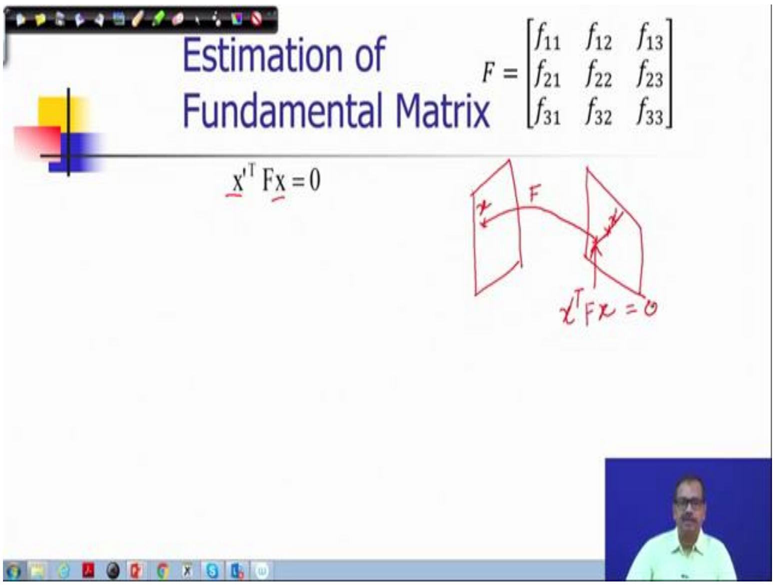
Computer Vision
Prof. Jayanta Mukhopadhyay
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 19
Stereo Geometry Part – IV

We are discussing about Stereo Geometry and we discussed how a fundamental matrix of this geometry characterizes the system. The in this lecture we will consider how to estimate a fundamental matrix given a pair of images of a stereo imaging system. So, you consider the relationships with the corresponding pairs of points for a fundamental matrix.

(Refer Slide Time: 00:53)

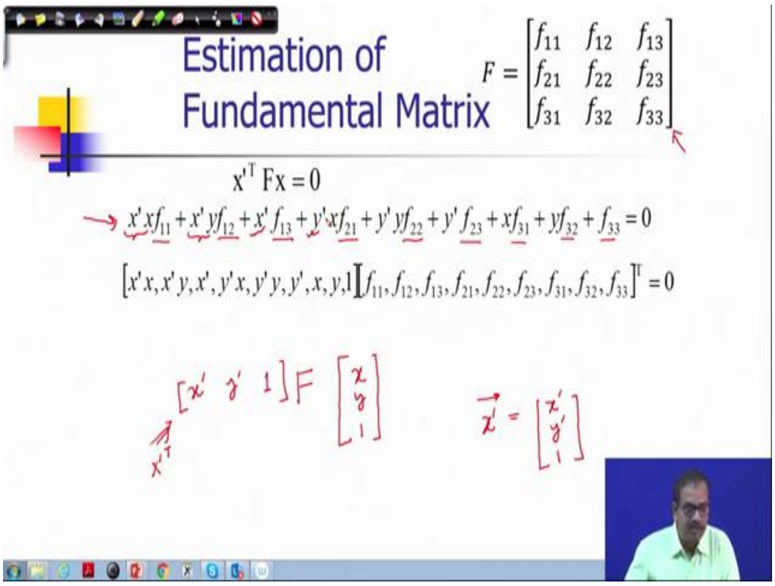


And as we have already discussed that you have two pair of points that is x'; that is in the right stereo image and x in the left which is a reference stereo image. Then the relationships with the fundamental matrix between and also this pair of points can be described in this form; that is $x'^T Fx = 0$; if I consider two images of a stereo system as I mentioned. And consider a point this is x; this is the reference camera system or reference image plane of the stereo system and its corresponding point x'.

Now, the task of a fundamental matrix or the property of a fundamental matrix is that; it transforms a point into the epipolar line. So, it transforms a point; if I apply Fx, it will transform the line where x' will occur. So, this line is given by this equation Fx and the applying the point content print relationship of projective space; since x' lies on these lines; so $x'^T Fx = 0$; so that is how we have derived this relationship.

So now we will see that using this property how we can estimate a fundamental matrix given a pair of image points; so for that we need to know say a few corresponding pairs of points, so this is a problem now.

(Refer Slide Time: 03:16)



So, this is the expansion of this particular equation. If I consider the element of the fundamental matrix is given in this form; it is a matrix element and using the notation of index indexing notations of an element of a matrix that element of an ith row and jth column is given by fij.

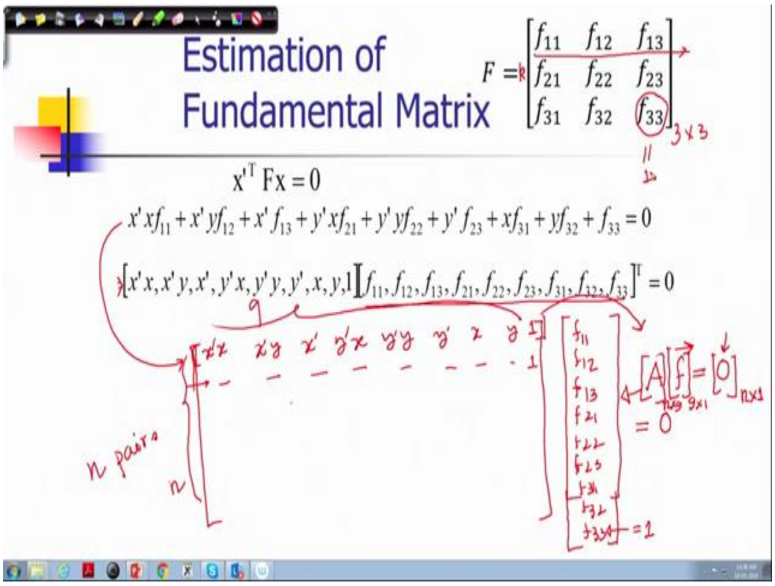
$$F = \begin{matrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{23} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{matrix}$$

And also you consider the column vector notation of a point for example, x' = [x' y' 1]. In our convention we consider a point x', but its x coordinate is x also denoted by this symbol x'; only thing is that it is scalar quantity of an x coordinate, then y coordinate and 1. So, this is the presentation of the point x' there. So, if I we can write it in this form; [x' y' 1], this is what is x' transpose. And then multiply with fundamental matrix F and also the point x in the reference image corresponding point which is also given by this column vector [x y 1].

So, now you consider multiplications of these matrices; if you expand, you will get this equation; And again it would be convenient for us to represent in the form of a matrix because in that case a set of linear equations can be represented by the matrix operations. So, a single equation is given in this way and in the matrix notation, here our objective is to compute the fundament elements of fundamental matrix; so, these are unknown variables or unknown parameters in our case.

And given the observation these values we already because that is what is given to you; I mean we will discuss later on how we can also compute these corresponding pairs in my next topic. But for the time being; let us assume that these points are given to you, maybe even through manual observation in a crude way you can get a few such matching points. So, these values are given to you; that means, these coordinates are given to you. So, now this equation in the matrix form is written here.

(Refer Slide Time: 06:42)



Let me write it in a proper notation which is more convenient to understand. So, we will see that these are the coefficients of those equations.

$$x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + f_{33} = 0$$

So, I am converting this equation in the matrix form which is written here also. So, because this is for the sake of convenience a column vector in this slide is shown by a row vector by using transpose operation; so, you are computing all these elements of fundamental matrix. So, you can see that all the rows are again represented as a column vector and they are; they are concatenated in this form.

$$[A]_{nX9}[\vec{f}]_{9X1} = [0]_{nX1}$$

So, this equation is formed with only one curve pair of corresponding points; as I explained.

Suppose you know a few more points so; that means, I can have; I can represent all these equations in a general matrix form. So, all the coefficients of those equations will come in this matrix. So, this is for the first pair of points; for the next pair of points once again I will put these elements observations and I can write this equation. So, in this way if I know n pair of points; n pairs of points each one will give me one equation. So, as you can see what is the number of columns in this case? You have here 9 unknowns because fundamental matrix the dimension is 3 X 3; so, there are 9 unknowns. So, in this form we have I mean there are 9 columns and number of rows would be n and there are 9 columns.

So, this vector can be shown as a column vector f and this matrix let me write it as an A. So, $[A]_{nX9}[\vec{f}]_{9X1} = [0]_{nX1}$; so 0 is also a column vector because each equation is giving me 0. So, if there are n equations; so 0 will be a column vector of dimension n X 1 and as we know this should be 9 X 1 and this dimension should be n X 9. So, given n pairs of points corresponding points; you can form these set of equations.

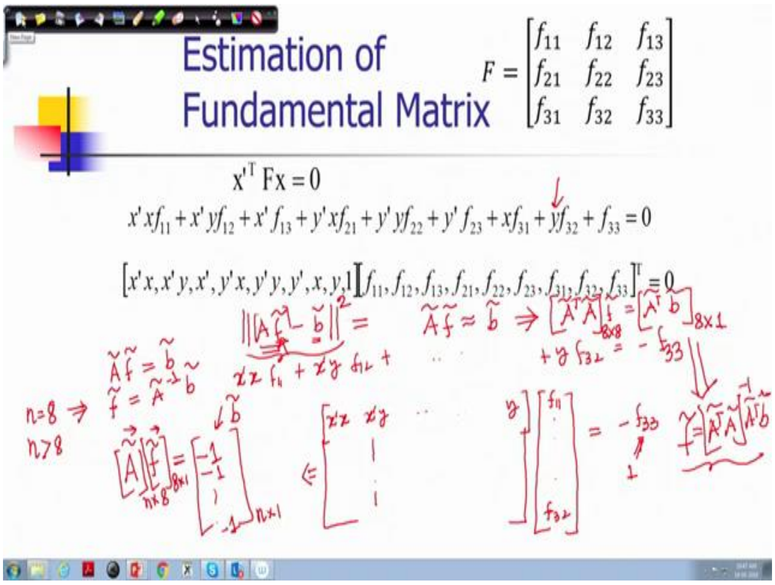
So, now your task would be to solve this equation to get f and you can see this is a homogeneous equation; a set of homogeneous equations, again you can apply the least square error estimate to solve this; however, you can make this equation as a non homogeneous set. Suppose I consider the; I said this f33=1; like we did earlier also. So, one of the value let us consider say to certain particular value because I; I mentioned that though there are 9 elements; number of independent parameters in a fundamental matrix is 7.

First thing that it is an element of a projective space. So, if I multiply all the elements by is constant k still this relationships it holds; that means, that is also a solution of this system. So, fundamental matrix multiplied by any scalar constant they are all equivalent that we discussed earlier. So, one of the element could be considered as a scale factor say if I take this equal to 1; this would be taken care of. So, in proportion to that value all other values are expressed that is one particular feature property; so, that it is at least one reduction of the number of independent parameters.

The other constraint in a fundamental matrix that we also discussed that it is a singular matrix. So, if I take the determinant of this matrix; it will be 0, so you will get another equation. So, if I expand determinants you will get equations in terms of all these elements and that should be equal to 0; so that is another constraint. So, that is why there are number of independent parameter is 7.

Now, for the time being let us consider that now we will be solving for 8 parameters. We will later on we will see that how that properties of singularity could be enforced; for the time being let us let us approximately estimate fundamental matrix where the singularity constraint is waived. So, in that case as I mentioned that one of the parameter you can set to some known value; some given value. So, let us check to this value 1 and then you can convert this equation in this form. So, let me rub this part.

(Refer Slide Time: 13:24)



So, once again; I can write this equation as

$$[x'x, x'y, x', y'x, y'y, y', x, y, 1][f_{11}, f_{12}, f_{31}, f_{13}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}]^T = 0$$

so this is a transformational. So, now with this equation becomes non homogeneous equation because this part is not 0. And in this way same for all the n corresponding pairs; you can get n such equations.

So, let me consider that matrix say A' matrix. So, A' into let me use also f prime; these are the dimensionally reduced matrices because of this conversion that should be equal to minus f 33. For example, if I set minus if set f 33 equal to 1; so this is the equation we will get ah; actually you will get a column vector of minus 1 because every question will have minus 1.

So, the dimension of this should be n cross 8; for dimension of this vector would be 8 cross 1 because there are 8 unknowns and this is n cross 1. So, applying the solution for system of non homogeneous equations; you can perform the least square estimate and you know that. But if I consider this column vector as say b; so I would be solving

$$\tilde{A}\tilde{f} = \tilde{b}$$

And since there are more number of equations and usually; you should have more number of equations at least minimally you require 8 equations to solve it then exactly you get a solution. Because this matrix would be 8 X 8 and if it is not a singular matrix you perform $\tilde{f} = \tilde{A}^{-1}\tilde{b}$ that would give you; so, when just you have 8 corresponding pairs of points I can simply write $\tilde{f} = \tilde{A}^{-1}\tilde{b}$

So, that would be the solution in that case and that is a requirement you should have 8 numbers of corresponding points that is the minimum requirement. And when it is n greater than 8; then what you need to do? You need to perform a least square estimate; that means, you will have to minimize the error.

So, error is defined in this way

$$||Af - b||^2 = 0$$

so the deviation of the vector b from vector Af. These dimension of Af would be also n X 1 and b would be n X 1. So, if I take the norm of this vector which is nothing, but the sum of the square of its elements and you can take the square root also or sum of the square of its elements. And that should be I mean you can use also square just to show that square of the norm; that should be equal to that you have to minimize this.

So, you have to find out that f which will minimize this norm, can be solved by least square error method. There is a nice derivation we can perform by applying almost like matrix operations. But we should use the theory of minimization in this case by taking the derivatives of these expression and equating with 0; we have to solve these equations then you will get these solutions.

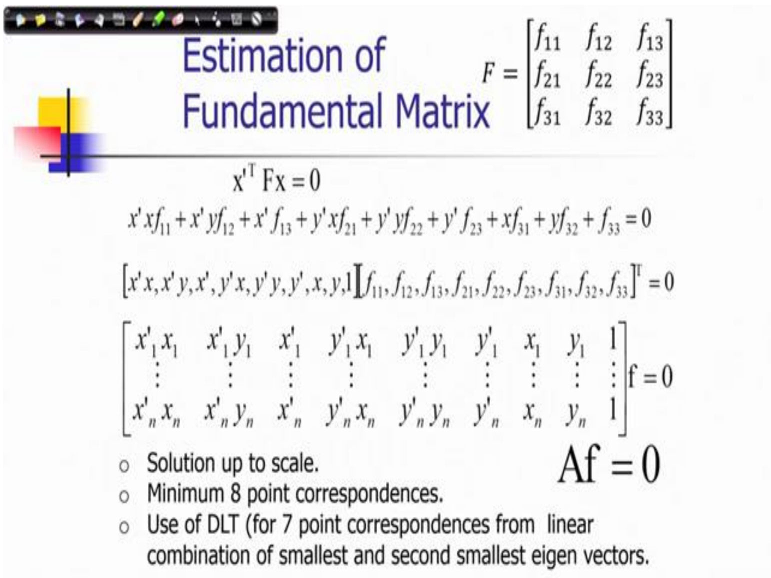
Just for the sake of your easy this aspect

$$\tilde{f} = [\widetilde{A^T}\tilde{A}]^{-1}\widetilde{A^T}b$$

So, now you see that this becomes a square matrix. So, this is this square matrix is in the form of 8 X 8; A is n X 8; so a transpose is 8 X n into n X 8. And of course, the whole thing this would be 8 X 1.

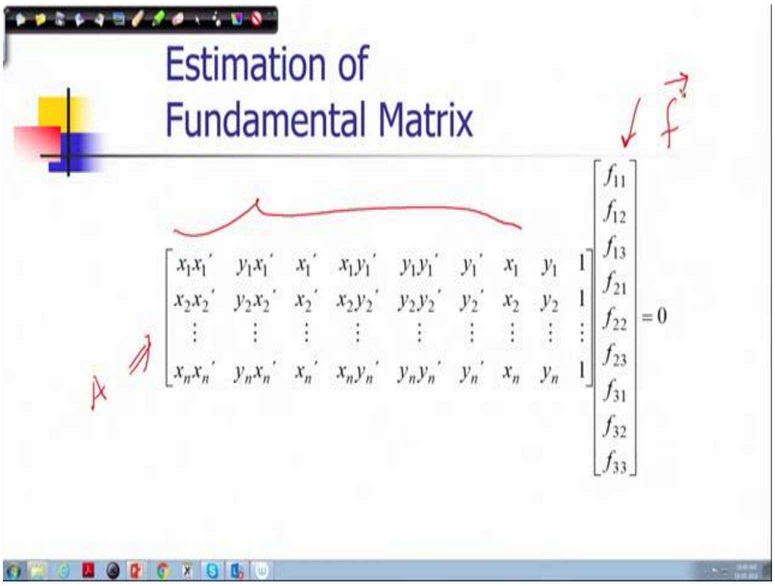
So, this is your least square error solution that is for the set of non homogeneous equations. So, let me summarize these expressions because I have also the slide set for this representation.

(Refer Slide Time: 20:26)



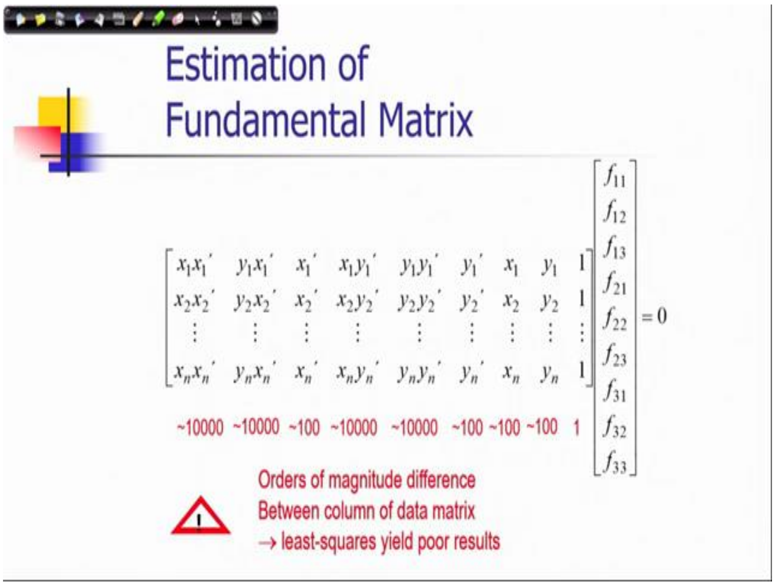
So, as I mentioned this is how we represent the system of equations given n pairs of corresponding points. And in our discussion we also mentioned that the elements of f; they can be retrieved up to scale; that means, whatever solution you get if you multiply by any scalar constant; that would be also a solution of this equations. And also we discussed how minimum 8 point correspondences are required. There is a technique which we will discuss later on that even you can do its 7 point correspondences; just to mention that because as I mentioned there are 7 independent parameters. So, we will discuss this technique in my next slides.

(Refer Slide Time: 21:27)



So, this is a representation of the set of equations with n pairs of corresponding points and you can see this is that matrix A; I was referring at. So, this is matrix A and this is the f that is a column vector of consisting of elements of fundamental matrix; that is the unknown and this is a set of equations.

(Refer Slide Time: 22:06)

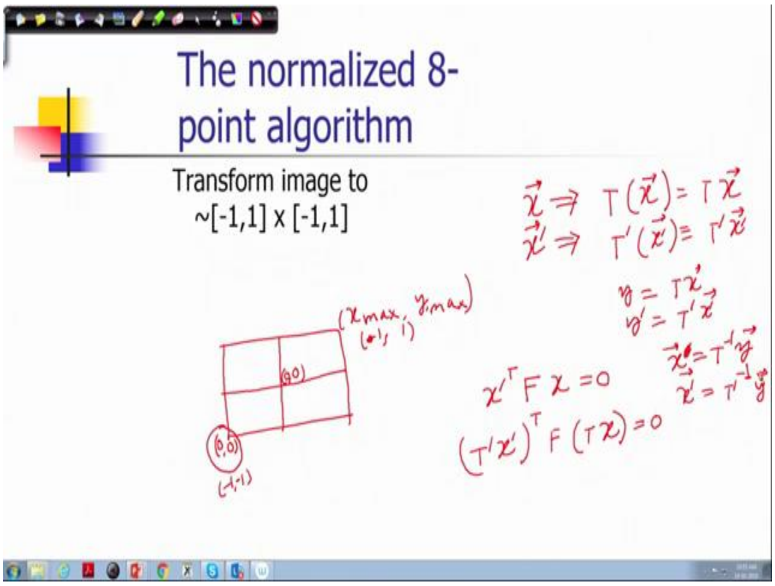


One of the problem with these presentation is that the coordinates of these points; they have a very large range. And as you are taking products in some cases; you understand product of two large numbers will be also very large; so, the dynamic range of these coefficients of this equation of the unknown parameters; that widely vary.

You can see that for the products; the ranges could be in the tune of minus in sorry in the tune of 10000; this could be magnitudes that is the variation ranges. It could be negative positive whatever because the coordinates could be usually in the images all the coordinates are positive; so, let us assume all values are positive. So, so this is the range 10000; 10000 and whereas, the coefficients with single variables range would be smaller 100; assuming the images dimensions are in that 100’s only. So, 100 X 100 image, but then you get all the; you know coordinates within 100.

So, that is why since the orders of magnitude to difference it is; there is a lot of difference in that orders of magnitude. So, numerical stability is less in these computations and it yields poor results. So, we will see how to handle that part let us proceed.

(Refer Slide Time: 23:47)



So, one of the technique to take care of this particular fact that why not you transform the coordinates within a smaller; within this unit range of [-1, 1], then you can see even the products of coordinates also they have equivalent equal ranges of the range of the single coordinates; all the coefficients they will have this.

So, one of the technique is that you can transform these coordinates between [-1, 1] X [-1, 1]. That means, suppose you have your image coordinates it varies; if I assume say this is 0, 0 and say this is the width and height of the image say (xmax, ymax). So, these are the coordinates within this ranges all the coordinates are represented. So, now we I would like to transform this coordinate.

So, the variations are there; that means, exactly in the middle you have the centre point is represented by origin (0, 0); so this is how coordinate is transformed. And you can see that now if with transformations you get transformations in both the images independently, still you can recover the fundamental matrix that also let us understand. Say you have an image as I mentioned say any point x; now it is transformed to a point, this transformation is given by this in the say left image; that means, a reference image.

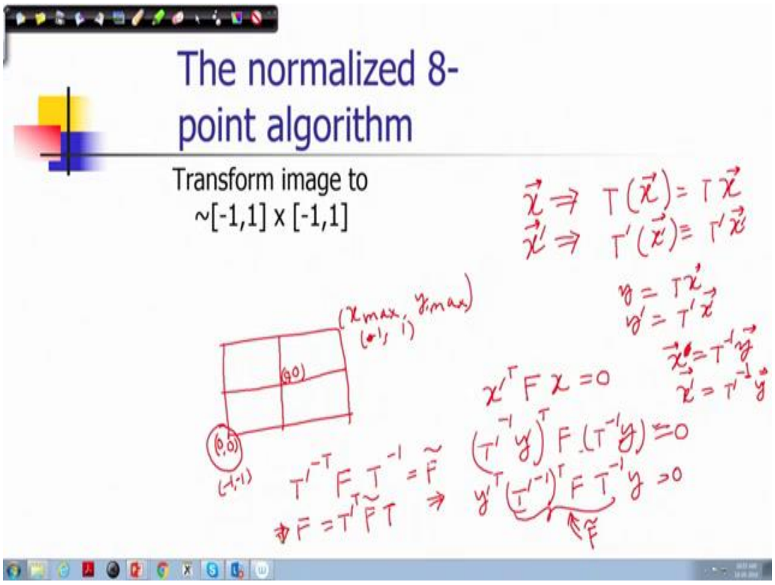
For the right image; let us consider that another transformation we are performing; so we are expressing in this. So, these are linear transformation;

$$(T'x')^T F(Tx) = 0$$

So, if I consider the equation of the fundamental matrix; involving fundamental matrix; so, this is the equation that we are getting; so now you consider apply this transformation; so this is T’.

So, what I will do? I will rather write see transformed point; if the transform point is y. So, I will have y =T x’ and y’=T’x’ so $x = T^{-1}y$. So, this is a requirement the transformation should be invertible and x prime is T prime inverse y prime; so now this should be written.

(Refer Slide Time: 28:13)



So, now x prime is written as T prime inverse y prime, transpose F and x is T inverse y that is equals 0. And performing this matrix transfer transposition operation; so y prime transpose T prime minus 1 transpose; then F that is T T; T prime inverse transpose, we can write it also T prime minus T in our previous notation; T inverse y equals 0.

Now, you can see this is also a 3 cross 3 matrix and this is the transformed matrix after transformation you get this matrix; you get this one. So, this is the fundamental matrix associated with y transformed points. So, what you can do? You can, you can compute the fundamental matrix between these by using this transformed points; corresponding transformed points where all these coordinates in this ranges will be used. And then from F; F tilde from this whatever you get; then again transform it back to get the fundamental matrix to get the original fundamental matrix; that relationship also we can use. Say this is T prime transpose then F T inverse equals F tilde.

So, now we will multiply at the both end; so both end we will multiply with T. So, it become identity and in this case it would be T prime transpose. So, that would be equal to F. So, this is how you can get; get back the original fundamental matrix. So, with this let me stop here; we will continue this discussion of estimation of fundamental matrix in my next lecture.

Thank you very much.

Keywords: Fundamental matrix, least square solution, normalized algorithm