Discrete Structures
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Lecture - 46
Algebraic Structures

Today you will read the Algebraic Structures. Mainly in this study of algebraic structures we will read how we can define the algebraic systems, their properties and then, we try to identify that between two algebraic systems what are the properties shared and so that they are structurally they can be identical or what relations are there in respect to their algebraic structures we will read.

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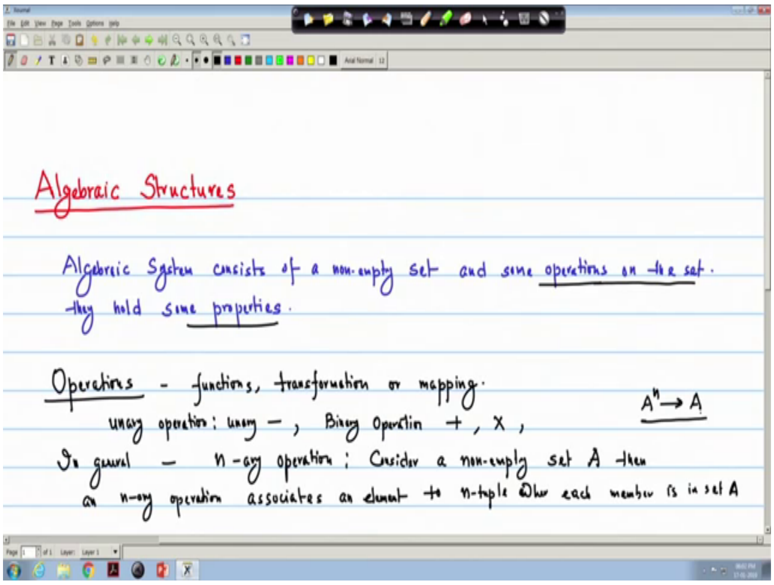

So, first we define the, what do you mean by Algebraic systems. Normally algebraic systems they are the examples are that we know that these are group ring field and the more simplest form semi-group monoid like that.

So, first we try to define that algebraic systems. So, it is consists of a set. So, algebraic system consists of a set or rather we call non-empty set and some operations on the set and they hold some they hold some properties and based on these properties, we will try to identify the different systems. So, mean systems and then, they are proper properties then we will try to identify the structures they hold.

So, here the main important thing is the operations that on the set that we define and the properties also. So, first we define that, what are these operations, ok. So, these operations normally this can be some functions some transformation or sometimes we called some function means sometimes we called it is a mapping. So, these operations that can be unary operation like unary operations like or subtraction unary minus we called the unary operation is unary minus, then our binary operations where the operation that it operates on the two operands like plus our simple multiplication division all these.

Now, in general I can tell in general that it is some n-ary operation n-ary operation. So, this in n-ary operation I can define that in general I can n-ary operation we can define that. If we consider a set on empty set A, then an n-ary operation associates an element of this set to an n-tuple where each member each member is off is in is in set A. So, normally we call a into a. So, this is in general the n-ary operation. Now, this can be a binary operation. This is a it a cross a 2 a and our addition is a binary operation like that.

So, now we can formally we can define our algebraic systems.

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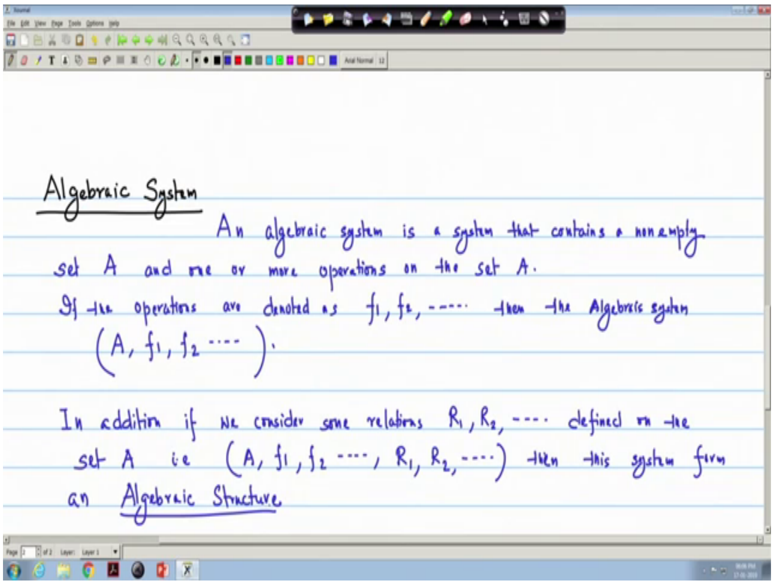

So, now if we define our algebraic systems since we know the set we know now the operations on the set s. So, we define algebraic system. So, an algebraic system or sometimes we called is simply an algebra is a system that contains a non-empty set is set A and one or more operations. Just now we have defined the operations on the set A.

Now, if the operations are denoted as f 1 f 2 same this, then the algebraic system is A we define as A f 1 f 2 in this way.

Now, I can give that otherwise it can be now in addition in addition if we add some relations with this algebraic system, so in addition if we consider some relations say R1 R2 defined on the set A that is I can denote A f 1 f 2 all these operations and R 1 R 2 with all these relations, then this form an Algebraic Structure, then this system form an Algebraic Structure.

So, in summary I can tell that algebraic system is a set, non-empty set with some operations on it and if now we define some relations on this set, then this set non-empty set A along with the functions and the relations we can tell that these are my Algebraic structure. Now, first we see some general properties. So, we read some general properties.

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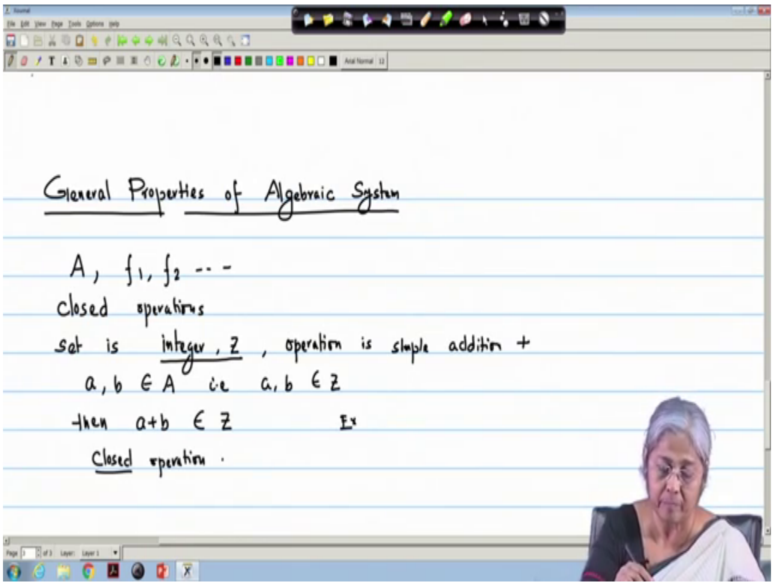

Because based on these properties first we will try to identify these properties and will be defining the different algebraic systems or algebra.

So, some general properties of now first we see that the operations that just now we have defined the algebraic systems, the set non-empty set A and the different functions or operations f 1 f 2 etcetera. Now, these operations that we call these are closed operations. That means, what do you mean by closed operations that see if I consider that set, my addition the set is the set is integers say integer normally we denote as z and the operation is simple addition operation is or simple addition.

Now, if I take element a and b belongs to a, then a plus b, this is now here it is z. So, that is here a b belongs to z integer, then a plus b also belongs to z because z is set of set of integer. So, on the operation under the operation plus that the element we get a plus b that also belongs to z. So, this is it calls the property of closure or these operation is closed. It is the closed operation. Now, we see another see example we say we take another example of set.

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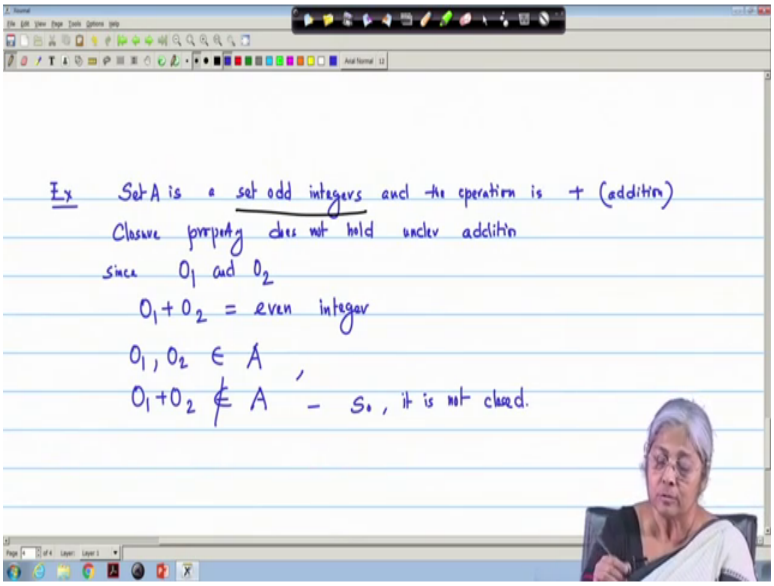

We take the set is A set of odd integers set odd integers and the operation is say addition.

Now, the now the addition or this it is not closed, it is not a closed operation or closure property does not hold. You can write the closure property does not hold under this addition for the set of odd integers since here if I consider that even, i toward integers like consider that O 1 and O 2 are two odd integers, then O 1 plus O 2 always this will be equal to the even. This is some even integer and hence I.

So, we have considered that O 1 O 2 belongs to a A set of odd integers a set of odd integers, then O 1 plus O 2 O 1 plus O 2 does not belongs to A since it is even. So, it is not closed. So, it is not closed. So, now we start that we write the all the general properties of Algebraic System. So, since now we know the closed operation or we can even write the first property is the closure property.

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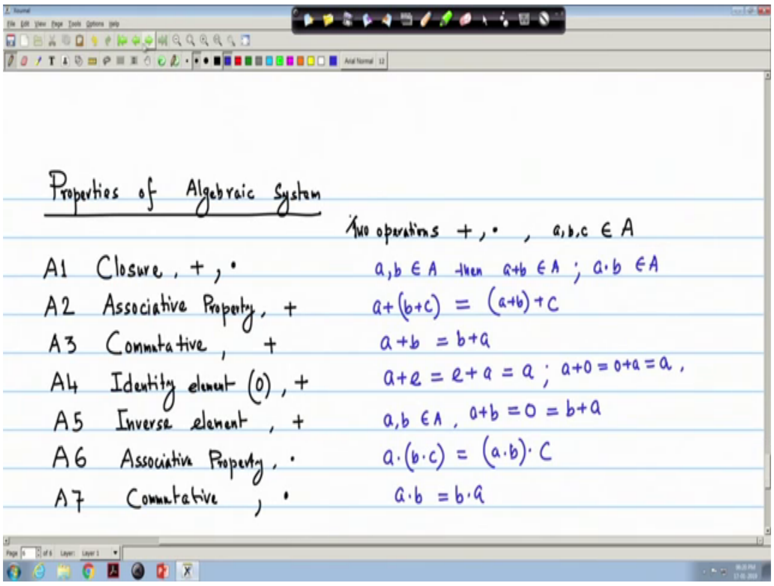

So, the properties, so we give that we give some name that this is A 1. This is a closure and closure under say addition. That means, we here we consider that two operations when this system we are considered we consider two operations plus and dot. With these two operations, we try to identify the properties.

So, I can write closure with plus and dot. Now, I give the property of A 1 A 2 I consider the property of associative property with respect to addition. Now, if I consider say element a b c, so we will consider set A and where a b c at the belongs to A. Now as just now we have defined the closure means that I can write that some a b belongs to a, then a plus b belongs to a, area a dot b belongs to A.

Now, associative property I can write that a plus b plus c equal to a plus b plus c. Now, I write property A 3 which is commutative with respect to addition. That means, a b belongs to A, then a plus b equal to b plus a. I give property A 4 is my identity exist some identity element say here 0 is the identity element with addiction. So, identity is that I get a plus 0 normally if e is the identity elements a plus e is e plus a equal to a only.

So, in respect to addition we know that a plus 0 is 0 plus a is a. So, 0 is the identity element. So, here e equal to 0 now I get some inverse element. So, if there exists an element in the set A such that if a b belongs to A, such that a plus b equal to the identity element. So, here it is 0 equal to b plus a. So, then b is the inverse of a. So, additive inverse of a. So, same a minus c is the additive inverse of a. So, additive inverse exist here.

Now, I get now if I consider the property or with respect to my multiplication say dot. So, again this I can consider the associative property associative property with respect to dot. So, I can write similarly that then multiplication that a dot b dot c is a dot b dot c, then A7 is the property is called the commutative with respect to multiplication and we can write a dot b equal to b dot a. Now, we write the property A 8.

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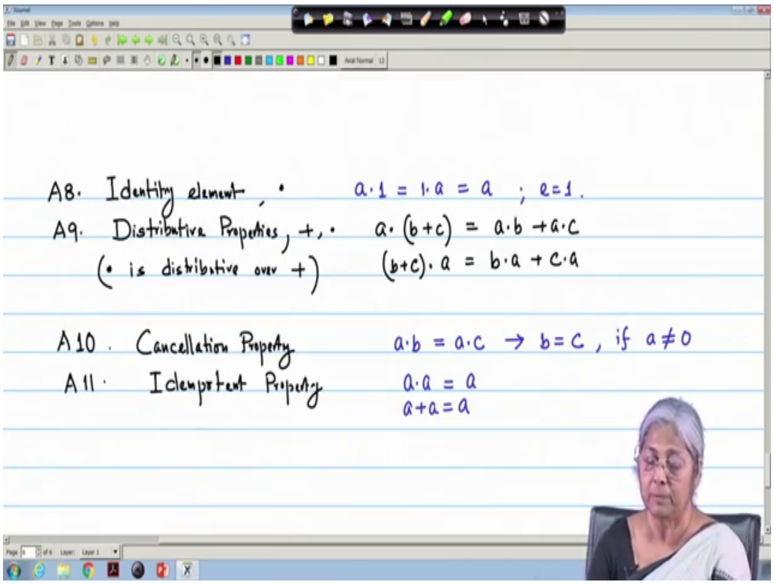

So, the identity element identity element with respect to dot and we know that one is the identity element since a dot 1 is 1 dot a equal to a. So, 1 is the identity element. Now, I can write 9 that is my distributive properties. Now, we have two operations. So, I can write two operations plus and dot. So, distributive properties with respect to addition and multiplication.

So, I can write a dot b plus c is a dot b plus a dot c or b plus c dot a, I can write b dot a plus c dot a. So, here I can write that multiplication dot is distributive over addition. Now, we can write two other properties that this is called the cancellation property that I can write that if a dot b equal to a dot c implies b equal to c if a not equal to 0 if a not equal to zero. Now one another is idempotent property that is a dot a equal to a and a plus a equal to a.

So, normally these are the properties and two operations a plus and dot we define, then these are the properties we will study on the set A. Now, we take some example that how actually there these properties are hold and identifying those properties we can define some algebraic systems.

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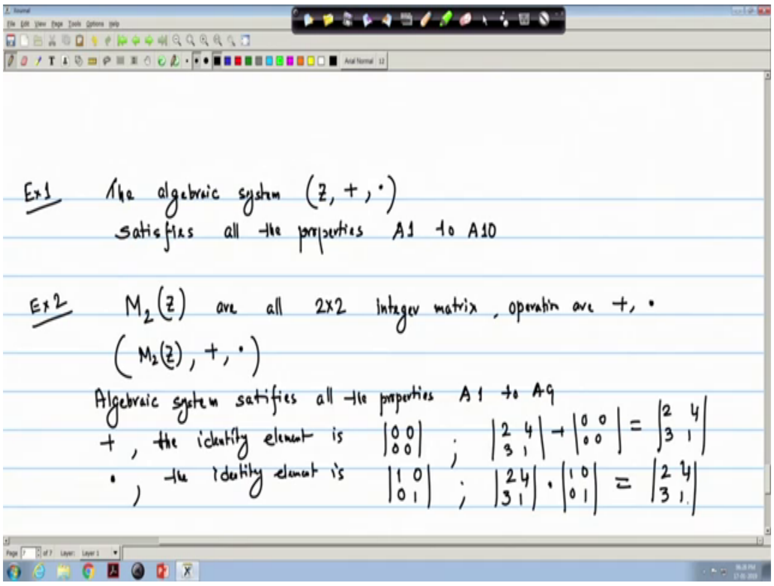

So, one simple example is say the algebraic system say set of integers and the operations are simple addition and multiplication,.

Now, we can see that this satisfies these algebraic system satisfies all the properties that a 1 to a 10 all the properties a 1 to a 10, but not the idempotent property. Now, we consider another one that another example we consider which it is the, we consider the set of met 2 by 2 matrices, ok. Let me denote say 2 by 2 matrix on integers values M 2 z are all 2 by 2 integer matrix and the operations are operations are simple addition and multiplication. That means, my system is M 2 z plus and dot.

Now, you see that addition under addition all the properties that means it is closure associative commutative identity elements is also there that. So, it will be up to A5. We see that M 2 z if I consider plus, that means this algebraic system satisfies all the properties that up to A9, the distributive element because I am reading all the properties

A1 to A9 here with respect to addition the identity element is element is 0 because if I add if I take say 2 4 3 1 plus 0 0 0 0 I will be getting element wise addition.

So, this is 2 4 3 1 then with respect to multiplication the identity element is 1 0 0 1. Since if I multiply 2 4 3 1 to 1 0 0 1 simple matrix multiplication, we will be getting 2 4 3 1 now the inverse also exists.

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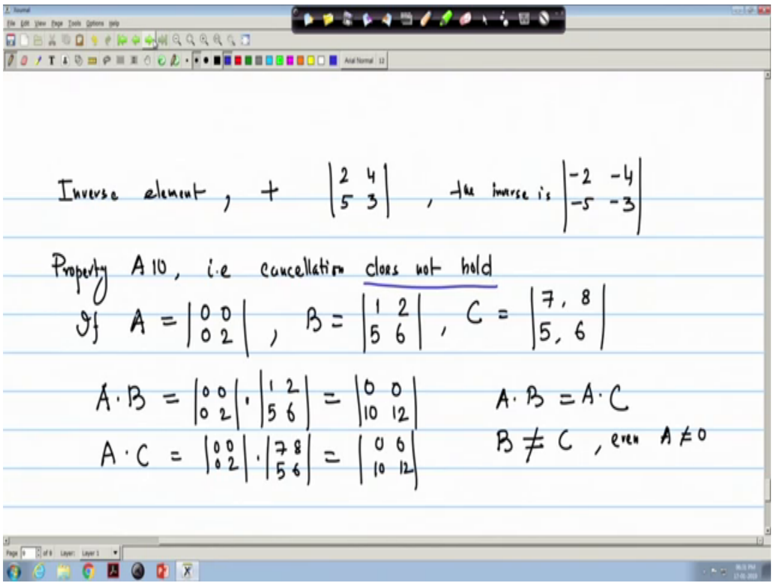

The inverse element with respect to addition this is say if it is say 2 4 5 3 the inverse is minus 2 minus 4 minus 5 minus 3 because this is the my additive inverse.

Now, in this way we can see that it is commutative it is under addition and some multiplication, but only the cancellation does not hold. Cancellation means our property in it. Why that property it in that is cancellation does not hold since if say if a is 1 2 by 2 matrix say it is 0 0 0 2 and I consider b is something called 1 2 5 6 and my c is 7 8 5 6.

Now, see that if I multiply a b, then this a b is 0 0 0 2 is 1 2 5 6 and this equal to this is 0, then this is also 0, this is 0 2. That means, 0 into 1 2 into 5 10 doing then this is 10 and 12. Similarly if I get a dot c, then it is 0 0 0 2 dot 7 8 5 6, then I also get 0 0 and 10 and 12. So, I get a dot b equal to a dot c, but we see that then also b not equal to c even my a is not equal to 0 a not equal to not equal to 0.

So, cancellation property this cancellation property does not hold here and idempotent does not hold.

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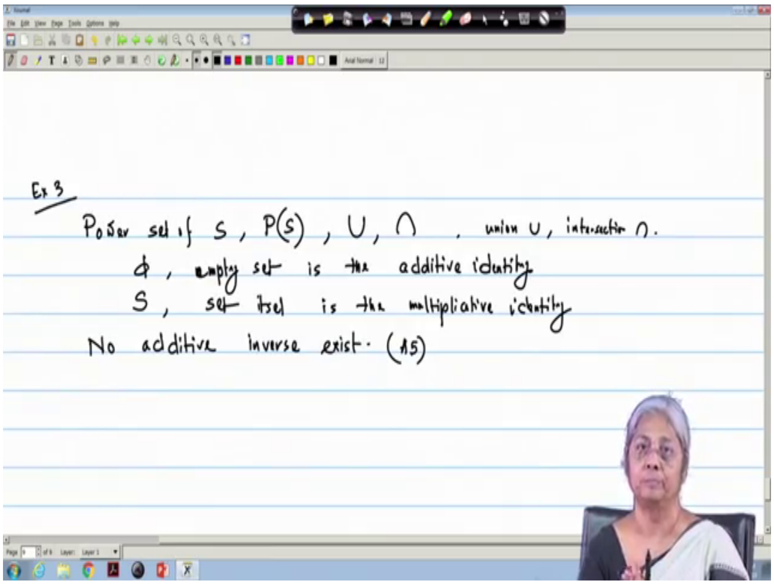

So, if I take another example say the if we consider the power set of S, we know the power set of S we denote P S and the two operations we said the union and intersection, the sum and product. So, the union and the intersection union and intersection, then our phi the null set or empty set. Empty set is the additive identity and the set itself S is the set itself is the multiplicative identity.

However here there is no additive inverse exist. That means, these additive inverse is the inverse element that A5. So, there is no A5. So, in this way we can actually, so this is that A5 is not there. So, in this way we can identify the different algebraic systems and what are the properties they hold.

So, in the next lecture we will try to identify that our many common and popular algebraic systems and we will see that what are the general properties they hold. Actually the properties they are satisfying based on these that actual the algebraic systems are defined and then, we will study their properties also.