Computer Vision
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Lecture - 20
Stereo Geometry Part – V

We are continuing our discussion on estimation of fundamental matrix and in the last lecture we discussed how the equations can be formed using the pairs of corresponding points in relation to the elements of fundamental matrix. And, by solving those equations you can get those elements of fundamental matrix. One of the things we are discussing that we require to perform coordinate transformation as the dynamic ranges of coefficients of those equations they widely vary because of the nature of the equations.

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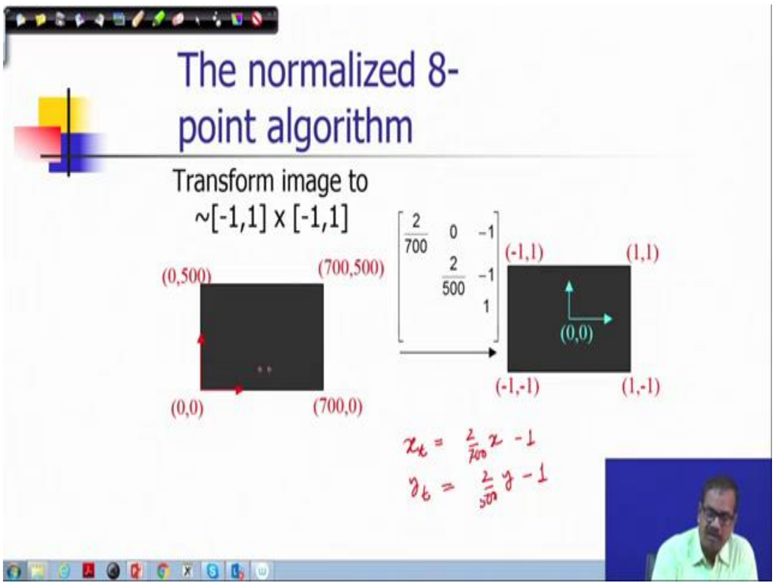

So, we will continue that discussion and as we discussed that we will map the coordinate points to these ranges and bring them into these ranges. An example is shown here. Usually that is a very common case that no image coordinates vary between say 0 to 700 in this example and 0 to 500 there are other values. So, it is a typical range and there you can see product of those coordinates will have a very large dynamical range compared to single variables of coordinate points, single points. So, the example of a transformation will convert these coordinates into these ranges as given here.

$$Transform\ image\ to \sim [-1,1]X[-1,1]$$

You can see that this transformation is given in this form, that if I multiply this matrix with x', I will get the transformed point.

So, let me denote the transform point as say xt coordinate; that means,

$$x_t = \frac{2}{700}x - 1$$

$$y_t = \frac{2}{500}y - 1$$

So, given (x, y) will be converted into this value. So, that would translate the origin to the (0, 0) in the middle of this (-1, -1) has disc affect. Anyway as discussed that you have to apply the corresponding computations to apply that transformation back into what you get as a fundamental matrix out of this computation. Then the inverse transformations are to be applied to get their original fundamental matrix in the respective coordinates of the image planes.

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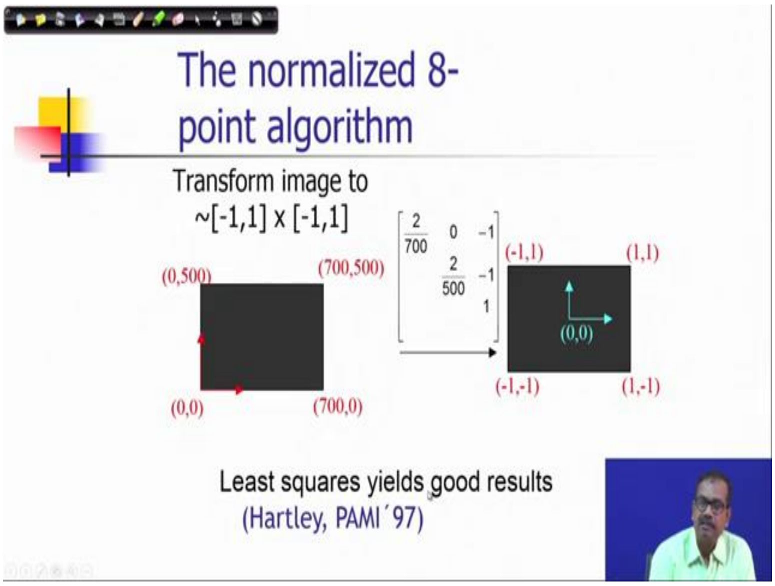

We will now continue, we will see what other aspects are there. So, as we mentioned that this is a paper where actually this particular technique is discussed and it has been shown that least square error method yields good results when we perform this transformation.

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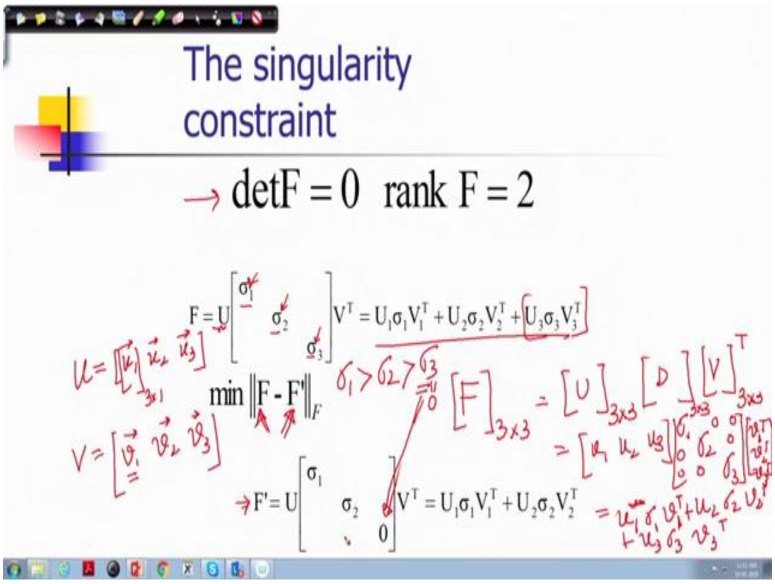

So, now we will consider the other aspect of this estimation, I mentioned that in the previous case that we have not applied the constraint of singularity on the fundamental matrix. So, the solution what you get that is not truly giving you a fundamental matrix unless it satisfies that singularity constraints. So, how to enforce that singularity constraint? So, that is what we will be discussing here. So, here you see that this constraint is expressed in the form of that the determinant of the matrix F should be equal to 0. And, also we know since its rank deficient, it should be less than 3 and in this case it would be 2.

$$\det F = 0 \; and \; rank(F) = 2$$

So, there is 1 zero particularly of this fundamental matrix we discussed also, that there is a epipole that is left a epipole. Epipole in the reference image reference plane image plane that is a 0 of this fundamental matrix that is a right 0. Similarly, there is a left 0 which is a right epipole; Now if I perform singular value decomposition of a fundamental matrix. So, we present singular value decomposition in this form. So, you see that U is a matrix which is orthogonal or we can make it also orthonormal which means if I take the column vectors.

$$U = [u_1 \; u_2 \; u_3]$$

So, let me represent U as a set of column vectors in this form, I will use this notation. So, each vector is a column vector which is of 3 cross 1 dimension; that means, U is a 3x3 matrix.

And, in singular value decomposition all these column vectors would be orthogonal or you can make it orthonormal also as I mentioned. Similarly, V is also another matrix whose column vectors are also orthogonal. So, you have V1, V2, V3 so, this is also orthogonal. So, any square matrix or even any matrix not only square can be decomposed into this form, into this particular structure in using singular value decomposition. So, in this case particularly I will discuss with respect to square matrix only for in this context so, it can be decomposed into this form.

$$F = u_1\sigma_1{V_1}^T + u_2\sigma_2{V_2}^T + u_3\sigma_3{V_3}^T$$

So, U as I mentioned 3x3 then there is a diagonal matrix, and then V which is also a matrix, but we have to take transpose of this V. So, I can write also like [u1 u2 u3] and this diagonal matrix there in this case it is also a 3x3 matrix. You can see that if I multiply I will get a 3 x3 matrix. So, diagonal matrix as we know that all off-diagonal elements will be 0 except the diagonal elements which should be non-zero not necessarily, that it should be non-zero. It could be 0 also and we can apply no sign laws it is, it we can adjust the signs in both the sides so, that no one can make it all positive.

So, if I shuffle these columns similarly shuffle the rows of [v1 v2 v3] in the same order still we will get the same matrix F. So, I will shuffle in such a way that

$$\sigma_1 > \sigma_2 > \sigma_3$$

which means I will take those column vector those pairing U1 and V1 whose sigma is those singular value. So, σ_1 this is called as singular values whose singular value is the maximum here, then the next maximum then the minimum. So, this is how the singular value decomposition works and this is how you can see the matrix can be considered as a super positions of 3 rank 1 matrices.

So, you are just summing up 3 rank 1 matrices so, each one is a 3x3 matrix. So now, what you can do that you have to estimate a singular 3x3 matrix which should be very close to your estimated F matrix, now that is a computational problem. So, F you have estimated from the data, but that is not singular. So, your objective is to get that solution F' which should be very close to your estimated F which means you have to minimise the Frobenius norm.

That means, element wise if you take the square of their differences and add them and take the square root of the sum that is a Frobenius norm. It is as good as saying that this is L2 norm when you are representing a matrix in the form of a vector by concatenating all its rows or its columns in whatever way. So, the solution lies in the fact that this should be minimum that theory says that when you set the minimum singular value to 0; that means, you are setting to 0 so, you are ignoring this part. So, once you set it 0, now then this F prime is a singular matrix because that is one of the conditions one of the signature of a singular matrix.

That if you perform singular value decomposition, then at least one of the singular value should be 0 and if I reorder them all those singular values in order of their increasing values. And, as I mentioned that we enforce that singular values should be positive we can enforce positive point 0 so, the minimum values should be 0 in that set. So, in this case number of 0s or number of non-zero singular value will also provide us the rank of that matrix. So, since the rank of this fundamental matrix is 2 so, we are keeping two non-zero singular values and only setting the minimum singular value to 0.

So, this becomes your solutions now and you have applied the singular constraint in this fashion. So, this is that we are computing singular value decomposition.

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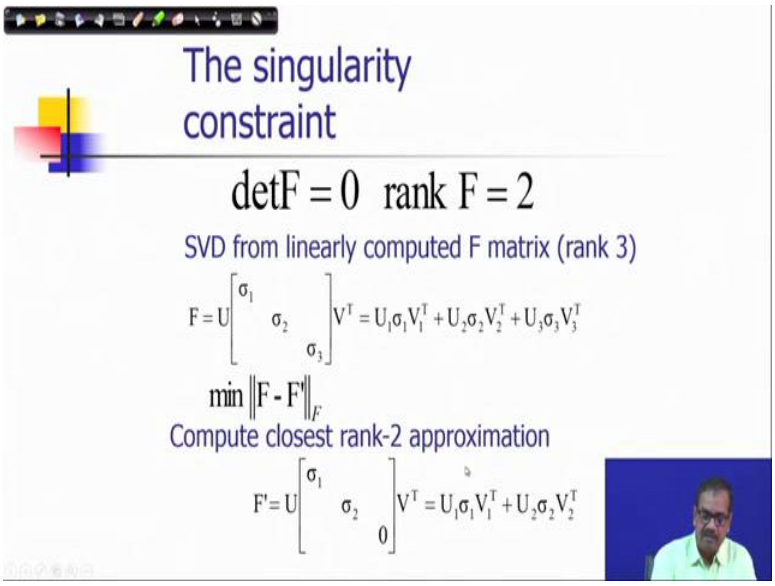
$$Closest\ rank\ 2\ approximation, F' = u_1\sigma_1 {V_1}^T + u_2\sigma_2 {V_2}^T + u_3\sigma_3 {V_3}^T$$



And this is the closest rank 2 approximation from that estimated F matrix.

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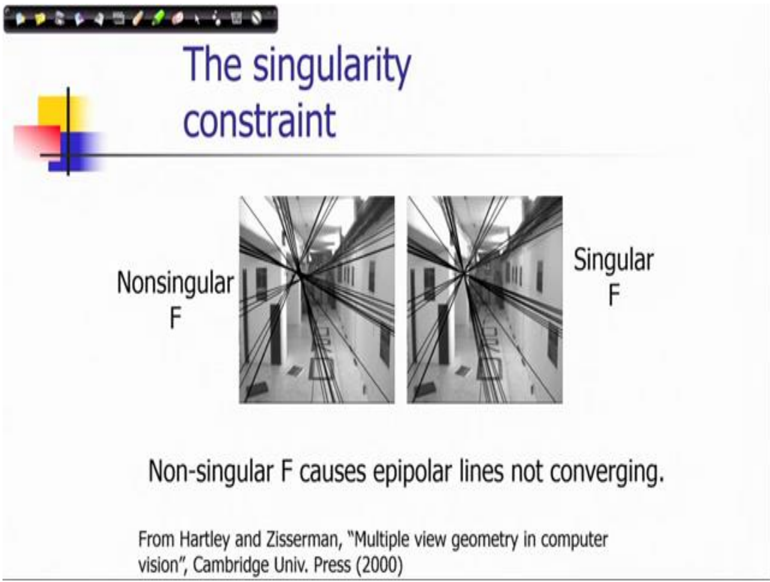

So, one of the nice example that I could get from the book of you know Hartley and Zisserman that is "Multiple view geometry in computer vision". You can see there are the effect of singular F and non-singular F by showing how epipolar lines behave in a particular image plane. So, this is a right image plane with respect to a left image plane which is a reference plane and we are drawing the epipolar lines of all the points corresponding points of the left image planes in the right image plane.

It is expected that all epipolar lines should be converging to a single point and that is the right epipole of that stereo system. So, you can see that if it is non-singular then they are not converging. Only their convergence is ensured if it is a singular. So, that is why geometrically also this property needs to be satisfied and that is why this constraint is so important in this respect.

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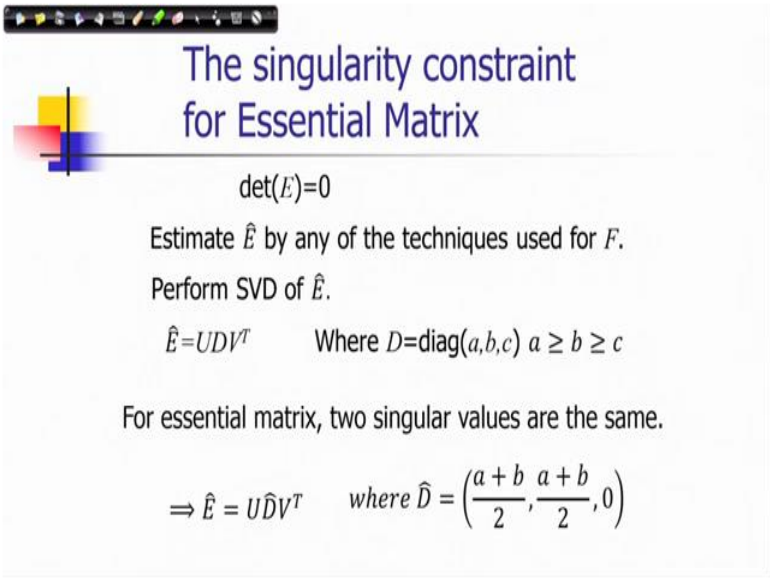

Now, if I consider the estimation of essential matrix there is another interesting feature, I told earlier also the number of independent parameters in essential matrix is 5. So, which means there are few more constraints here. So, let us find out what are those constraints.

$$\det(E) = 0$$

So, just remember that essential matrix is the same fundamental matrix when your camera is calibrated which means the calibration matrix of the camera. And, you can convert all the image coordinates into the canonical form and express the fundamental matrix just by using the rotation and translation of the word coordinate to camera coordinate system those parameters. So, extrinsic parameters are the only constraints and since there are 6 parameters and also the scale is another constraint so, there will be 5 independent parameters in this case. So, suppose you have a calibrated camera; that means, you know the calibration matrix. So, you can take and make that conversions and then you apply the same estimation technique. And, whatever fundamental matrix you have estimated that is essentially an essential matrix because, you get this canonical coordinates and you get in that form of the fundamental matrix. So, there also we need to perform singular value decomposition to enforce the constraint of singularity.

But, there is one interesting property of essential matrix that the two singular values that you get here should be also equal.

$$\hat{E} = U\hat{D}V^T \text{ , } \hat{D} = (\frac{a+b}{2}, \frac{a+b}{2}, 0)$$

So, the way you can enforce is that you consider the average of those two values (a, and b) and set them to the σ_1 and σ_2 in my previous technique whatever we discussed. You just make these modifications when you are estimating an essential matrix.

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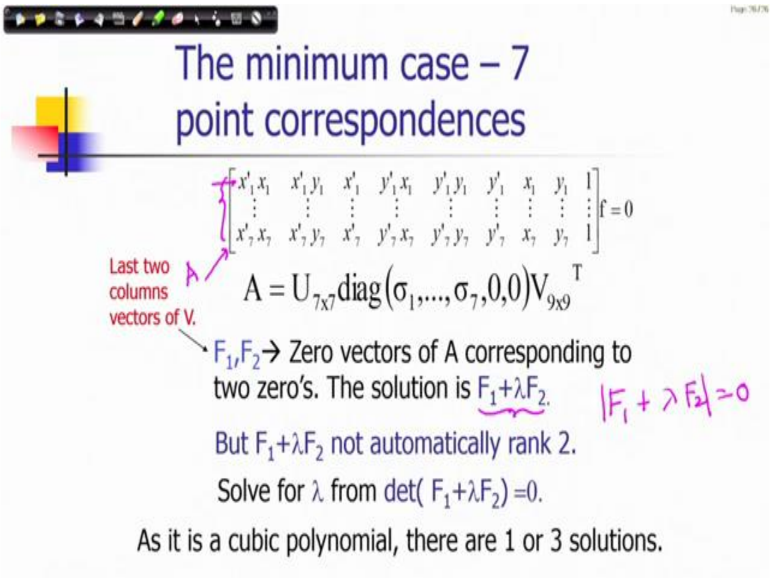

Let us discuss now how to estimate fundamental matrix using 7 point correspondences which is the minimum number that is required as we mentioned earlier. So, if you have 7 point correspondences there would be 7 such equations involving the elements of the fundamental matrix as we also discussed in our previous slide. So, we can see that each row is also formed in the same way what we considered earlier. And, there are 7 such row corresponding to each of the 7 pairs of corresponding points. So now, if you would like to solve this as you can see this is a set of homogenous equations and since the number of elements in F is 9 and you have 7 equations.

So, the rank of this matrix is 7 and here we can use the singular value decomposition of this matrix A. So, we consider this matrix is A and if I perform a singular value decomposition then we will be naturally expecting there will be two 0s in this case. And, the U and V you can consider know their dimensions in this form and V there will be corresponding 0 vectors of this particular matrix A. So, those 0 vectors are the last two columns of V and they can give you the solutions of the equations. Because, 0s are the solutions of A are they are the solutions of these equations and they are the corresponding fundamental matrix.

But since there are two such zero vectors so, actually the solution is a linear combination of these two zero vectors. So, we can express the solution as a linear combination of two zero vectors F 1 and F 2 as it is shown here and this linear combination is given in this form. So, where lambda is any scalar constant, but you have to determine exactly which linear combination which lambda value.

$$|F_1 + \lambda F_2| = 0$$

Because, the fact is that the determinant of this should be equal to 0, it should be rank deficient matrix which means if I take the determinant of this matrix, if I take the determinant that should be equal to 0. And, since the determinant it is not automatically rank 2 we have to solve for lambda which will give you the 0. And, it will be a cubic polynomial because it is a 3x3 matrix and in cubic polynomial we would like to get a real valued solutions of lambda. And, there are two situations either there will be one real valued solutions or there will be three such solutions. So, a maximum we have three such possibilities of fundamental matrices in this case that is a technique we discussed.

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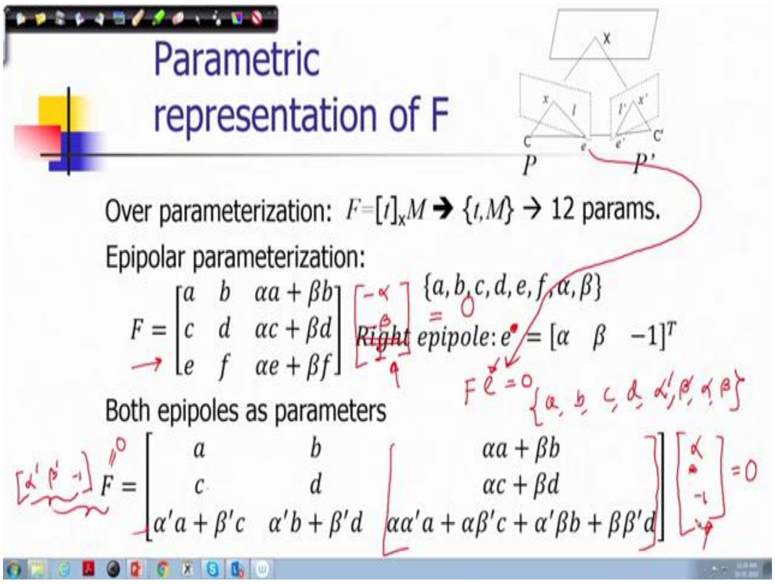

So, the essence is that you have to perform singular value decompositions or rather eigen decomposition of the A^T matrix here. And, and solve for the lambda which is the coefficient of linear combination of the eigen vectors by enforcing the determinant of the combination linear combination should be equal to 0. So, let us consider now the other aspect of the fundamental matrix. So, how fundamental matrix can be represented in different parametric form. So, we have already seen how many number of independent parameters should be there, but not necessarily your representation will have that intrinsic characteristics.

So, it you can represent with more number of parameters, but then you have to enforce the constraints; that means, the independencies among those parameters needed to be enforced there. So, in the estimation process and this is not always ensured and a post estimation those operations are taken care of as we discussed that how we can make the determinant 0 for F. So, over parameterization representation could be in this form; that means,

$$F = [t]_X M \rightarrow \{t, M\} \rightarrow 12\, parameters$$

$$e = [\alpha\ \beta - 1]^T$$

where t is a epipole. And, in this form you see that there are more number of parameters like you have epipoles t and M.

$$F = \begin{array}{ccc} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ e & f & \alpha e + \beta f \end{array}$$

So, M is 9 and t is a 3x1 so, we can write 12 parameters. Now, we know essentially there are seven independent parameters. One of the interesting representations which follows this parameterization is this one, you can see that how many independent parameters are there a c e b d f 6 then alpha and beta. So, there are 8 independent parameters. The interesting part is that the singularity constraint on f; that means, the determinant that is equal to 0 here because if I take the linear combination of these two columns; the third column is a linear combination of these two columns.

$$F = \begin{matrix} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ \alpha' a + \beta' c & \alpha' b + \beta' d & \alpha\alpha' a + \alpha\beta' c + \beta\alpha' b + \beta\beta' d \end{matrix}$$

So, we know that in this case this is a rank deficient matrix and one of the column is dependent of others. So, also its determinant should be equal to 0 that is the property of determinant. And, if I multiply this F say with$[-\alpha - \beta \ 1]$, you will find that you are getting 0. So, you perform minus alpha a minus beta b plus alpha a beta b that is 0 and in this way. So, what does it mean? It means that the left epipole; that means, 0 of F is this 1 so, which means it is an epipole. So, alpha and beta has an interpretation in this form or you can see alpha beta minus 1 so, it is because you are multiplying F e should be equal to 0.

So, I think this is fine. So, it should not be right epipole e, it should be the left epipole e it should be this one ok. So, let us consider the next one that is both epipoles as parameters. So, you have this number of parameters a b c d e f alpha beta. Now, this is another representation you can see that how many number of parameters you are having here. You have a b c d then alpha prime beta prime and alpha and beta; that means, here also you have 8 parameters a b c d alpha prime beta prime and alpha beta. So, a b c d alpha prime beta prime alpha beta.

Now, in this case you can find out that if I multiply F with once again with alpha beta minus 1. So, here also you can see that the third column; that means, this column this is a linear combination of first two columns which is alpha into this and beta into this will give me the third column. And, alpha in yeah, and also the third row is a linear combination of the first two row which is alpha prime into first row and beta prime into second row will me the third row. So, if so alpha beta minus 1 that should be equal to 0 like the previous one. And, then if I perform on the other way if I perform the multiplication alpha prime beta prime minus 1 with F that should be also equal to 0; that means, this becomes your right epipole and this becomes your left epipole so, in this set up.

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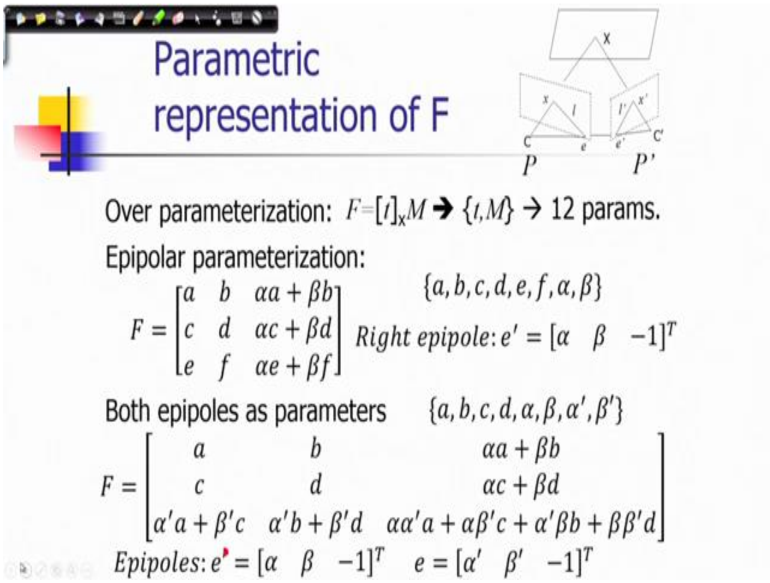

So, once again now we have the same problem of representation. So, e should be alpha beta minus 1 and e prime should be so, here also you have a problem. So, please note this corrections e should be alpha beta minus 1 and e prime should be alpha prime beta prime minus 1 as per my diagram.

So, let me stop here and we will continue our discussion in the next lecture.

Thank you for your listening.

Keywords: fundamental matrix, singularity constraint, essential matrix, parametric representation.