Discrete Structures
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Lecture – 38
Counting Techniques and Pigeonhole Principle (Contd.)

So, we have learned the Pigeonhole Principles, the simple form and last lecture we have read some of the applications; the different type of applications that we can solve using pigeonhole principle. Today again, we will continue to solve problems using pigeonhole principle in strong, in simple form. So, we see again some simple applications that we can solve.

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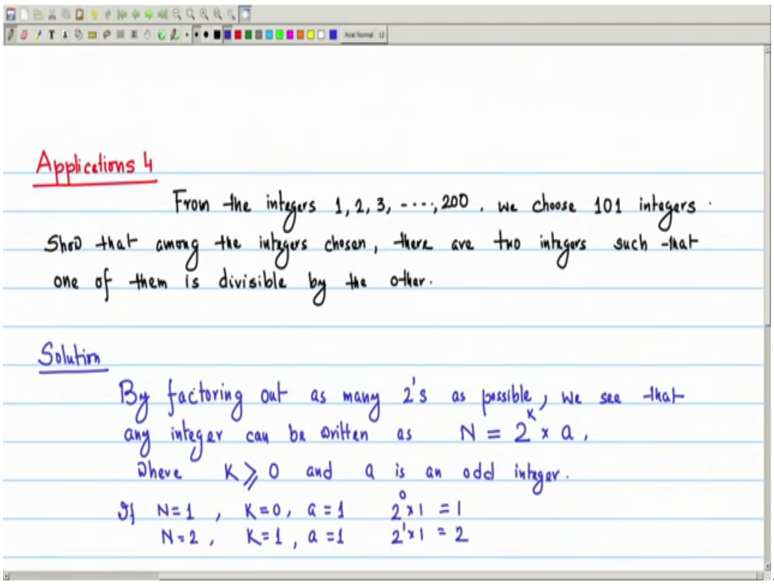

One such applications; mainly we are discussing the numerical problems so, see some applications. So, write the problem that again this is a number theoretic problems that from the integers 1 to 200; we choose 101 integers.

Now, we have to show that among the integers chosen, these are 101 integers that we have chosen. There are 2 integers such that one of them is divisible by the other. So, again I repeat the statement, problem statement that we have 200 integers 1 to 200 and we choose any 101 integers. We have to show that among these 101 integers, there are always 2 integers such that one will be divisible by the other.

So, we will try to solve this problem using pigeonhole principle in simple form. So, first we see one property of elementary number theory. So, by factoring out as many tools we can; we know that property is that any integer can be represented, we see that any integer can be written as, see if the integer is N. Then this is 2 to the power K into a; that means, K number of 2's we have taken from N; from the factor of N and a is one odd integer.

So, here where K greater than equal to 0 and a is an odd integer. So, first we see that for small integers, what are the values of K and a. So, if N equal to 1; then, I can write that K equal to 0; a is 1, since 2 to the power 0 into 1 is 1. Similarly, if N equal to 2, I can write that K equal to 1 and a equal to 1 since 2 to the power 1 into 1 equal to 2. So, it is valid.

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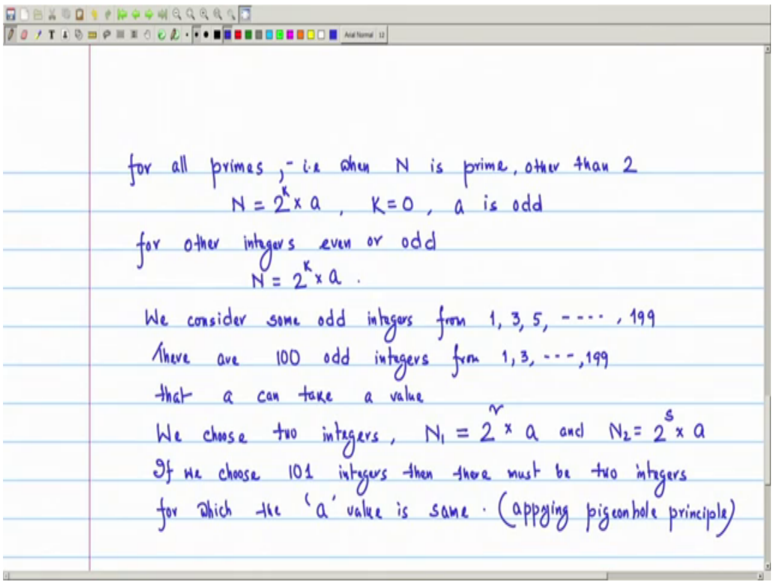

Now, for all primes; that means, when N is prime number, we get that is when N is prime that N equal to 2 to the power K into a. That means, here K equal to 0 and a is odd. So, N is prime, I should write other than 2 because other than 2. So, a is odd number and for all even numbers. So, for other integers even or odd, other integers even or odd, we can write or we can represent N as 2 to the power K into a ok; where, a is odd number for even. We must get at least 1 2; that means, power of case in that case greater than equal to 1.

So, now which we consider some odd integers from 1, 3, 5 to 199 since my numbers given are 1 to 200. So, how many odd integers are there in 1 to 199? So, there are there are 100 odd integers from 1 to 199 that it can take any value of these from 1 to 199, I can take a value. To say we choose 2 numbers 2 integers say N 1 equal to 2 to the power r into a and N 2 equal to 2 to the power s into a.

Now, since there are 100 odd integers, now if we choose 101 integers; then, there must be 2 integers for which the a value is same. Here, we applying pigeonhole principle.

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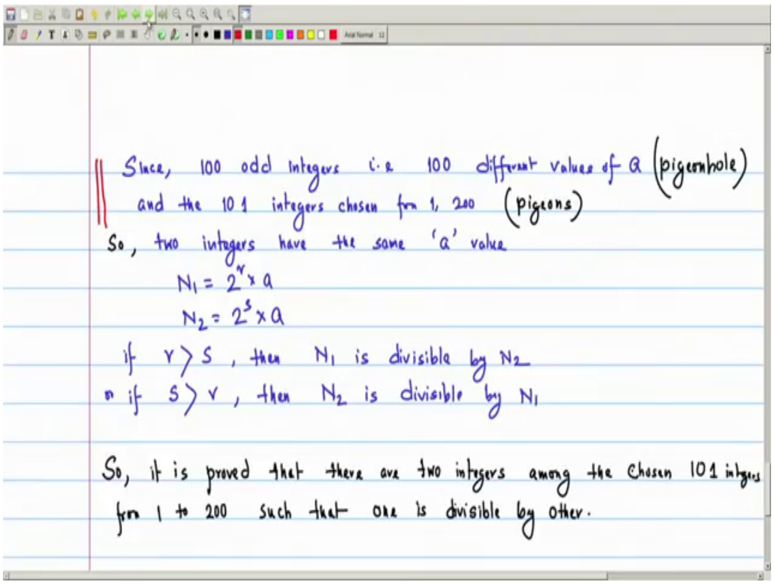

Since we have 100 odd integers; that means, 100 different values of a that can be treated as a pigeonhole and the 101 integers chosen from 1 to 200 that we can take as the pigeons.

So, 2 integers have; so 2 integers have the same a value and that 2 numbers we have taken N 1 equal to 2 to the power r into a and N 2 equal to 2 to the power s into a. So, if r greater than s; then N 1 is divisible by N 2 or if s greater than r; then N 2 is divisible by N 1 so, we prove that there exist. So, it is proved that there are 2 integers among the chosen 101 integers from 1 to 200 such that one is divisible by other and we have applied the pigeonhole principle. This is where we have applied the pigeonhole principle to show that thing. So, this is one class of numerical problems that we can solve using pigeonhole principle.

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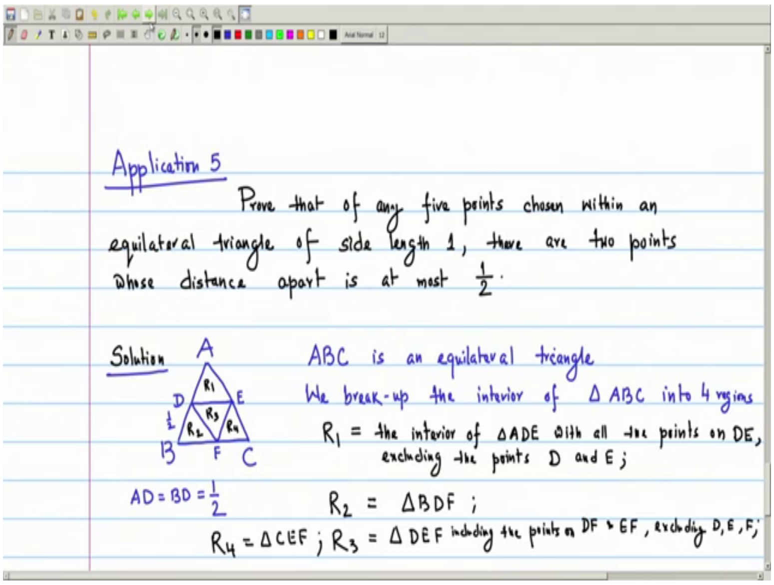

Now, we see another type of applications, where that some geometrical problems where we can apply normally the pigeonhole principle to solve or to show some statements to be true. So, we take a very simple problem that prove that any 5 points chosen within an equilateral triangle of side length 1; there are 2 points whose distance apart. That means, the distance between these 2 points is at most half. So, this is a geometrical problem. We have a equilateral triangle and we have to show that always there are 2 points within these equilateral triangle that the distance between these 2 is at most half. So, we see the solution. I draw a equilateral triangle. So, ABC is a ABC is an equilateral triangle and we have to prove that there are 2 points in between in the interior of ABC that whose distance at most half ok.

So, now we break up the interior of triangle ABC into 4 regions, say give DEF. So, these are the DEF at the midpoints of AB, BC and AC. So, this AD equal to BD equal to half and for other sides also this is true. Now these regions I can write the 4 regions like R 1 is the interior of ADE of triangle ADE. So, this is my R 1 with the points on DE; with all the points on the side DE, but excluding the point 2 points D and E, excluding the points D and E.

Then, I take R 2 is triangle BDF. So, this is my R 2 triangle BDF. Then, I take triangle R 3; again, this is the interior of triangle DEF, all points including the points on DF and EF; DF and EF and excluding the point DEF excluding DEF. I have another region R 4; I take another region R 4. I write here R 4 is CEF triangle CEF. So, we get 4 congruent equilateral triangles; that means, the interior of ABC now becomes 4 regions; R 1, R 2, R 3, R 4. So, if I consider any 2 points.

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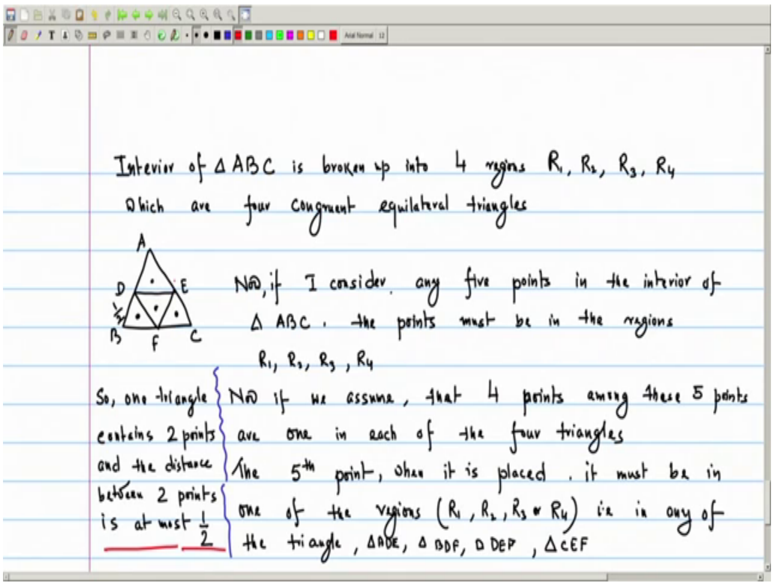

Now, I can write that interior of ABC, triangle ABC is broken up into 4 regions R 1, R 2, R 3, R 4 which are 4 congruent equilateral triangles. Now, if I consider any 5 points in the interior of triangle ABC.

So, they must be in the points; must be in the regions R 1, R 2, R 3, R 4. Now, if we assume that 4 among these 5 points, 4 points among these 5 points are 1 in each of the 4 triangles. That means, that 4 points we assume that 1 point is here; 1 point is here; 1 point is here and 1 point is here. So, the remaining points, the 5th point now if we put that must be in 1 of the triangle of this 4 triangles.

So, the 5th point when it is put or it is placed, it must be one of the, must be in one of the regions R 1, R 2, R 3 or R 4; that means, that means in any one of the triangle that ADE that BDF, DEF and CEF; that means, one triangle must contain 2 points. So, one triangle contains 2 points and the distance between these 2 points is at most half since my BD DE, these are half and these triangles are also the equilateral triangle. So, the distance is half.

So, we prove that there are always 2 points we get in the interior of equilateral triangle ABC whose distance is at most half. Now we consider one-third application.

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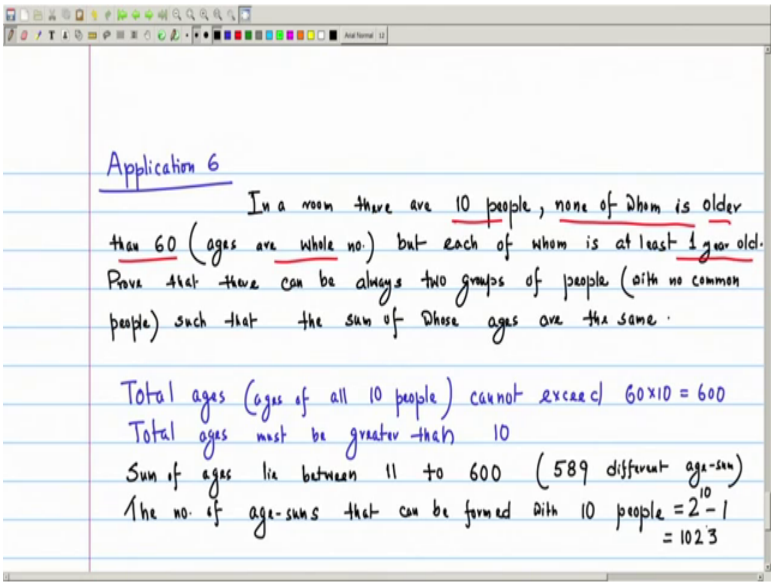

So, this is a problem. So, I write the problem that in a room there are 10 people, none of whom aged is a older than; how can I write? No, none of whom is older than 60 and we assume that ages are whole number; ages are whole number.

So, everyone is none of whom is older than 60; that means, every ones age is less than equal to 60, but the age each of whom is at least 1 year old. Now we have to prove that there can be always 2 groups of these people with no one, no common people such that the groups of people such that the sum of whose ages are equal, ages are the same. That means, I have 10 people in a room and ages are each one is not older than 60; none of them is older than 60.

So, we can always make 2 groups. So, the sum of the ages of 1 group is same as that some of the ages of the other group, we have to prove these thing. So, now if we since 10 people are there and none of whom is older than 60; 10 people are there and none of whom is older than 60. So, I can write that the total ages cannot exceed if I take the sum. So, total ages; that means, ages of ages of all 10 people cannot exceed 60 into 10 equal to 600 and similarly the total ages must be greater than 10 because everyone is at least 1 year old; so, greater than 10.

So, since we are taking the ages as whole number. So, the ages the sum of ages lie between the sum of ages between 11 to 600, which is greater than 100 and we are taking a whole number. So, I can consider that 11 to 600 and that will be 590's of 5 there are 589 different sums; age sum I can take the age sum.

Now, if I consider that as if 10 people as if the 10 numbers. So, what will be the total number of sums? The number of sums I can write the age sums since, I am taking the sum of ages, number of age sums that can be formed since there are 10 people 2 to the power 10 minus 1 that can be formed. With 10 people is 2 to the power 10 minus 1 this is equal to 1023. So, now, I can apply the pigeonhole principle.

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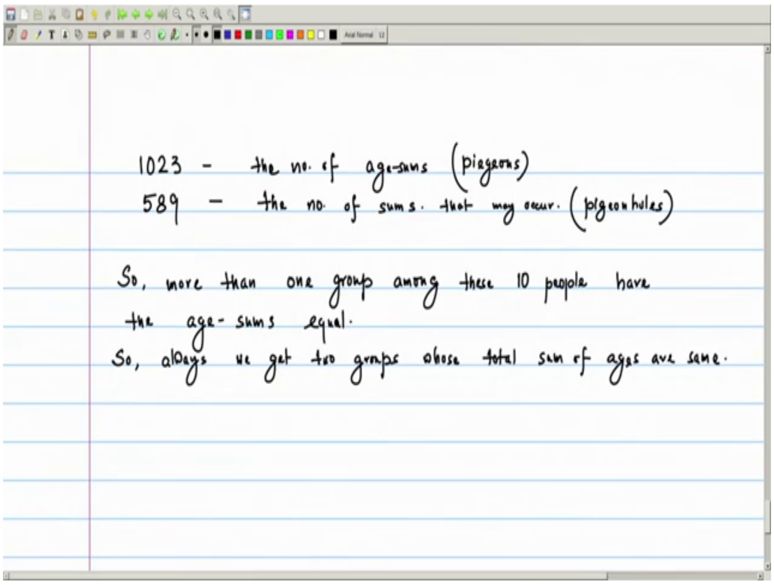

So, these 1023 as if the number of the number of age sums, number of sums or number of age sums I am telling these are actually my treated as the pigeons. And the sums of ages these 589 are the number of sums that may have occur, number of sums that may occur. So, and these are treated as a pigeonholes.

So that means, among these 1023 that we can get more than 1 group; so, more than 1 group since 589 is less than 1023. So, more than 1 group among these 10 people have the age sums equal. So, we get always at least 2 groups whose total sum of ages are same. So, always we get 2 groups whose total sum of ages are same.

So, see this is a different type of problem and we can apply pigeonhole principle and we can prove that or we can see that we can solve the problem or we can prove the statement. Now, we have read the pigeonhole principle in simple form and the mainly the applications that we can solve using this Simple form of pigeonhole principle. In the next lecture, we will read the Strong form of pigeonhole principle.