Introduction to Automata, Languages and Computation
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 30
Arden's Theorem

So, we are talking about regular expression. So, today we will discuss this theorem, Arden Theorem, which is which will give us a solution for equation in regular expression ok. So, before that let us just recap the properties of the regular expression or we say this, these are the identities which you have already discussed.

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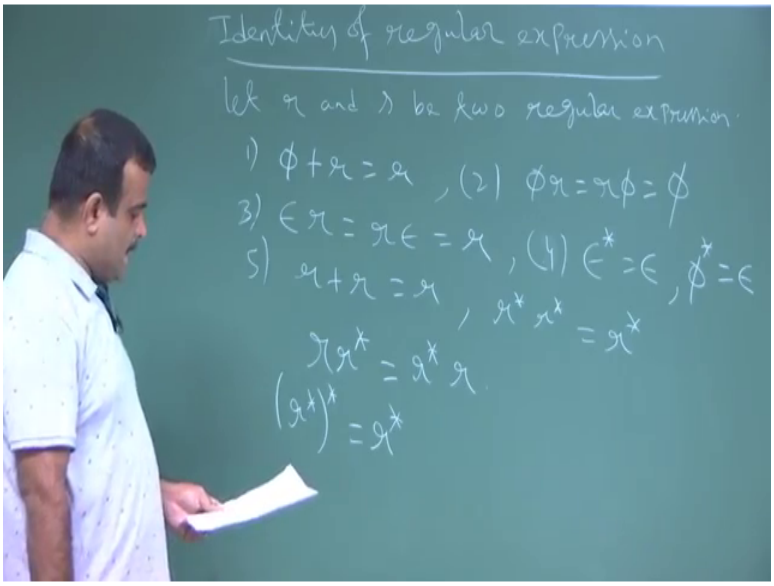

Identities of regular expression, this were going to use to prove the Arden theorem. So, let r and s be 2, a regular expressions which is coming from some language or r and s corresponding to some language.

Then we know this properties like phi plus r is r phi r is equal to r phi is equal to phi, epsilon r is equal to r epsilon is equal to r, they are equal in the sense that they are corresponding to the same language. Then we have epsilon star is equal to epsilon, but phi star is equal to epsilon and 4 1 2 3 4 and 5 is r plus r is r then r star. These are the identity, we are going to use for this to prove the theorem which we are going to discuss.

Now, next property is r r, they are commutative r r star equal to r star r. Now, r star r star is equal to r star. So, r star r star is equal to r star. Next one is say epsilon plus r r star is equal to r star which is same as epsilon plus r star r. This we will use directly.

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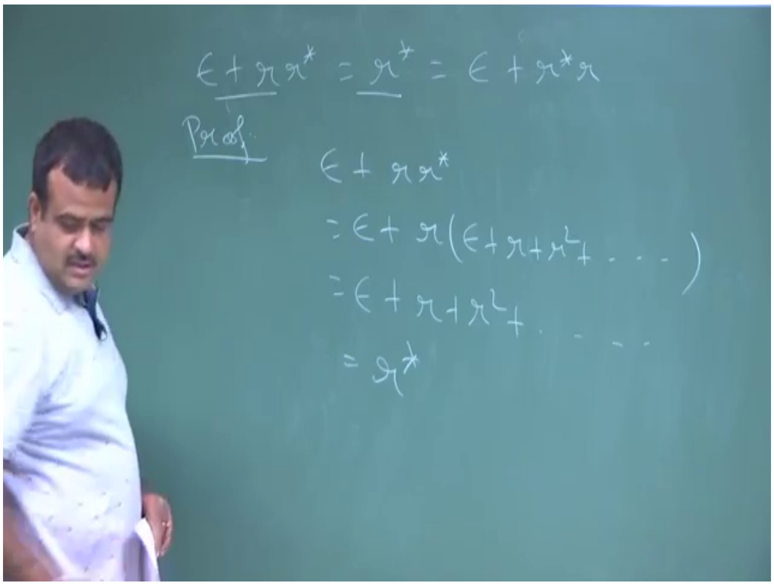

Now, how to prove this? To prove this we can take this, this is informally r plus r r star. So, this is nothing, but r star is meaning is your r star means; r then epsilon plus r plus r square dot dot dot and so on.

So, this is nothing, but epsilon plus r plus r square plus r cube and so on. So, this is nothing, but r star. This is informally, because this is a infinite series. So, unless the convergence we show, we assume this, we cannot write like this, but we can prove it more formally like this say, we want to prove this equal to this; that means, this is in language two states are same. So, this will say this is say r 1 and this is r 2, we have to show r 1 equal to r 2.

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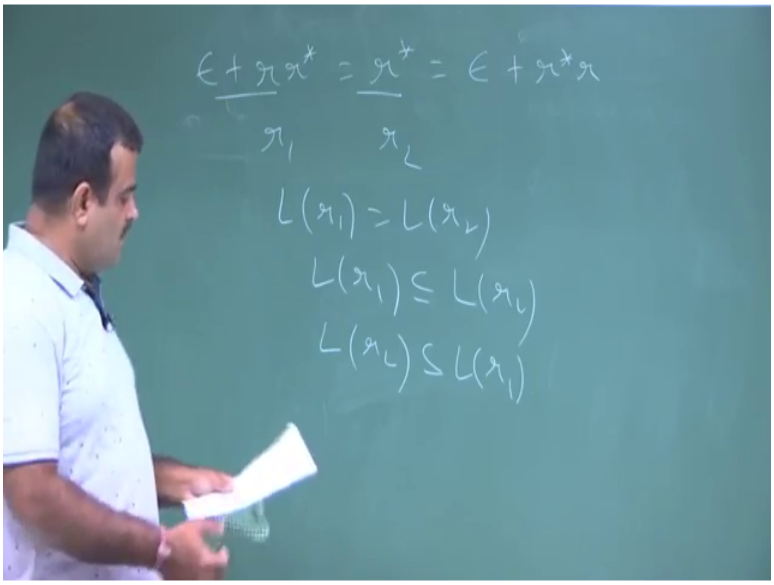

So, for that what we need to show, we need to show the language of r 1 equal to language of r 2. So, how to show this? We have to show this is subset of this and this is subset of this ok, then we can say that these two sets are same anyway. So, this result we are going to will be using in the, in this lecture. So, another property is, r s star r, which is same as r s r star. This also we can prove.

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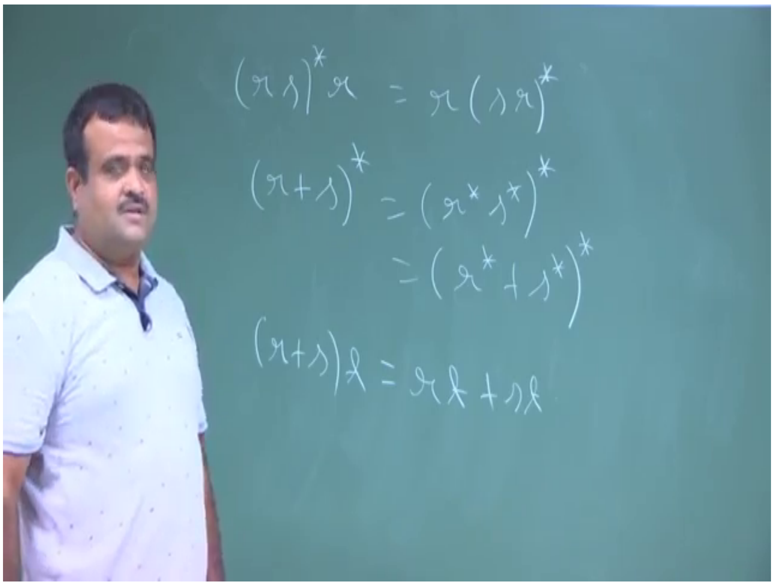

Another property is r plus s star, this is same as r star s star star, which is same as r star plus s star star and the next property is the associativity sorry, distributed law r plus s into t concatenate with t is nothing, but r t plus s t and this is right distributed, we can have lib distributed property.

So, these are the properties, we have few of them, we have already seen. Now, we going to use this property to prove the Arden theorem. So, what is Arden theorem?

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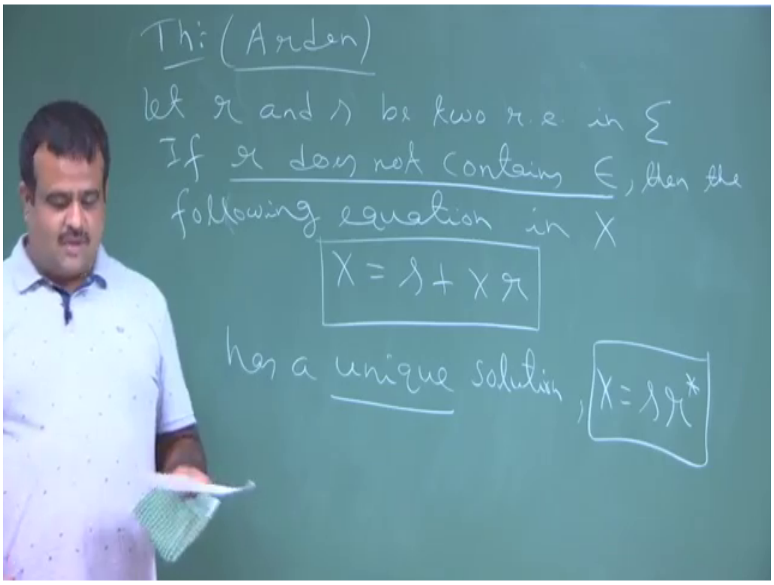

So, it is related to a equation in the regular expression. So, suppose let r and s is be 2 regular expressions in sigma. Suppose, sigma is the input alphabet about sigma and if r does not contain epsilon, does not contain epsilon then only we can apply this, then the following equation then the following equation in X, X equal to s plus X r. This is the equation, where X is a regular expression, r and s are the regular expression and r does not contain epsilon.

So, this equation has a solution, unique solution. So, that is the theorem, has a unique solution and that solution is given by X equal to s r star X equal to s r star. So, this is the equation in terms of X, where X is a variable which is a regular expression, X equal to s plus X r and if r does not contained any epsilon then this equation has a unique solution.

So, we up to this theorem is called Arden theorem. So, this you have to prove it and then after proving this we will have some application on this to a given, a DFA. How we can get a regular expression ok. So, let us try to prove it. So, first of all we need to show the existence of the solutions, then you can show the uniqueness. First, will show the solution exist.

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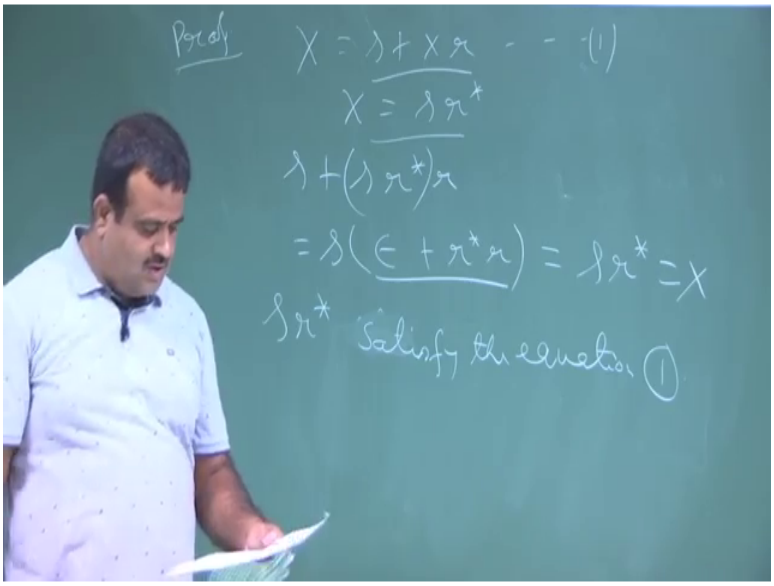

So, this is the rough idea of the proof ok. So, what is the expression? The equation is X equal to s plus X r. Now, we are claiming that this is the solution X equal to r star. So, we will put this in here, right hand side of 1.

So, X plus s r star r. So, this will take it in bracket. So, now, we can take s common, this will be epsilon plus r star r. Now, just now, we have seen the identity, this is nothing, but r star. So, this is s r star which is nothing, but X; that means, this is a solution, this satisfy the equation, satisfy the equation 1; that means, this is a solution. Now, we need to show the uniqueness. This is only the solutions ok. So, to show these, we need to so, let us write the equation again, so X equal to s plus X r.

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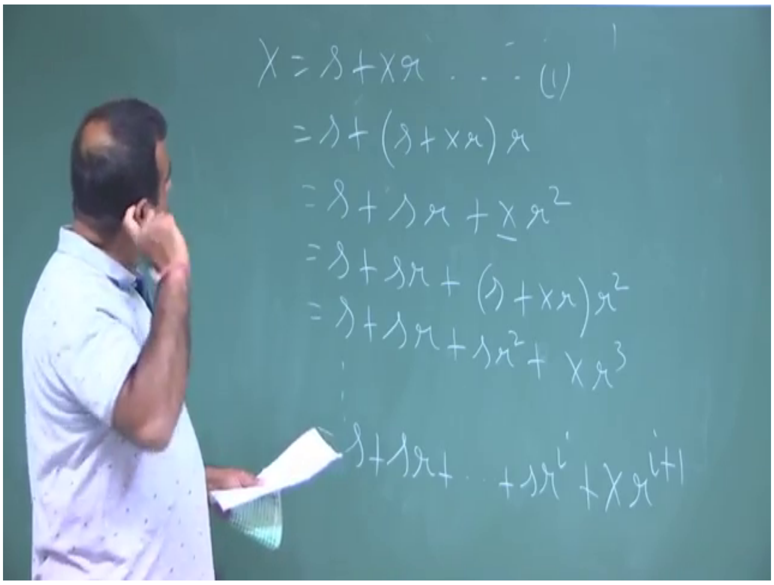

So, now this X in the right hand side, we again replace by s plus X r. So, this will be s plus X r into r. So, this is nothing, but s plus. This is the distributed law plus X r square. Now, again this X will be replace by s plus X r. So, s plus X r plus s plus X r then r square, this again we can write s plus X r plus s r square plus X r cube.

So, if you continue like this, we will be getting this in the form of s plus X r plus dot dot dot plus s r to the power i plus X r to the power i plus 1 and this i can be any value greater than 1 ok. So, if i can be any value ok. Nowso, now, what we have?

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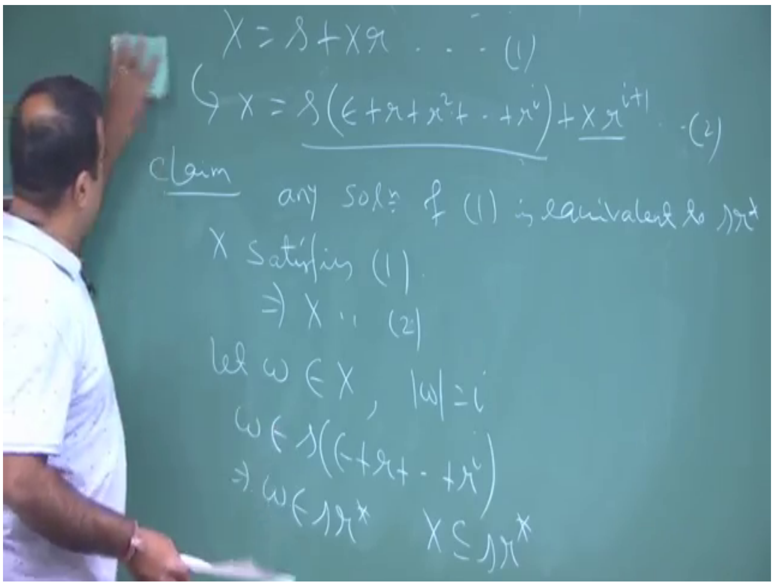

So, you have two equation; this is one and another one is this one, from the given this we have this another equation X equal to this one s, we can take s common epsilon plus r plus r square r to the power i plus x r to the power i plus 1. This we name the equation number 2, this is coming from here.

Now, we want now, our claim is any; so, this is our claim. Claim is any solution of 1 is equivalent to s r star any solution 1 is equivalent to s r star. So, how to prove this? To prove this suppose X satisfies 1, this implies X satisfies 2. So, X satisfies 2; that means, we can we take a w from X with the length of w is i.

So, then X satisfy to mean the r does not contain the epsilon. So, w will not be here. So, w must come from here. So, w must belongs to the language of this regular expression epsilon plus r plus r to the power i ok. So; that means, w is, w belongs to s r star so; that means, X is a subset of s r star. Now, the other way round, we have to prove X r star is a subset of, X r star is a subset of X that also you can prove similar way. This is our equation number 2.

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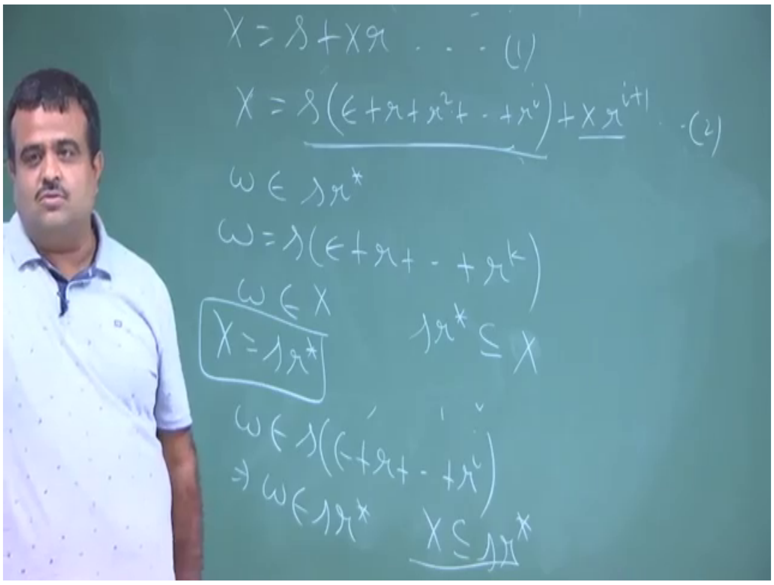

Conversely suppose w belongs to s r star; that means, w will of the form like this epsilon plus r r to the power k for some k; that means, it is coming from here. So, that this implies w belongs to x ok, which is the solution of the equation 1 so; that means, s r star is a subset of w X and we have seen s is subset; that means, these two are same X equal to s r star.

So, this is the uniqueness proof. This is the rough idea of the proof ok. So, this is called Arden theorem. So, now, using this we will, we have some applications to get the regular expression of the DFA. So, we will see that how we can get the regular expression of a DFA application of Arden's theorem ok.

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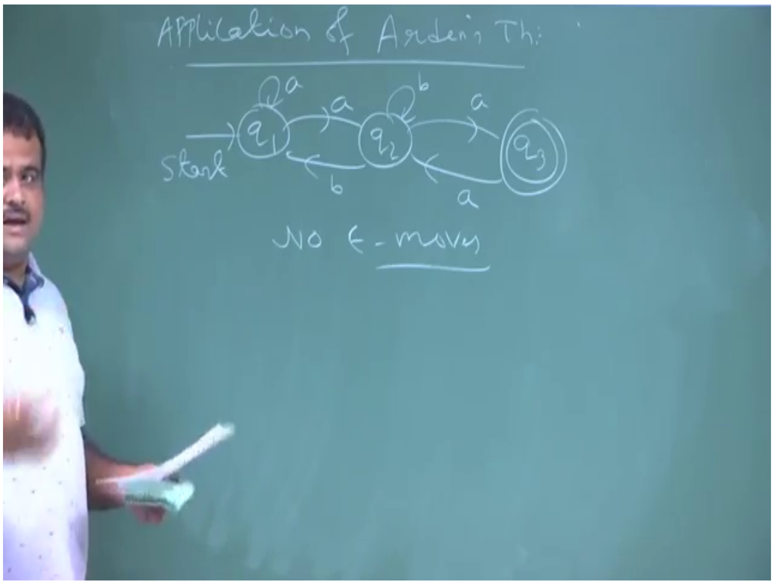

Now, suppose we have given a DFA for example, this one q 1 and there is a condition this DFA should not contain any epsilon move and there is only one starting state that is always therefore, also for our automata ok.

So, this is going to a, going to a, this is b, this is b, this is a, this is a. Suppose, this is a given DFA and it is not having any epsilon move. So, Arden's can be apply, if there is no epsilon move. We will write a general form, but let us explain this through an example ok. So, there is no epsilon move and also only one initial state, this is the starting state or initial state only one initial state.

So, we can apply Arden's. So, we can have equations in the regular expression. So, we can have the corresponding equations for the q vertices. First, let us take this vertices. So, suppose the variable will take q 1.

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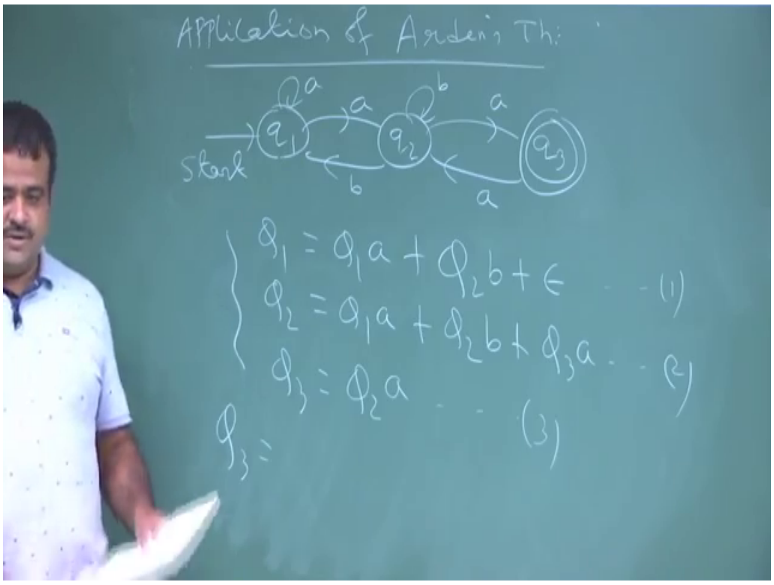

So, q 1 will be written as; so, q 1 is coming from q 1 by a plus there is a move from q 2 to q 2 to q 1. So, q 2 b plus epsilon, because q 1 is the starting state and with the epsilon we can be at q 1. So, this is equation 1 and q 2, this is you take this q 2 we are trying to get the regular expression for three vertices, three nodes or three states. So, for q 2; so, q 2 we have a move from q 1 to q 2. So, q 1 a and q 2 b plus q 3 a.

This is equation 2 and with q 3 we have q 3, we are going only from q 2. So, q 2 a this is equation 3. So, from these three equation, we have to solve. We will repeatedly apply the that theorem, Arden theorem and we will get the final regular expression for the final state that is only one final state q 3. So, if you get some regular expression for q 3 that is the answer, that is the regular expression corresponding to this language ok. So, how to solve this. So, this is the equations. So, we can just equation 1 and 3. So, you can just replace this here.

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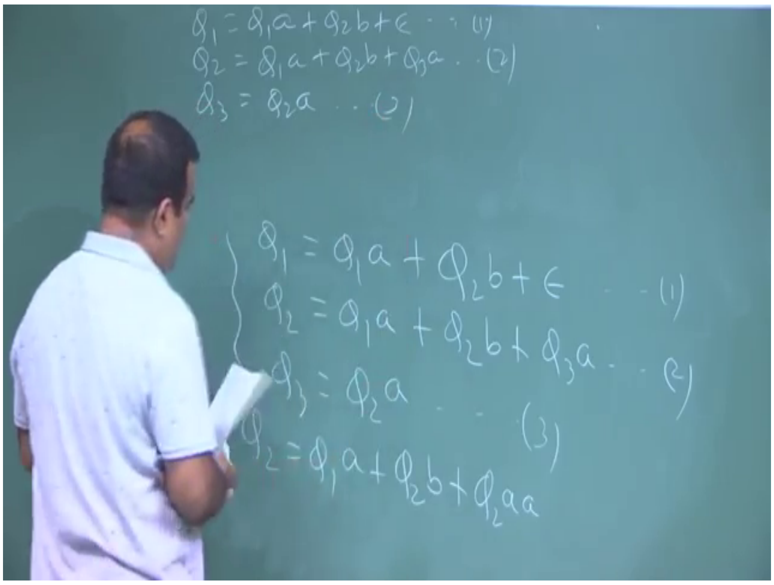

So, we got q 2 is equal to q 1 a plus q 2 b plus q 3 a q 3 is nothing, but q 2. So, q 2 a. This is my equation 1 and 2. So, we can write the equation here. So, we can rewrite the equation in a smaller form q 1 q 1 a plus q 2 b plus epsilon, this is 1 and q 2 is nothing, but q 1 a plus q 2 b plus q 3 a, this is 2 and q 3 is nothing, but these 3 a. Now, from 2 and 3 we get this. 2 and 3 what we got? We got q 2 is equal to q 1 a plus q 2 b plus q 2 a a ok.

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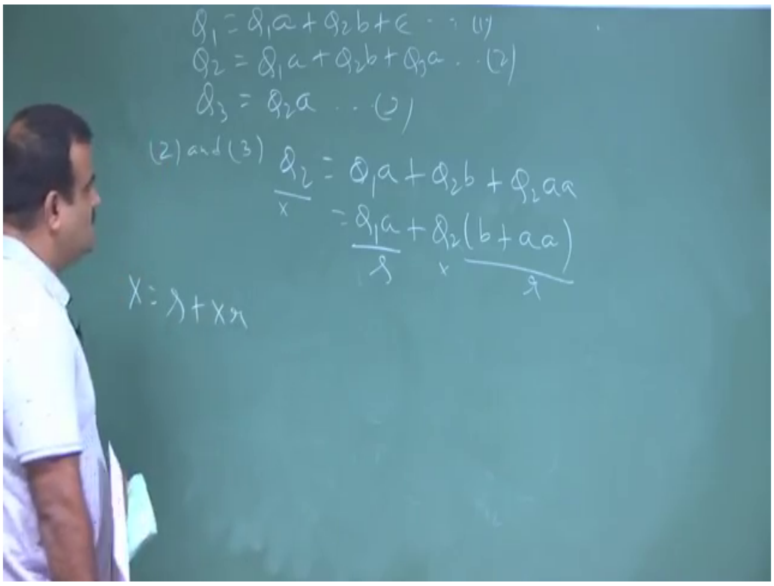

So, these are all regular expression. So, these we can write as q 1 a plus, we can take common q 2 b plus a a. Now, we want to apply the Arden's theorem. So, further we need to identify X equal to s plus X r.

So, this is our X q 2 is unknown. So, if you take this is our s and this is our r. So, this is nothing, but the. So, this is X this is X, X equal to X plus s plus Xr. So, if we apply Arden's then it is nothing, but X equal to s r star.

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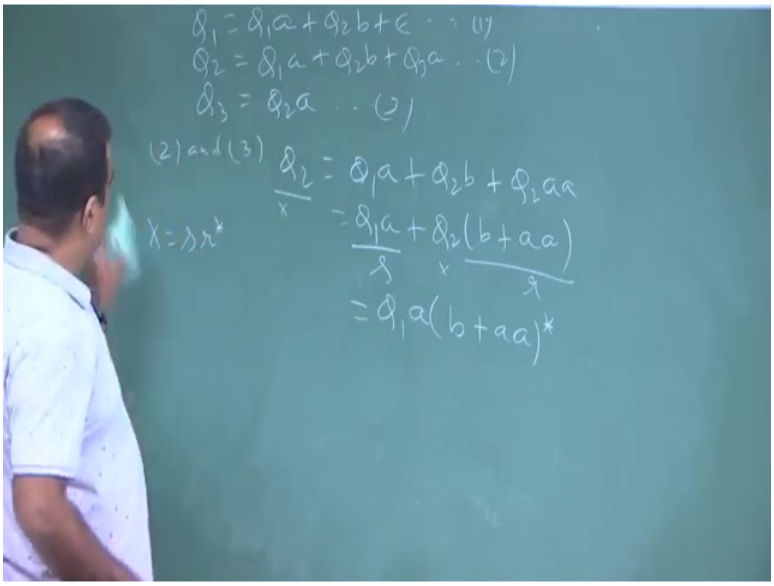

So, if we apply that it will be q 1 a which is s then r star r star is b plus a a star ok. So, this is by applying the Arden's theorem. So, you got another equation, which is basically q 2 is equal to q 1 a b plus a a star.

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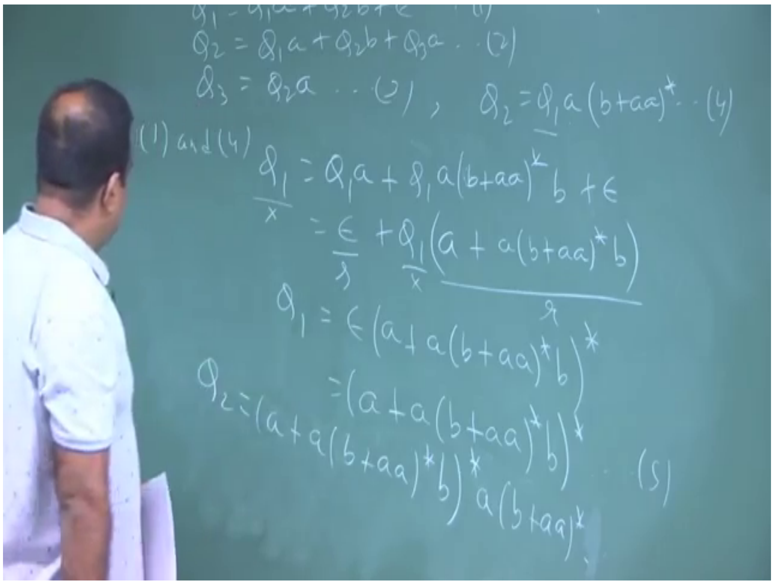

So, this we referred as equation number 4. This we refer as equation number 4. Now, we can block 1 and 4 yeah, we can just use from 1 and 4, what we have? So, q 1 is equal to q 1 a and this q 2 we can replace. How we can replace q 2; q 2 we can just replace by this q 1 a b plus a star and then we have a b and we have epsilon over here.

Now, this we can write as epsilon, we can bring here plus q 1 we can take common then a plus a into b plus a star into b. Now, this is also in the form of Arden's theorem. So, because this is extra X, this is our s, this is X, this whole thing is our r and r does not contain epsilon ok. So, that is one condition for the Arden's. So, if we apply this again ah; so, this is a. So, this will give us X is equal to s r star. So, q 1 will be s then r star r star means this star.

So, a plus a into b plus a a star b star. So, this is same as a plus a b plus a, a star b star. Now, if you simplify this will be getting this. So, this is equation 5, we can say you can simplify, but later on we can say, because what we are looking for?

We are looking for the regular expression corresponding to q 3 and even we can get the regular expression for all of this by, this is the regular expression for q 1 similarly once we get this. So, you already got the regular expression for q 2, because we have regular expression for q 1, we can replace that and we get the regular expression for q 2, but we need the regular expression for q 3.

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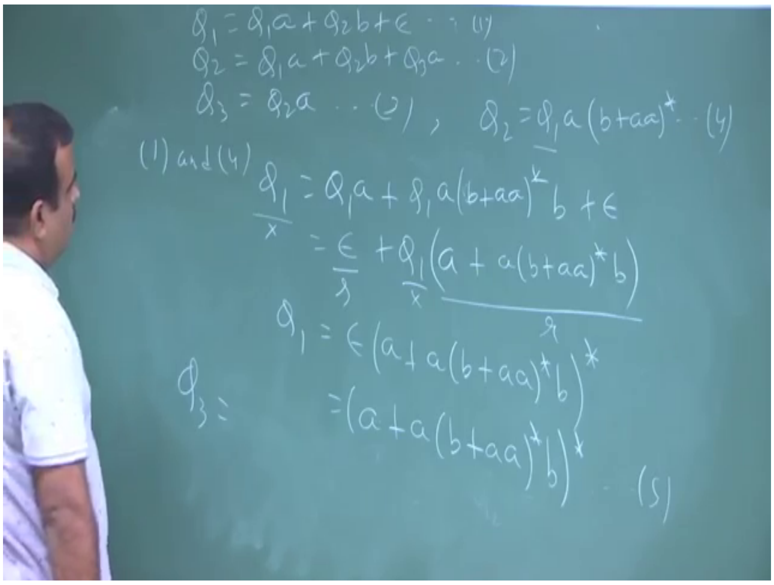

So, for that; so, what is, let me just write this regular expression for q 2 we just put this over there and we got a plus a b b plus a a star b star and then we have a b plus a a star b sorry, not b a b plus a star. So, this is for q 2. Now, once we got q 2 we can get, we can get q 3 from here.

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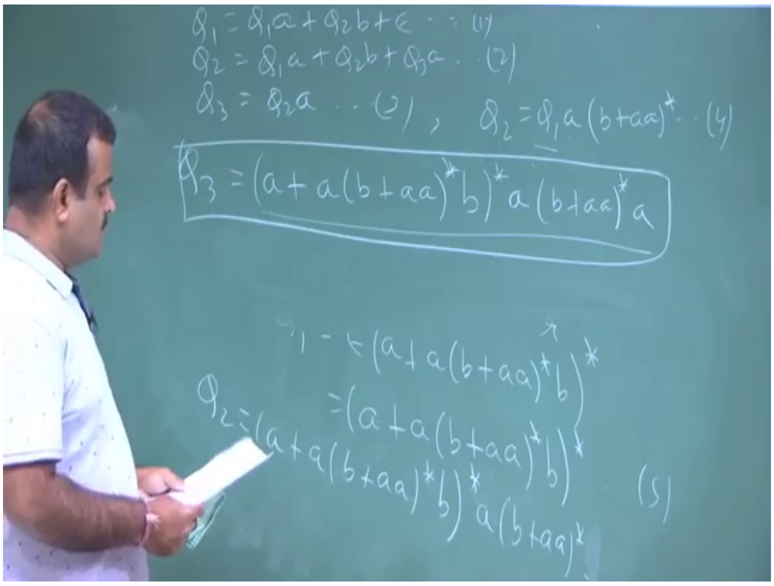

We are looking for q 3, regular expression for q 3, because q 3 is the corresponding to this final state. So, q 3 is nothing, but q 2 a. So, q 2 is this one. So, a plus a b plus a a star b star a b plus a star then a. So, this is the regular expression we are looking for, this one.

This is the regular expression corresponding to that D F A. So, this is the application of the Arden's theorem. So, we can generate this is through an example, we can generalize that ok. So, we need to have the following assumption.

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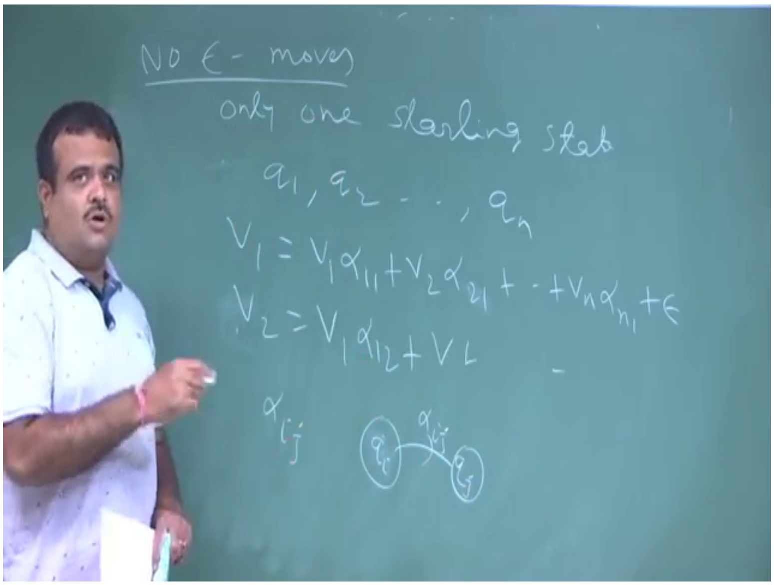

So, there should not be any epsilon move, no epsilon move and only one starting state starting state. Then what we can do? We can write the regular expression for each of the states like this say V 1. So, suppose you have a state q 1 q 2 q n. Suppose, there are n state then we can write this as the regular expression V 1 is corresponding to the regular expression q 1.

So, V 1 alpha 1 1 what is alpha 1 1 alpha 1 1 is the alpha i J I will write that. So, V 2 alpha 1 2 sorry, alpha 2 1 plus V n alpha n 1 plus epsilon then V 2 is equal to V 1 alpha 1 2 plus, if there is no move so, this is the alpha. This is the alpha i J alpha i J is the regular expression from V i to q i to q J. If there is a i mean alpha i J is the move like this, we will write and then by the Arden's we will try to solve this equations. There are n equations and we will keep on apply the Arden's theorem to solve this equations.

Thank you.