 0 Gaussian and mean should be 0.

So, we extend this kind of observation to various other shapes in a 3 dimensional surface and you can explain the kind of a features or observations those are provided in this particular slide. And in fact, this is a paper which discusses this particular properties this is a paper by Paul J Besl and Ramesh Chandra Jain in 1986 it was published and it is very pioneer in paper in characterizing the or in finding out the local topology of a range images from the range image from the range data.

So, this shows how we can compute this particular characteristics local properties and this feature again you can use for your purpose is later on for example, segment in surfaces etcetera.

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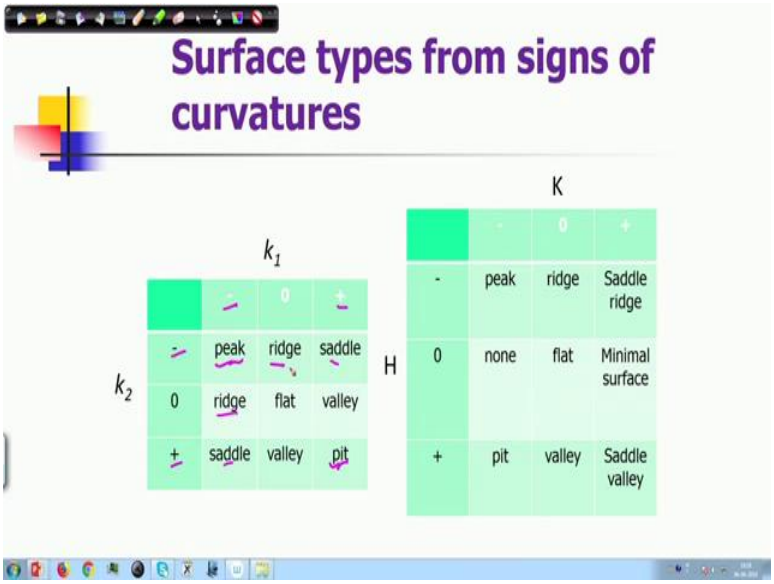

So, just to summarize we can characterize the this local topology by the signs of curvature, it could be signs of principal curvatures or it could be signs Gaussian and mean curvatures. So, you can see that for the principal curvatures as I mentioned when both are negative we have peak when both are positive we have pit and then there are combinations of negative positive, positive negative. So, they have almost like a symmetric relationships you can you can find out in this case.

You have say ridge, ridge, saddle, saddle because of these see it does not matter in which direction it is negative or positive, but the local surfaces they will have symmetric this property can be described in the form of a symmetric matrix form you can say, it has certain symmetry with respect to this. Whereas for Gaussian and mean curvatures the characterization is a bit more elaborate that you can see here and that is the reason why in the previous work of Besl and Jain they have used Gaussian and mean curvatures, it is interesting they have and we identify these characteristics.

So, there are regions like saddle ridge which was earlier could not be characterized a saddle valley minimal surface. So, these are few other kind of characterizations other kind of topologies also they have considered in their analysis.

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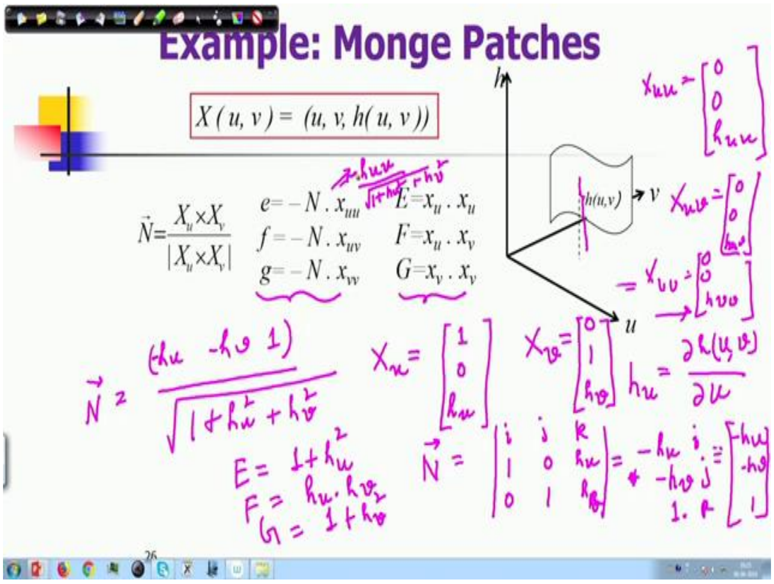

So, we will be considering a special kind of a surface the description and which is akin to our range data and in fact, there is a term of this kind of surfaces surface patches when you consider the parameters u and v, they itself describes the x coordinate and y coordinate and the z coordinate is a function of those two parameters like h(u, v) has been shown. So, this is a familiar range data representations what we have discussed in previous in our previous lectures and one example is shown here in the as a figuratively.

So, this is a surface patch and this is u and v like it is equivalent to x and y axis and you get the corresponding height of the surface point from that plane which is a z directions. So, this is how this is described and already we know in general in for any description of parametric surface we can compute the surface normal and then all the elements of first fundamental form and second fundamental form matrices in this form. So, this is note that this is second fundamental form matrices. So, these are all double derivatives they are related to curvatures.

$$e = -N \cdot x_{uu}$$
$$f = -N \cdot x_{uv}$$
$$g = -N \cdot x_{vv}$$

So, this is the second fundamental form matrices and

$$E = x_u \cdot x_u$$
$$F = x_u \cdot x_v$$
$$G = x_v \cdot x_v$$

this is the first fundamental form matrices, these are all derivatives and they are related to magnitude of the tangents or actually this is a dot product of tangents along two directions two different directions for F otherwise they are also of the same directions. So, this quantities can be computed now easily I mean they are they have a special form in this particular aspect, because if I compute X_u then if I take that gradient and let me represent them in the form of a column vector so, you get $\begin{bmatrix} 1 \\ 0 \\ h_u \end{bmatrix}$

That means it is a partial derivative of function $h(u,v)$ with respect to u, $h_u = \partial h(u,v)$

similarly we can compute X_v as $\begin{bmatrix} 0 \\ 1 \\ h_v \end{bmatrix}$. So, from this two vector we can compute E. So, what should be E? $E = X_u \cdot X_u$ which will give you $1 + h_u^2$, then $F = h_u \cdot h_v$ and, G is $X_v \cdot X_v$ that magnitude itself it is $1 + h_v^2$. So, this is about the elements of the first fundamental form.

What about the elements of second fundamental form? Again from this description so, let me first compute the normal vector because it requires to computation of normal and for that I need to take the first cross product of X_u and X_v. So, I will use that familiar cross product to know computation. So, this is I will expand this determinant and this could be written as. So, this is minus h u i plus, this is minus this should be v, this is v minus h v you know this is minus h v j and k is 1.

$$\overrightarrow{N} = \begin{vmatrix} i & j & k \\ 1 & 0 & h_u \\ 0 & 1 & h_v \end{vmatrix} = -h_u i + (-h_v j) + 1.k \equiv \begin{bmatrix} -h_u \\ -h_v \\ 1 \end{bmatrix}$$

So, this vector N can be considered as this vector is minus h u minus h v 1 and if I have to normalized it. So, N is given as minus h u minus h v 1 divided by root over 1 plus h u square plus h v square . $\vec{N} = \frac{(-h_u \quad -h_v \quad 1)}{\sqrt{1+h_u^2+h_v^2}}$

and from there you can compute now the other thing. So, if I consider X_{uu} which means I have to once again take the derivative with respect to this and we will get $\begin{bmatrix} 0 \\ 0 \\ h_{uu} \end{bmatrix}$ and

$X_{uv} = \begin{bmatrix} 0 \\ 0 \\ h_{uv} \end{bmatrix}$ and $X_{vv} = \begin{bmatrix} 0 \\ 0 \\ h_{vv} \end{bmatrix}$.

So, your value of e should be dot product of N and x_{uu} and only the you know z component. So, you have $\frac{-h_{uu}}{\sqrt{1+h_u^2+h_v^2}}$ e and the similarly I mean for F it is $\frac{-h_{uv}}{\sqrt{1+h_u^2+h_v^2}}$ and for G it is $\frac{-h_{vv}}{\sqrt{1+h_u^2+h_v^2}}$. So, this should be minus. So, so this is how these elements are computed just I have described and let us see the result how we can get it.

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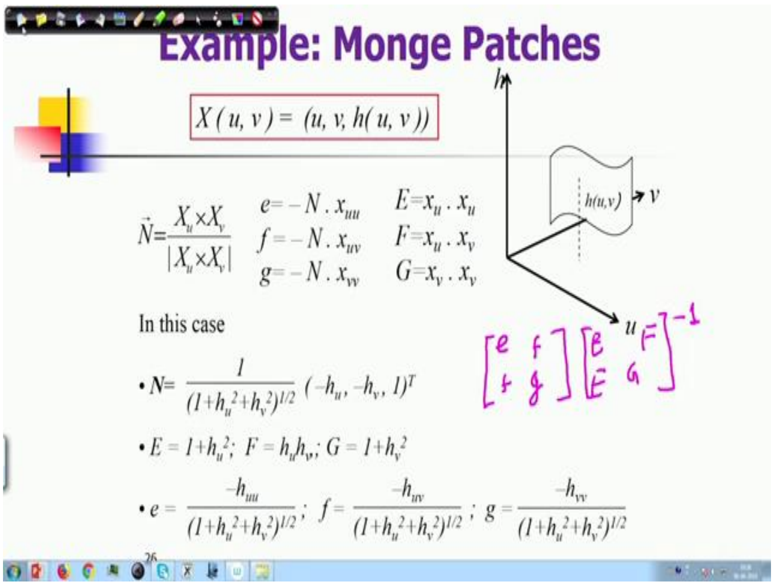

So, you see that what I have described those are the thing mentioned here and once you have computed this then you can compute the Gaussian curvature and mean curvature using this element. So, linear map is given by $\begin{bmatrix} e & f \\ f & g \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}$ So, you can compute those quantities and then take the trace of the matrix half of the trace and determinant of the matrix then you will can get the Gaussian curvature and mean curvature and from there also you can then take also principal curvature. So, I will show you the expressions of Gaussian curvature and mean curvature as a result of this operation.

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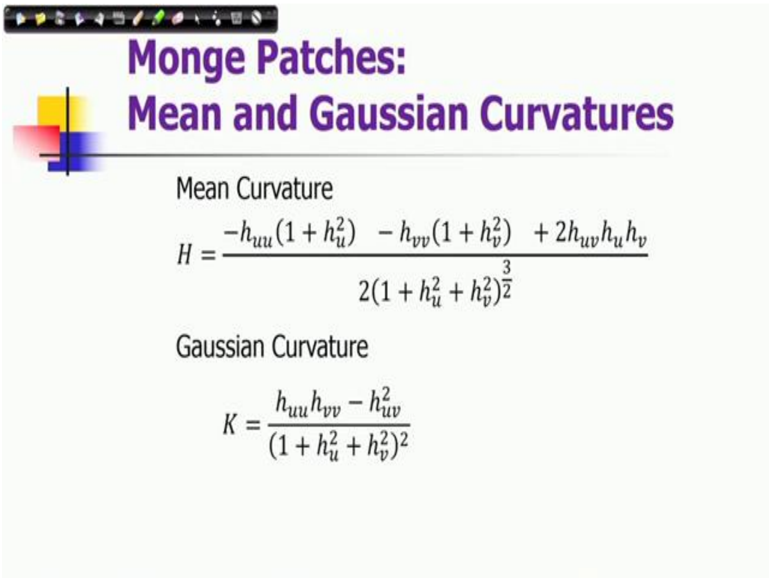

So, this is how we will get you will get mean curvature this is the expression

$$H = \frac{-h_{uu}(1+h_u^2) - h_{vv}(1+h_v^2) + 2h_{uv}h_u h_v}{2(1+h_u^2+h_v^2)^{\frac{3}{2}}}$$

and also Gaussian curvature this is expression.

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1+h_u^2+h_v^2)^2}$$

So, when you have a range data then you can compute these curvatures by using this particular these expressions because range data as I told that is a simplified form of surface description where the parameters are described itself by the x and y coordinates in those directions.

And the depth value is a function of x and y coordinate which is z coordinate and by computing them you can compute the derivatives using the masks as we have discussed earlier in our lectures. In the very first lecture we discussed how derivatives could be computed even the double derivatives other gradients could be computed of a function of a 2 dimensional function that we have discussed.

So, use those masks to compute this derivatives partial derivatives and also double derivatives and then from there, you can compute the mean curvature and Gaussian curvature then look at the signs of those curvatures and that would give you the topology of the local surface. So, with this, let me stop here we will continue this discussion of surface geometry local surface geometry in the next lecture.

Thank you very much for your attention.