Discrete Structure
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Lecture – 40
Counting Techniques and Pigeonhole Principle (Contd.)

Today we will read, an important counting technique called the Inclusion Exclusion Principle. That we our last lecture, we have read the pigeon hole principle and today we will see that, how inclusion exclusion can be used to count the number of sets or number of elements in a set or some other real life problems.

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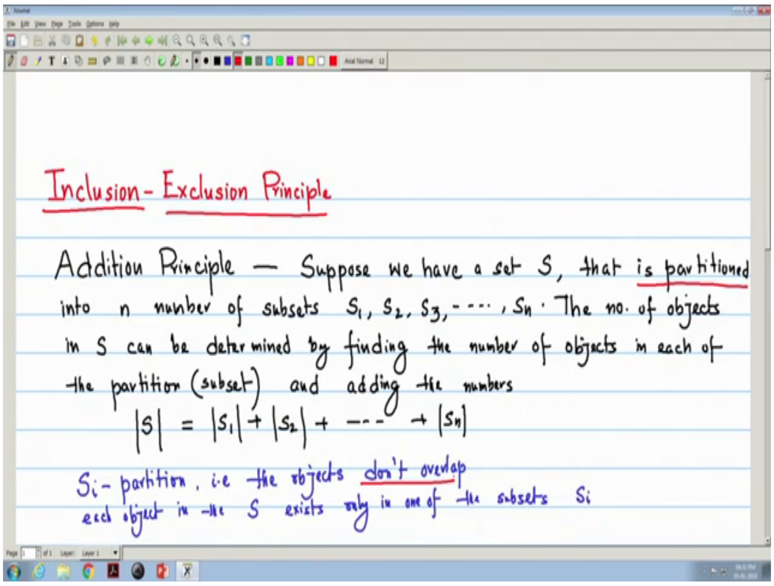

If we recall the addition principle of counting, then suppose we have a set is so, we recall the addition principle. Suppose, we have a set S and that is partitioned into n number of subsets, say S 1, S 2, S 3 up to S n, these are the n number of subsets.

Now, the number of objects if I want to count, number of objects in S, that can be determined by simply adding the number of objects in each of the subsets, by finding the number of objects in each of the partition or subset and adding the numbers; that means, the cardinality of S; the number of objects in S is the cardinality of S1, cardinality of S2 up to the cardinality of S n.

Now, this is true, because the addition principle works. Since, I have considered these Si, that it is a partition, that is it is partitioned; that means, each Si is a partition. And the definition of partition is such that, they do not the objects do not overlap partition; that means, the objects do not overlap. So, one object in the set S will be in only one of the subset Si.

So, this principle works because each object in the set S exists only in one of the subsets one of the subsets say Si. So, the main point is that here, the objects do not overlap and simply addition principle will work.

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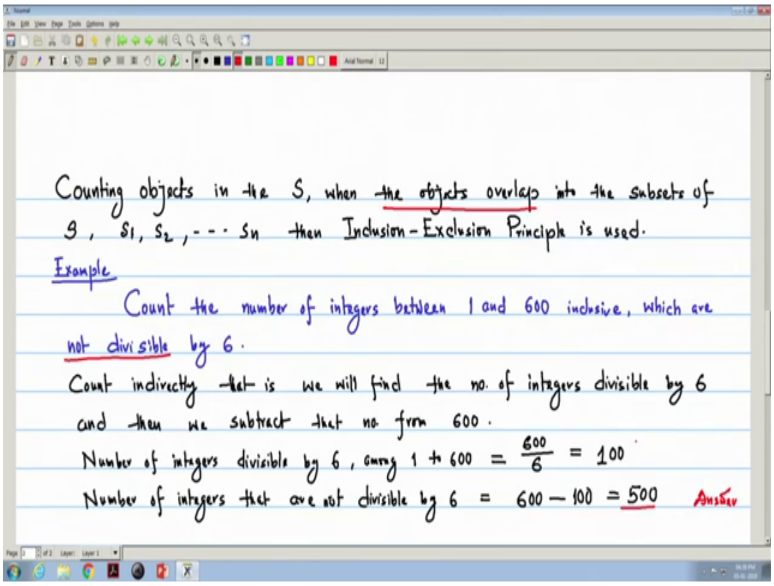

Now, the inclusion exclusion principle comes from here. So, if the objects overlap; that means, when a set S is partitioned into a number of subsets, then I should tell that not partition, because partition definition is that, that the objects do not overlap; so, a number of subsets such that the object may overlap. Then, if we want to count the number of objects in the set S, then we will apply inclusion exclusion principle.

So, simply we write that, counting objects in the set S, when the objects overlap I am into the subsets of S, that is S1, S2, Sn, then we have will be, then inclusion exclusion principle is used.

Now, we see that, how it works. So, this is actually we can write this is a general circumstances of counting the number of objects in a set; so, here the main thing is the objects overlap ok. Now, we see one simple example before we give the theorem of inclusion exclusion I take one example of counting. Say the example is to count the number of integers between 1 and 600 is inclusive, which are not divisible by 6.

Now, we can solve this problem indirectly, what we can do? I have to count that, which are not divisible. So, if I can count that very easily, that which are the numbers are; which are the integers between 1 and 600 that are divisible, then simply from 600 that number if we subtract will be getting that, what are the number of integers that cannot be divisible by 6.

So, this is some indirect way of counting the numbers. So, we count indirectly that is, we will find the number of integers divisible by 6 and then, we subtract that number from 600. Since, from 1 to 600 there are 600 integers are there.

So, what are the numbers that are divisible by 6? So, number of integers divisible by 6 among 1 to 600, is simply 600 by 6 equal to 100. So, there are 100 integers among 1 to 600 that are divisible by 6; so, number of integers that are not divisible by 6 equal to total number of integers 600 minus 100, that are divisible by 6 equal to 500 and this is my answer.

So, this is some indirect way of getting the solution. And inclusion exclusion principle tells exactly this thing, this is a technique of indirect way of finding a solution.

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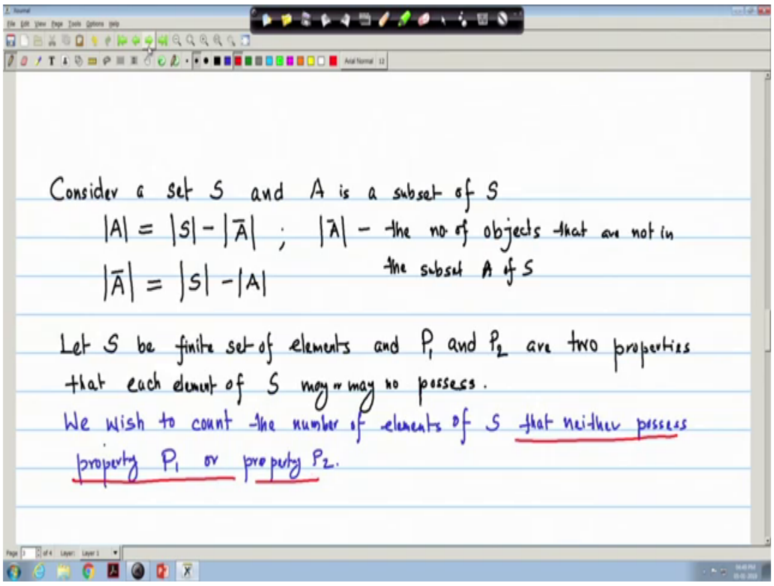

So, see we this we consider a we consider a set S, set S and A is a subset of S. Now, if we and A is a subset of S, if we try to find out the number of objects in A, then we can indirectly we can write that, the number of objects that are in A.

I can write, the cardinality of A is cardinality of S minus the cardinality of A complement; A complement is the or cardinality of A complement, I can write that, the number of objects that are not in A; the number of objects that are not in the subset A of S.

So, if I want the reversing; that means, that the number of objects which are not in A; that means A complement, I can write S minus A, the cardinality of cardinality of A and this simple principle is actually the is used in inclusion exclusion principle. So, we shall formulate the inclusion exclusion principle before that, we let the we define that one set having some properties.

So, let S is a set as we have already considered and P1 P 2. So, let is S is a, S be a finite set and P1, P2 are 2 properties, that each element of S may or may not possess. Now, say we wish to count the number of objects or elements in S that have neither that neither possess the property of property P1 or property P2 or property P2. With these examples or these illustration, we try to frame the inclusion exclusion principle.

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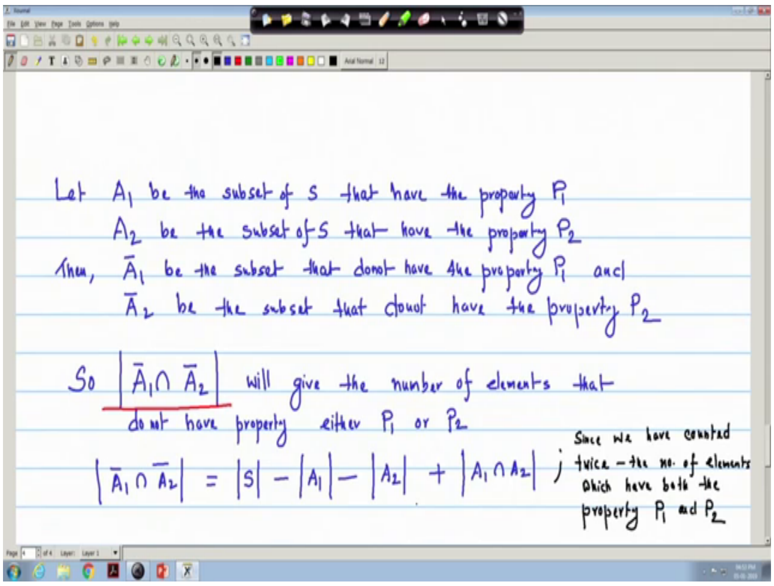

So, let A1 be the subset or S that possess; that means, each of the elements of A that each of the elements of A possess property P1. Similarly, A2 be the subset of S for each element of the property P2. So, then, A1 bar we can write that A1 bar be the subset that do not have the property P1 P1.

And similarly A2 bar be the subset that do not have the property P2. So, that A1 bar intersection A2 bar, the cardinality of this set will give us the number of elements or of S that do not have the property either P1 or P2; see, what we want to get our problem is that, we used to count the number of elements of S that neither possess property P1 or property P2.

So, clearly that A1 bar intersection A2 bar will give us that number of elements of S that have property neither P 1 or P 2. So, how we can get that thing? So, I can write that, A1 bar intersection A2 bar that is the total element S minus which have though the elements number of elements which are property P1, then number of elements which have property P2.

Now, there exist some elements which have both the property and we have counted twice that number, because once we have considered in A1, the number of elements which are property P1, in A2 also; we have considered number of property number of elements which are property P2.

So, some the numbers or the elements which have both the property, that we have taken twice; so, we must add that number that which are which are both property A1 and P1 and P2 and that means, that is A1 intersection A2.

Since I write, why we have done since we have counted twice the number of elements which have both the property which have both the property P1 and P2 P1 and P2 Once we have considered in A1 once we have considered A2; so, that is why we have again we have added the intersection of A1 and A2. So, and this is actually gives us the our inclusion exclusion principle.

So, this is indirect way of counting this is a indirect way of counting; that means, we count the number of elements which have property P1 and which have property P2 or which have property both. And then, we are finding that numbers we are getting that, the number of elements which do not have the property P1 or P2 that is, A1 bar intersection A2 bar.

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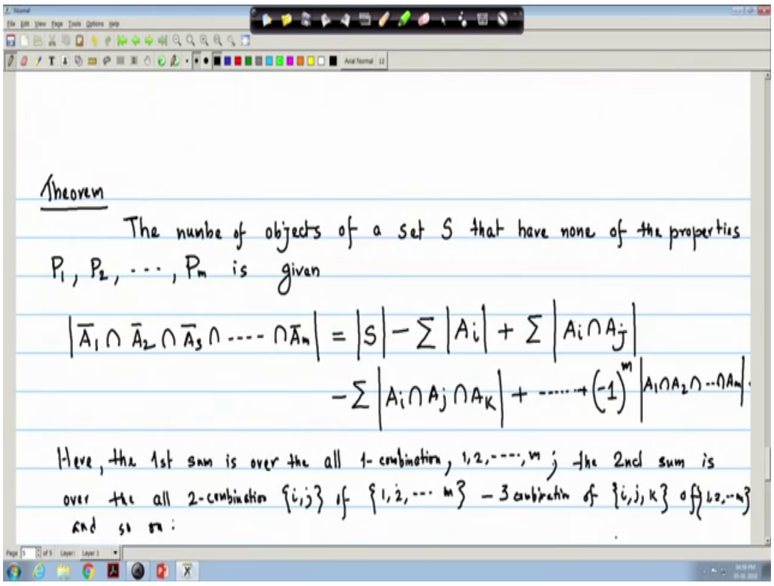

So, now, we write the statement of exclusion inclusion principle; so, I give the write as a theorem so, this is theorem of inclusion exclusion. So, we write the number of objects of a set S that have none of the properties P1, P2, Pn is given by.

So, just now the example are with only two properties we have seen, we are trying to generalize that principle. So, it will give us that A1 complement, intersection A2 complement, intersection A 3 complement is Am complement the cardinality of that equal to the cardinality of the set S minus, the number of elements having each property.

So, I can write that some of that Ai; i is from 1 to m, I am adding that, sum of the number of elements which have at least two property; that means, Ai intersection Aj and if I extend that the way we have done, then we should consider the summation when I consider 3 properties.

So, I consider that Ai intersection, Aj intersection, Ak and then you must have add the number of properties with 4 and in this way, if I continue if I continue then, I will be getting that minus 1 to the power n since, I am generalizing this thing. So, either it would be plus or minus and then all that is A1 intersection, A2 intersection up to A.

So, I can write that here the first sum, here the first sum is for all one combination is over the all one combination; that means, the number of elements which have only one property, either P1, P2 sum Pi, then the second sum 1 to m. The second sum is over the all 2-combinations all the 2 combinations say i and j; I have written i and j of this i, j are from 1 to m.

Third combinations the is i, j, k; I should write the 3 combinations of i, j, k of 1 to m and so on.

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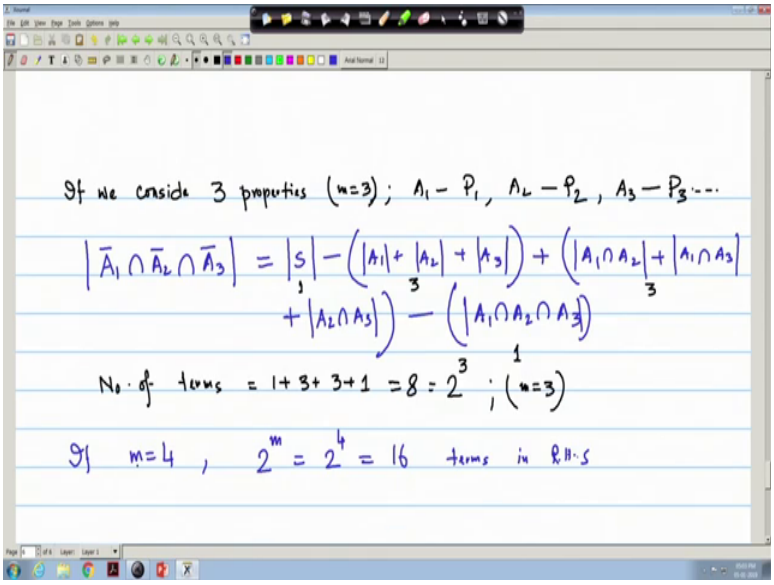

Now, if we see that for m equal to 3 so; that means, if my there are only we consider 3 properties so, if we consider 3 properties; that means, there are three subsets, that is m equal to 3 and say A1 set having property P1, A2 subset having property P2, A3 having property P3 and like so on and the only 3 I am considering.

Then, though my inclusion exclusion principle becomes that which have neither property P1, P2, P3 so, that is A1 complement intersection A2 complement intersection A3 complement. And this becomes S minus summation. So, this becomes A1 plus A2 one combination so, this is one combination of all A3. Then, I should take all 2 combinations so; that means, A1 A2 A1 A2, then A1 A3; A1 intersection A3, plus A2 A3; A2 intersection A3 and 3 combinations; so, 3 combinations are A1 intersection A2 intersection A3.

Now see we have how many terms in the right hand side, that we have 1, here I have 1 term, here I have 3 terms, here I have 3 terms, here I have 1 terms. So, number of terms is 1 plus 3 plus 3 plus 1 is 8, I can write this is as a 2 to the power 3. Since, I have considered m equal to 3 since my m equal to three or we have considered 3 subsets of these properties.

So, if we can generalize I can get that if I have, if we have, if m equal to 4, then similarly we can write or here that number of terms will be 2 to the power m equal to 2 to the power 4 equal to 16 terms in the right hand side.

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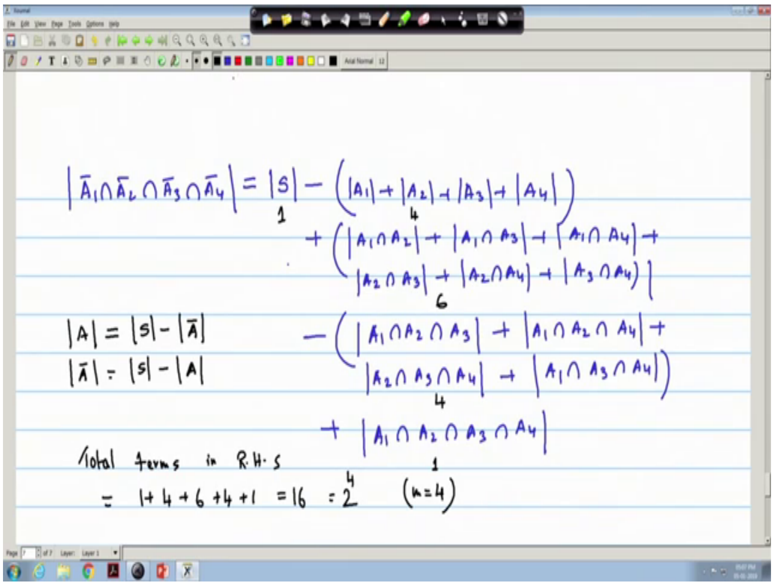

And if we can continue this way, I can write that A1 complement A2 complement A 3 complement A 4 complement; that means, which have neither property P1, P2, P3, P4 and I can write the cardinality of phase; that means, the total number of objects then I can write the A1, A2 though A1, A2, A3, A4. Then, I 2 combinations I have to add A1 A2, A1 A3, A1 A4, then A2 A3, A2 A 4 and A3 A 4.

I should take 3 combinations, then A1 intersection A2 intersection A3 plus A1, A2, A4; A1 intersection A2 intersection A4 plus A2, A3, A4 plus A1, A3, A 4. Then the 4 combinations 4 combinations are A1 intersection A2 intersection A3 intersection A 4.

So, I have here one term, here I have 4 term, here I have 6 term, here I have 4 term and here I have one term. So, total terms becomes is 1 plus 4 plus 6 plus 4 plus 1 is 16 is 2 to the power 4 since, m equal to 4. So, in this way indirectly we can calculate or we can find out the count or the number of elements in the subsets having sum properties or what earlier we mentioned even, it we can find out if it is that having not that properties; that means, either this is S minus A complement or it can be A complement is S minus A.

So, this is our inclusion exclusion principle and later when we will read the combinatorics, how to count the numbers in different for different types of problems, there we will apply this inclusion exclusion principle and this is a very easy way to count the numbers and this is a very important counting techniques.