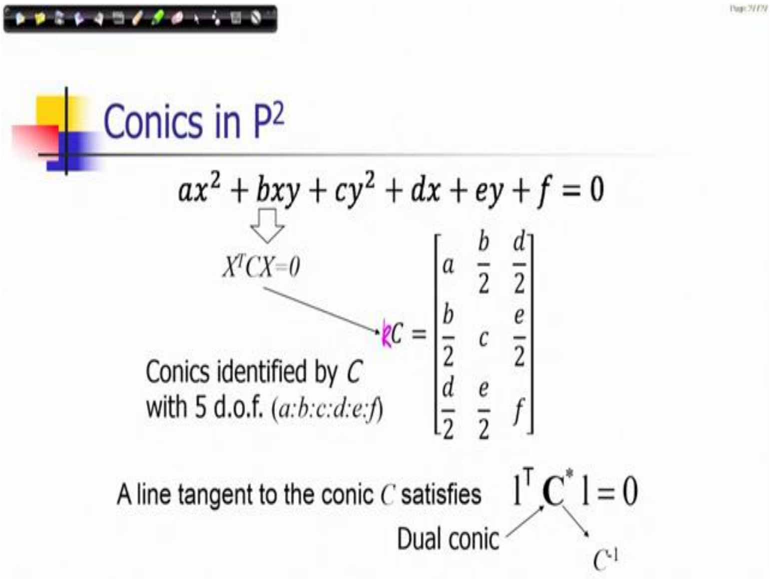
$ax^2 + bxy + cy^2 + dx + ey + f = 0$ **Computer Vision**

Prof. Jayanta Mukhopadhyay
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture – 10
Homography: Properties – Part III

We will continue our discussion on the Properties of Homography or projective transformation.

(Refer Slide Time: 00:19)



Now we will look at the properties of conics, how they appear in the transform space. Earlier we discussed about the general representation of a conics. In a non homogeneous coordinate representation the conics are represented in the following form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

whereas, in a homogeneous coordinate representation we can represent it in a simple quadratic structure $X^T CX = 0$

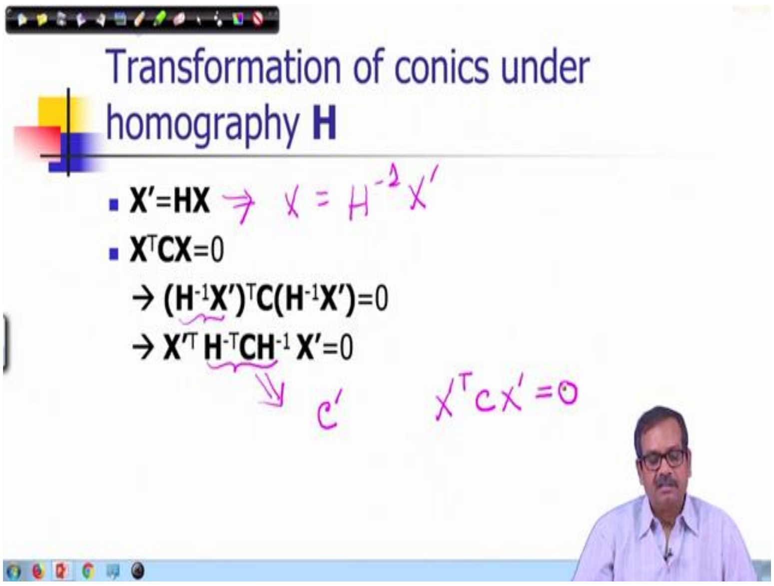
Where, C is given by the following parameters

$$C=\begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

It is 3×3 symmetric matrix and as you can see that there are 5 independent parameters because C is also an element of the projective space. So, if I multiply C with a scale k still it will remain the same conics and another interesting property of this conics is that, there is a dual representation of the same conics. it is a representation by their line tangents. So, take any line tangent given by l that should satisfy also the following property $l^T C^* l = 0$

where C^* is a dual conic representation and that is given as C^{-1}. We have discussed all this properties in our previous lectures.

(Refer Slide Time: 02:11)



So, let us see how a conic goes under transformation how a conics appear. So, let us consider a transformation H where a point X is transformed to X'. So, all the points which are lying on a conics and which are given by the equation $X^T CX = 0$.

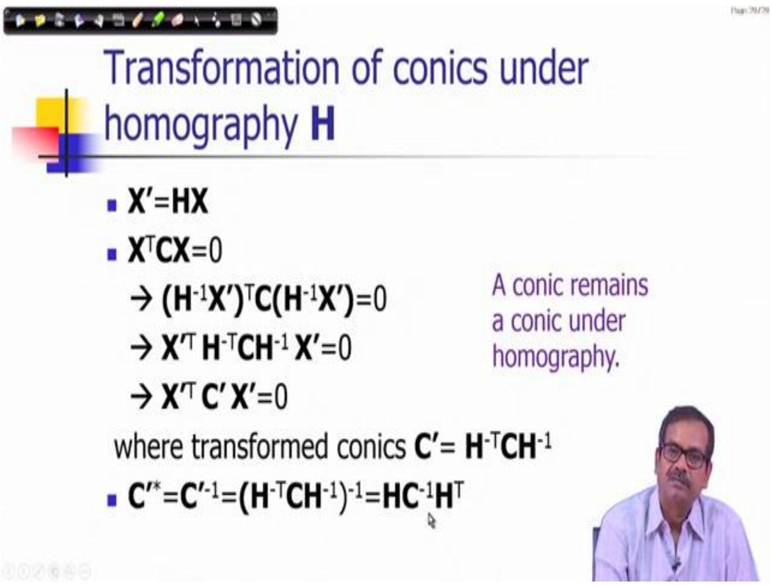
let us see what property they follow following the conics rule in the transform space . So, we can write this equation $X^T CX = 0$ in the following form $(H^{-1}X')^T C(H^{-1}X') = 0$

because X is replaced by $H^{-1}X^{'}$ this is coming from :

$$X^{'} = HX$$
$$\Rightarrow X = H^{-1}X^{'}$$

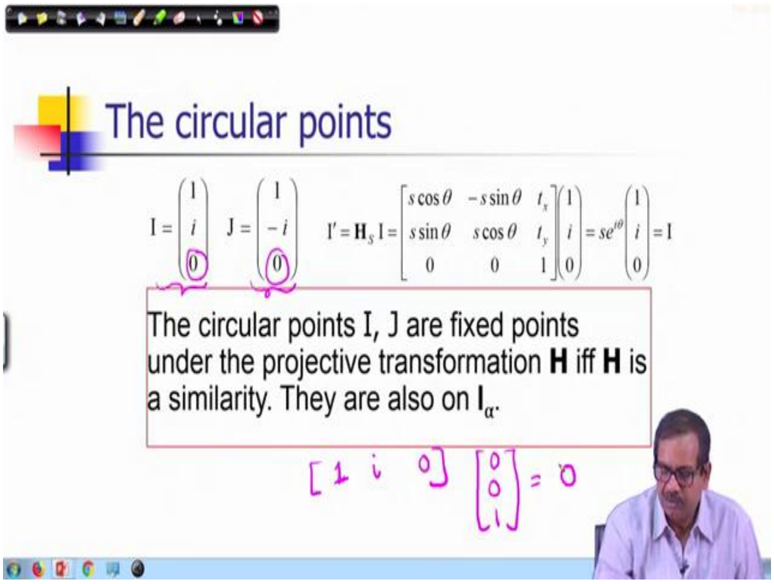
. So, X equals $H^{-1}X^{'}$ .So, if I replace it then I get this equivalent form $(H^{-1}X^{'})^{T}C(H^{-1}X^{'}) = 0$ we expand this equation once again; that means, this transpose operations can be expanded on the matrix multiplications which is $X^{'T}H^{-T}CH^{-1}X^{'} = 0$. So, you can see here there is a quantity $H^{-T}CH^{-1}$. It's a matrix multiplication and its a composite. It will provide you another matrix and say let me write this matrix as $C^{'}$ now this is also a 3×3 matrix and you can find out that you have a 3×3 matrix even in the transform space. and which satisfies the equation: $X^{'T}CX^{'} = 0$. This is nothing, but again another conics representation.

(Refer Slide Time: 04:25)



So, we get this particular. So, that transform conics is given by this $H^{-T}CH^{-1}$. So, a conic remains a conic under homography that is the essence of this particular mathematical exercise. So, conics is an invariant after transformation. As the dual conics is nothing, but the inverse of the conics in its point representation, So, if I perform the inverse operation. I will get its corresponding dual conics and that is consistent (Refer Time: 05:08) with respect to this transformations.

(Refer Slide Time: 05:11)



There is another interesting property of this transformation particularly in the conics. There is an interesting property of conics. So, there are two points which are known as circular points, We have already mentioned while discussing about the invariances of similarity transformation. You have seen that these points are invariant under similarity transformations. So, the definition of the circular points are given as

"The circular points I, J are fixed points under the projective transformation H iff H is a similarioty. They are also on I_α "

I mentioned that you need to consider a two dimensional projective space with the complex representation in this representation, where one of the axis is imaginary axis other axis is real axis and there is a scale axis.

So, these are the two points

$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

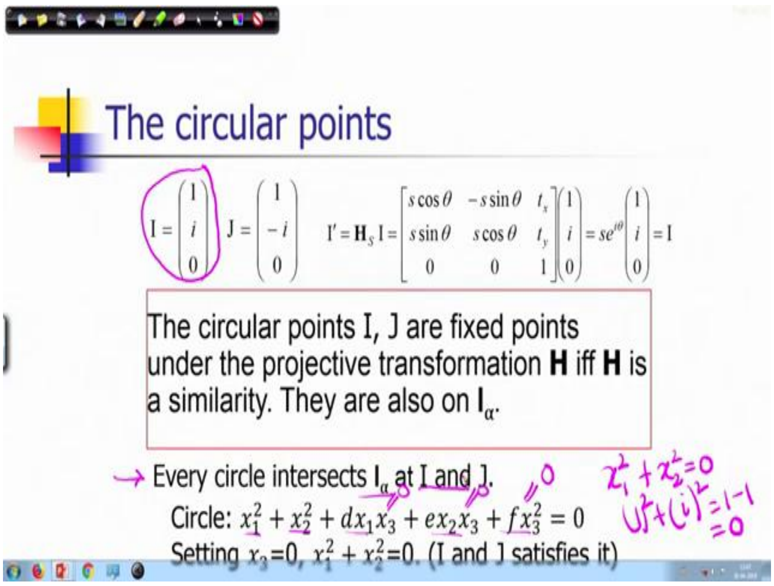
and if I apply a transformation once again you get a representation in the complex space itself. We have already discussed that under similarity transformation these points still remain the same under conics. So, they are the fixed points of similarity transformation. And this is the summary of this particular property that the circular points are fixed points under the projective transformation H, if and only if H is a similarity.

They are also on line at infinity because you note that their scale i,e, this third dimension is scale value which is 0. If I apply the point containment relationship with the line at infinity, it will satisfy that relationship which means the dot product of the column vector represented by circular points and line at infinities they should equals to 0. Just to elaborate this part. So, if I perform the following operation

$$\begin{bmatrix} 1 & i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

This is the point containment relationship.

(Refer Slide Time: 07:37)



So, every circle intersects line at infinity at I and J which is another property of every circle. So, let us elaborate this part, say you considered a representation of a circle in the homogeneous coordinate system.

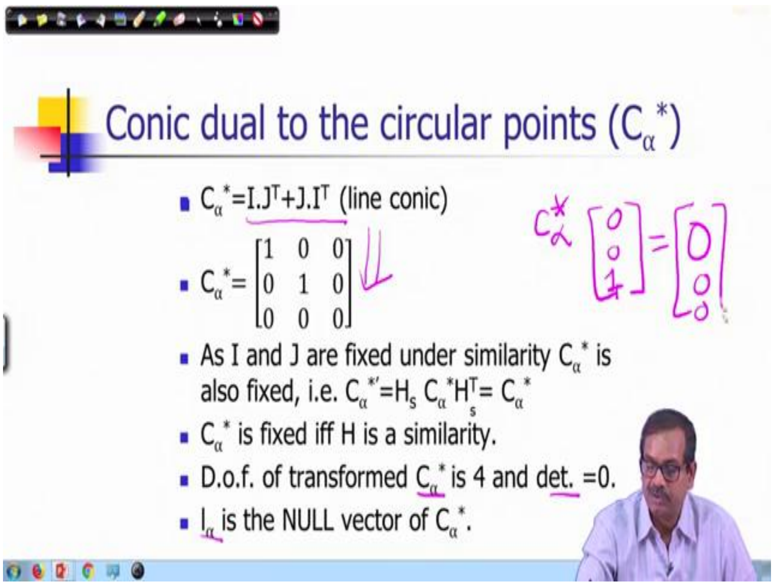
$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

So, the above equation is the representation of a circle. you can see that x_1^2, x_2^2 where the coefficients of x_1^2 x_2^2 remain, the same. In this case, we have normalized it with respect to those coefficients.

We require only three parameters d, e, f and that is the representation of the circle. And here if I set $x_3 = 0$ which means that you are computing the line at intersection of line at infinity I_α because any point whose third coordinate is 0 that is lying on the line at infinity. So, if I set $x_3 = 0$. the equation would be $x_1^2 + x_2^2 = 0$

And you can note that these two points I and J they satisfies this equation because if I take for I; that means, $1^2 + i^2 = 1 - 1 = 0$. i.e, the circular point I is lying on the circle and also it is lying on the line at infinity I_α. So, that is why the every circle intersects line at infinity I_α at I and J. So, this is also another interesting property of circular points.

(Refer Slide Time: 09:51)



So, since we have this two circular points, we can define also conic duals . Previously, we have discussed about this dual representation of conics by using two circular points.

Now, we can define a degenerate conic. So, we can define a degenerate conic in the following form,

$$C_{\alpha}^{*} = I.J^{T} + J.I^{T}$$

you can see that this is 3×3 form and which will be eventually in a simplified representation which will look like this.

So, you have

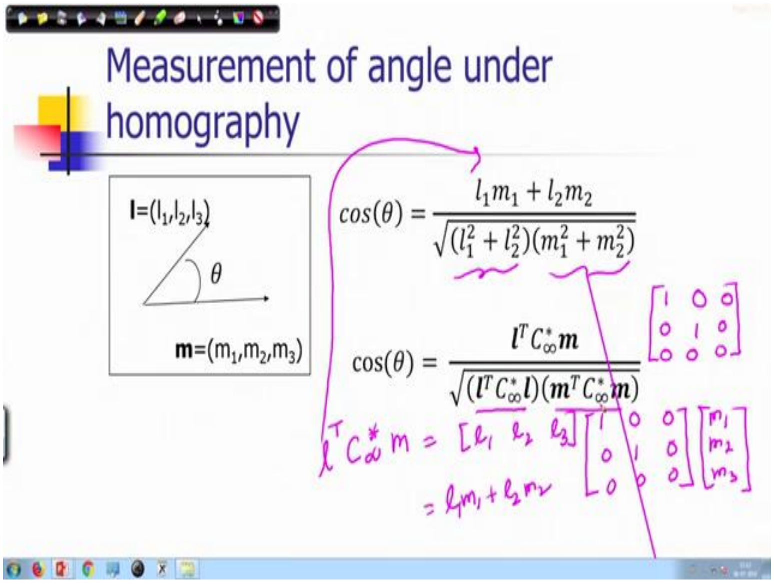
$$C_{\alpha}^{*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is a typical dual conic formed by the two circular points and this dual conic has a very interesting property since the circular points are fixed under similarity, this dual conics also remains fixed under similarity. So, this is another interesting feature. Earlier we mentioned that circular points are fixed under similarity, so, this dual conic representations using circular points is also fixed under similarity.

And degree of freedom of this dual conic at infinity is 4 and its determinant is equal to 0 . This is another property of this dual conics and other thing is that the line at infinity is null vector of this dual conic at infinity. Because if I multiply the dual conic at infinity that is the way we are you can call it if I multiply with the line at line at infinity you will get a 0 here; that means, $C_{\alpha}^{*}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

let us discuss how an angle could be measured under homography.

(Refer Slide Time: 12:09)



Suppose we have two straight lines l and m as shown in the figure which makes an angle θ and their directions of the straight line are given by l_1, l_2 and m_1, m_2 in the projective space we have this 3 vector representation, but we know that first two components of that 3 vector ,,they provide you the corresponding direction ratios. So, they could be used for measuring the angle between these two straight lines as shown in this expression that if you take the dot product of this vectors l_1, l_2 and m_1, m_2 and then if you normalize them basically dot product of unit vectors representing those two directions then you will get cosine of the angle that is $\cos(\theta)$.

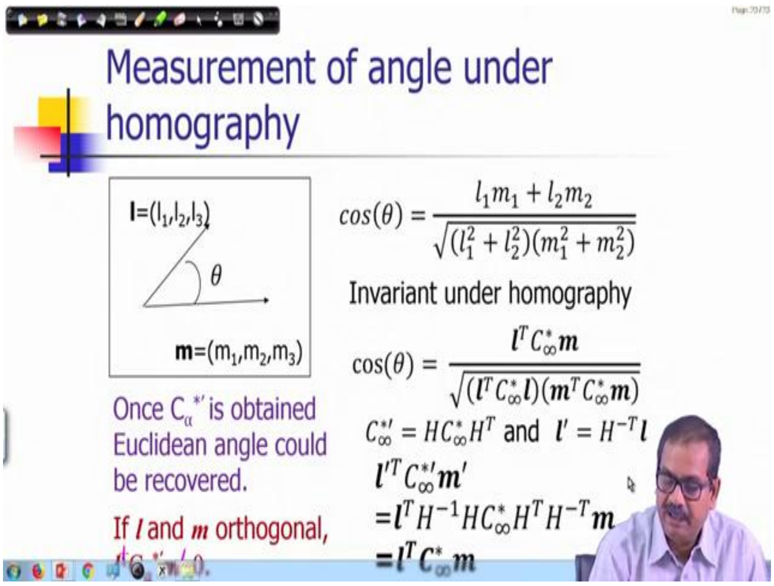
$$\cos(\theta) = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Interestingly, we can see that, this measurement can be also done even under homography. There is an in-variance which gives you this flexibility. So, let me discuss this property here you can see that even under homography. $\cos(\theta)$ can be expressed using the dual conic at infinity in the following form; $\cos(\theta) = \frac{l^T C^*_\alpha m}{\sqrt{(l^T C^*_\alpha l)(m^T C^*_\alpha m)}}$

that means, if I take $l^T C_\alpha^* m$ in the numerator, that expresses $l_1 m_1 + l_2 m_2$. Note here that C_α^* is that dual conic at infinity and it is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$l^T C_\alpha^* m = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = l_1 m_1 + l_2 m_2$ =. This computes the neumerator. Similarly, $(l_1^2 + l_2^2)$ and $(m_1^2 + m_2^2)$ can be computed as $(l^T C_\alpha^* l)$ and $(m^T C_\alpha^* m)$ respectively. Hence , $\cos(\theta) = \dfrac{l^T C_\alpha^* m}{\sqrt{(l^T C_\alpha^* l)(m^T C_\alpha^* m)}}$ it is computing the same quantity . The interesting part is that, there is an invariance associated with this measure; particularly the product $(l^T C_\alpha^* m)$. If I perform homography corresponding lines and corresponding dual conic transform dual conic at infinity under homography. They also preserve the same quantity. So, let me explain that.

(Refer Slide Time: 15:47)



So, this is invariant under homography and we use this property where the relationship between dual conic at infinity under transformation of homography H and with the dual conic at infinity in the original space is given by $C_\alpha^{*'} = H C_\alpha^* H^T$. And as you know that the line after transformation also they are related transformed line and original line in the original space. They are related by $l' = H^{-T} l$.

So, if I replace this quantity. So, let me show you the derivation. So, in this derivation we can see that we are. So, this remains the invariant. So, l' which is the transformed line under homography, it becomes now $l^T H^{-1}$ and then $C_\alpha^{*'}$ or dual conic under transformation becomes $HC_\alpha^* H^T$ and then this is the transformed line, this part is transformed line and as you know that $H^{-1}H$ would be identity matrix.

$$
\begin{aligned}
& l'^T C_\alpha^{*'} m' \\
&= l^T H^{-1} H C_\alpha^* H^T H^{-T} m \\
&= l^T C_\alpha^* m
\end{aligned}
$$

Similarly, $H^T H^{-T}$ also form identity matrix. So, what you essentially gets is equal to $l^T C_\alpha^* m$. So, this is how it becomes an invariant. So, the measurement of $\cos(\theta)$ is the same expressions even if I use this in a transformed space; that means, if I use all the corresponding transformed lines transformation of dual conics transformation of corresponding lines, then also we will get the same measure $\cos(\theta)$ that is how it becomes invariant.

So, once this is obtained. So, what you required to do that given a image under homography, you would like to compute the transformed dual conic at infinity dual line conic at infinity that C_α^*. And if you can obtain that then you can use this relationship to obtain the original angle formed by those two lines in the original space.

Which means suppose there are two perpendicular lines in the original space and even after transformation under homography, they will not remain perpendicular, but this relationship will still remain and we as you know the cosine of the angle between this perpendicular lines that is equal to 0. So, we can always say this orthogonality property of two lines, they are preserved using this dual line conic at infinity in the transformed space.

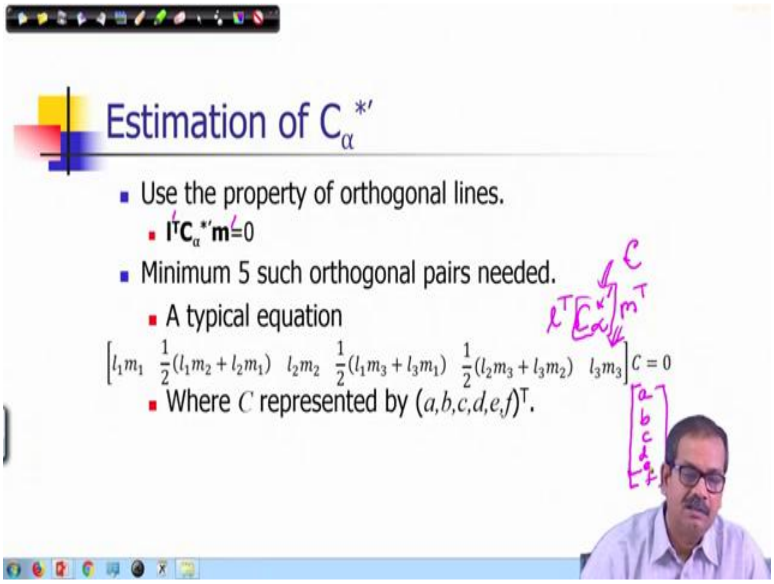
So, let us considered the following:

If l and m are orthogonal then

$l^{'}C_{\alpha}^{*'}m^{'}=0$

Note that $l^{'}$, $m^{'}$ and $C_{\alpha}^{*'}$, all the quantities are in the transformed space. Then actually $l^{'}C_{\alpha}^{*}m^{'}$ should be equal to 0. So, this could be used to for measuring angle and in fact, for recovering matrix properties as we can explain in the next slide.

(Refer Slide Time: 19:41)



So, first we need to estimate $C_{\alpha}^{*'}$; that means, the dual line conic under transformation at infinity dual line conic at infinity under transformation and we will be using this property of orthogonal lines which means I would consider that those lines are transformed lines.

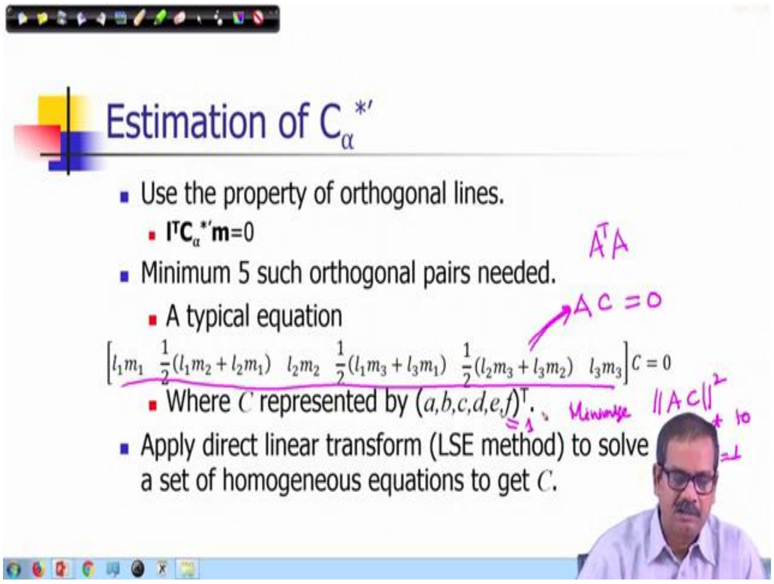
So, $l^{'}C_{\alpha}^{*'}m^{'}=0$. In this way that would give me an equation. Since there are there are five independent parameters in the conics. So, minimum 5 such orthogonal pairs are needed thatwould give five different equations to solve it. So, a typical equation will look like this. So, you consider a particular line say l and m we can consider l and m itself in that space. So, there is no problem in notation its a notational description only.

So, you will be using this equation. So, l is represented as $[l_1 \quad l_2 \quad l_3]$, m is represented as $[m_1 \quad m_2 \quad m_3]$ and you can observe that an equation $lC_{\alpha}^{*}m=0$ where C is a general conic. I will be estimating this and I am replacing by them corresponding conic variable

C and which is represented by the parameters $(a \quad b \quad c \quad d \quad e \quad f)^T$ which means I can write C as a column vector $\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$; you know these are the parameters and C can be represented as a symmetric matrix using those parameters we have discussed earlier how C can be represented using this parameters.

So, you have to obtain this parameters using this equation. As you can see that there are six parameters and there are five equations and in fact, its a set of homogeneous equation.

(Refer Slide Time: 22:25)



So, you need to use the method what we have used earlier for solving set of homogeneous equations, we can apply the direct linear transform method to solve a set of homogeneous equations to get C and which means that if I express it in terms of matrix say, set of five equations. So, in the following way,

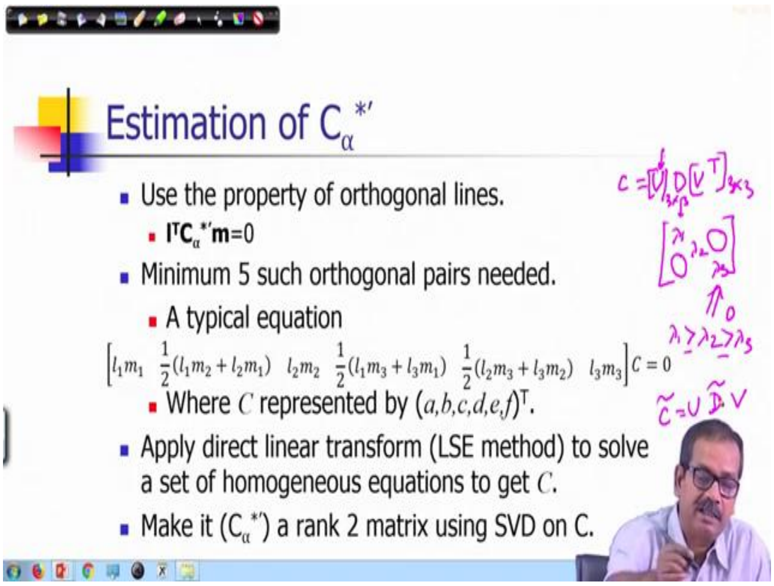
$$\begin{bmatrix} l_1m_1 & \frac{1}{2}(l_1m_2 + l_2m_1) & l_2m_2 & \frac{1}{2}(l_1m_3 + l_3m_1) & \frac{1}{2}(l_2m_3 + l_3m_2) & l_3m_3 \end{bmatrix} C = 0$$

I will I can represent the set of equations as a matrix multiplication where each row of A is formed by this particular row for every pair of orthogonal lines and if there are five there will be five such rows if there are more there would be more rows and you have to use an least square method.

So, you have to minimize this norms $\|AC^2\|$ subject to $\|C\|^2 = 1$ So, that is a problem statement in this context. We know that solution of this particular set of equations would be that you have to take the corresponding eigenvector of $A^T A$. So, the eigenvector of $A^T A$ which has minimum eigenvalue. So, that would be a solution of this matrix C.

Otherwise you can also convert it into a non set of non homogeneous equations when you are working with exactly five equations. So, set $f = 1$ once again the still that limitation is there suppose $f = 0$ in the representation of conic then this will not work, but let us assume that we can work with some non zero f in your particular given problem. Then you can use non homogeneous equations and you can use inversion by putting the values of f at the right hand side of the equations, we have discussed all these methods. So, with using this kind of technique you can estimate the conics and also estimates the dual conic at infinity dual line conic at infinity.

(Refer Slide Time: 24:57)



And then we need to do this operation because as we know that dual line conic at infinity is a rank deficient matrix. So, it should be rank 2 matrix. So, what we do the trick what we can you know perform here that we can do the singular value decomposition on C and set the minimum singular value as 0. So, let me explain this also suppose you get a C and you perform the singular value decomposition we know singular value decomposition sub C can be represented in this way.

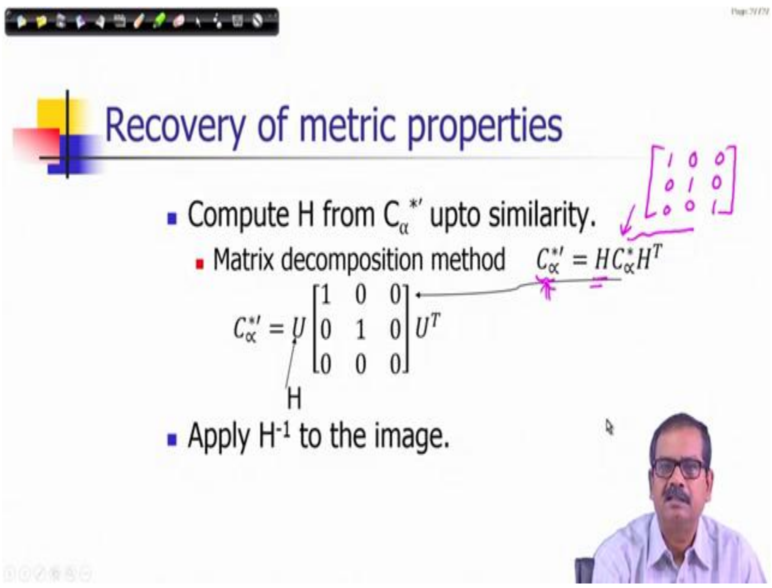
$C = UDV^T$

Where D is a diagonal matrix and its singular values are given as $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

and U is a 3×3 matrix and also V^T is also 3×3 matrix. So, what we can do? We can set $\lambda_3 = 0$

And suppose λ_1 has been rearranged in a way such that λ_1 is the maximum then λ_2 and then λ_3. So, set $\lambda_3 = 0$. then use that modified one as your estimate. So, use $U\tilde{D}V$ where $\tilde{D}$ is a diagonal matrix were λ_3 set to 0. So, that is, how we can make it a rank 2 matrix. So, in this way you can estimate the dual conic at infinity or dual line conic at infinity in the transformed space. Now you can perform a metric rectification using this. that I was mentioning. So, let us discuss that part also. So, how we can recover the metric properties?

(Refer Slide Time: 26:53)



So, what we will equate to do now we need to estimate the homography from C_{α}^{*} and we have seen that under the similarity transform this dual line conic remains preserved. So, what were we can do? Homography can be multiplied by any similarity homography, it will still remain the same and the properties will also remain the same. So, let us consider up to similarity will be computing this matrix, which means the ratios of the distances between egdes those will be preserve not the exact distance that you can estimate in this way.

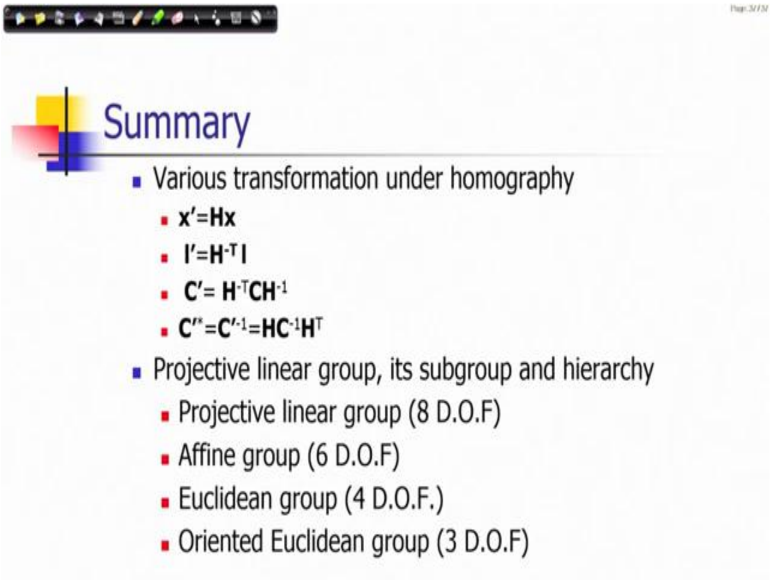
So, we can use matrix decomposition method to compute the homography, which means I need the homography matrix in such a way that I should get the transformed dual line conic what we have estimated dual line conic at infinity and note here this C_{α}^{*} is the dual line conic at infinity and which is given by this particular matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So, if I can decompose this matrix in such a way that we have this in this form and at the middle of this 3matrices. you have a matrix like this, then this could be an homography.

So, one such example one such method could be like this; $C_{\alpha}^{*'} = U\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}U^{T}$ that means, since this is a symmetric matrix because conic is a symmetric matrix we can decompose in this way, we can adjust the singular values in such a way that two quantities would be 1 1 and accordingly the columns of U and U^{T} would be adjusted.

So, now you see that if you use this kind of decomposition its a singular value decomposition exactly what we did but the diagonal matrix as to be adjusted. In such a way that both the singular values becomes one accordingly columns of us should be adjusted. So, as you see the middle 1 satisfies the structure of C_{α}^{*} or dual line conic at infinity and then we can take U as the homography and we can apply the inverse of this homography to the image to get the rectified image or to recover the metric properties as I mentioned you can get the ratios of the distances.

(Refer Slide Time: 29:39)



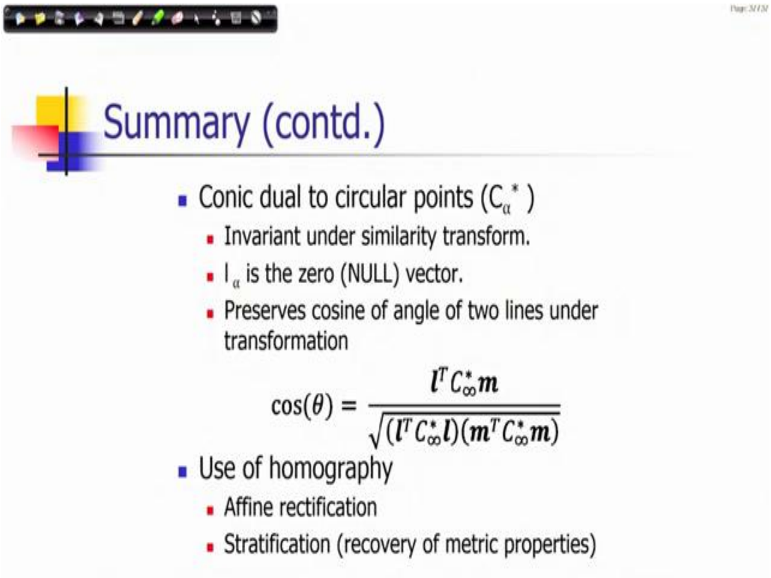
So, with this let me conclude this lecture and also this topic particularly with this summary that, we have studied various transformation under homography like we studied how a point could be transformed into another point with the transformation space in the transformed space how they related.

They are related by a nonsingular 3×3 matrix and which is called the projective transformation it preserves the co linearity of the points in the transformed space, it is invertible and correspondingly a line is also transformed into another line in the transformed space and the relationship between the transformed line and the original line is given by this $l' = H^{-T}l$ which is transpose of inverse of the transformation matrix.

Then, the conics they remain conic after transformation and their relationship have been summarized here. Similarly the dual conics representation also you know is valid in the transformation space and projective transformation they form various groups they form a group so, that is a projective linear group and there are different sub groups and their hierarchy that we have discussed like we have discussed about projective linear group having 8 degrees of freedom, then affine group having 6 degrees of freedom and in Euclidean group having 4 degrees of freedom and oriented Euclidean group having 3 degrees of freedom.

Incidentally Euclidean group is a similarity transformation what we discussed and oriented Euclidean group one of them is a isometric transformation as we discussed

(Refer Slide Time: 31:27)



Then we discussed about conic dual to the circular points and you have seen its various interesting properties like first thing it is invariant under similarity transformation, then line at infinity is a 0 vector of this particular conic dual, it preserves cosine of angle of two lines under transformation that is interesting. So, I should say it using the transformed dual conic, you can compute the cosine of this angles it implies that.

So, this is the particular property how cosine of angle could be computed using the transformed dual conic and homography is used in various tasks some examples are given here like you can perform you can use homography for rectifying images there are two types of rectifications we discussed, one is affine rectification the other one is stratification which is which helps you recovering the metric properties particularly proportionate distances that you can compute that you can recover underperforming that corrections. So, with this let me stop my lecture here and thank you for listening to my lecture we will continue our discussions on other topics from next lecture.

Thank you very much.