**Discrete Structures**
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**Lecture – 37**
**Counting Techniques and Pigeonhole Principle [Contd.]**

So, we are reading the Pigeonhole Principles and how this principle is used to solve the counting problems or the arrangement problems related with the counting. And very simple applications we have seen in the last lecture. Today we will see that how numerical problems can be solved using pigeonhole principle.

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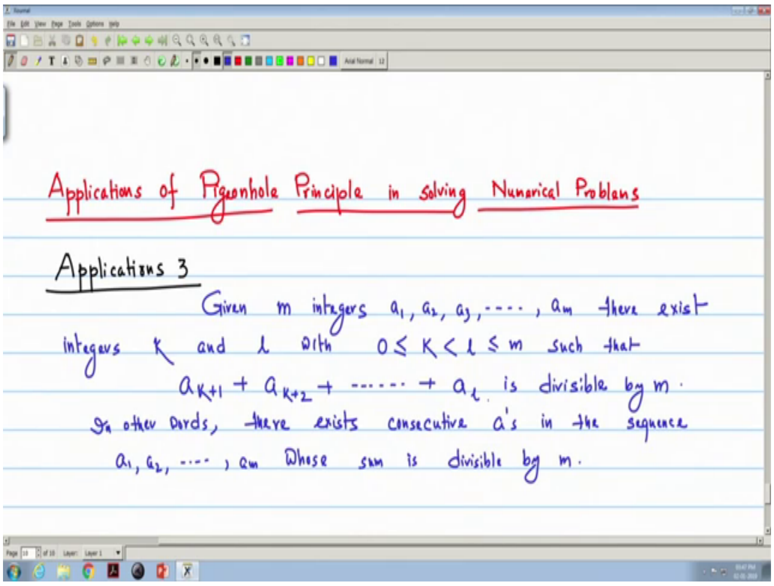


So, we will read the applications of pigeonhole principle and solving numerical problems. So, you see today another 1 type of applications we call the applications 3. We take some example and with these example we explain how pigeonhole is used here. So, one mathematical problems we take that given m number of integers say a 1, a 2, a 3 up to a m, we have proved that there exist integers k and l with the restriction that 0 less than equal to k less than n less than equal to m. Such that a k plus 1 plus a k plus 2 plus up to a l is divisible by m.

In other words, I can tell there exists consecutive is in the sequence a 1 to a m whose sum is divisible by m. So, this is a number theoretic problem that I have a sequence of numbers m numbers a 1 a 2 a 3 a m and they are exist some integers k and l. So, that a k plus 1 to a k plus 2 plus l; that means, there are consecutive numbers between k and l which is divisible by m, we have to prove this thing. And this we will see that if we use pigeonhole principle very simple way very easily we can prove these thing ok. So, how to solve? We have the solution procedure.

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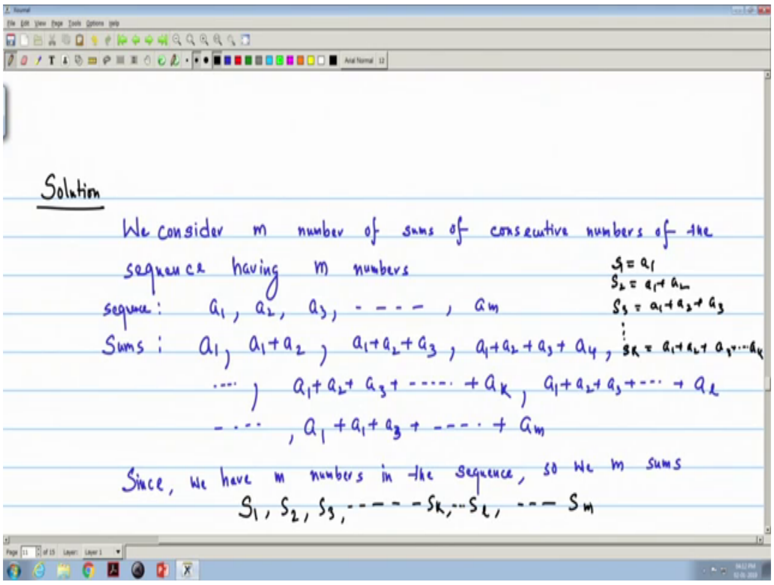


Solution: we consider the consecutive sums. So, consider I write m sums; m number of sums of consecutive numbers of the sequence having m numbers having m numbers. Since the sequence has m numbers say sequence is (Refer Time: 07:23) the sequence is a 1 a 2 a 3 a m. So, if I consider the consecutive sums then also now it will be m sums. So, sums will be say first sum is only 1 2 consecutive numbers now I have a 1 plus a 2, I have a 1 plus a 2 plus a 3. Then I have a 1 plus a 2 plus a 3 plus a 4 and if I consider that up to k, then this sum is a 1 plus a 2 plus a 3 plus up to a k, then it is a 1 plus a 2 plus a 3 a m; in these way if I go then a 1 plus a 2 plus a 3 up to a m. Since I have m numbers in the sequence so, I have m number m sums; since we have m numbers in the sequence so, we have m sums or sums of consecutive numbers.

Now, if we have to show that the consecutive sums of the numbers between k and l is divisible by m. Now we divide each of the sums by n.

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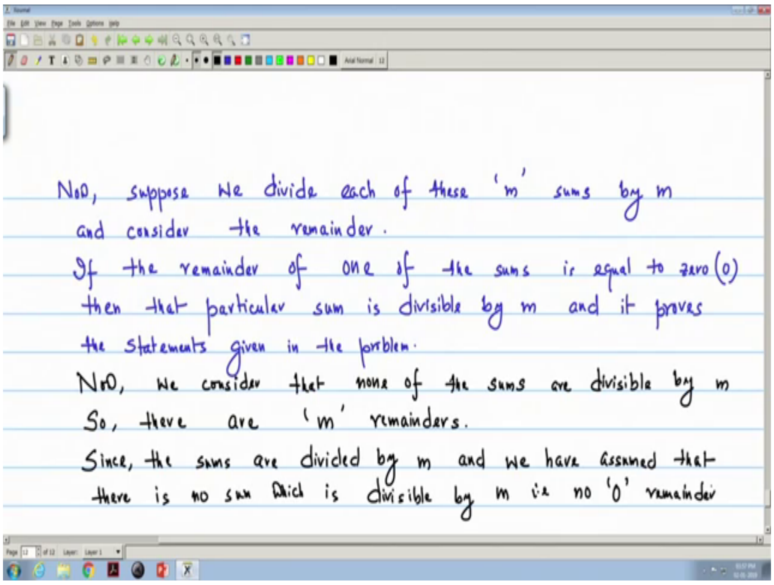


Now, suppose we divide each of these m sums by m and consider the remainder. If the remainder of one of the sums is equal to 0; that means, then that sum is divisible by m and it proves the problem statement.

Now, we considerthat none of the sums are divisible by m. So, there are m number of remainders so, I write there are m remainders. Now since we are dividing the sums; since the sums are divided by m and we have assumed that there is no sum which is divisible by m; that means, that is no remainder or no 0 remainder.

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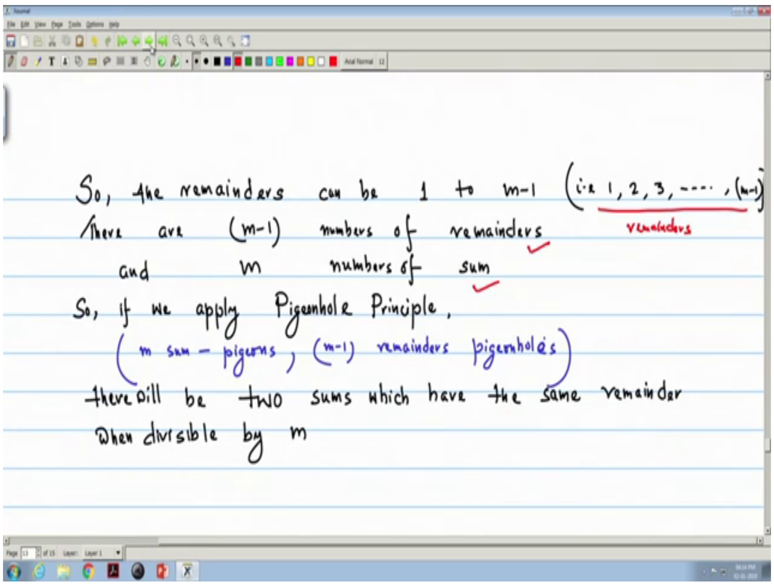

So, the remainders of this divisions when the m number of sums are divided by m the remainders can be 1 to m minus 1 that can be that is 1 2 3 up to m minus 1; that means, there are only there are m minus 1 number of remainders m minus 1 number of remainders. So, I have m minus 1 numbers of remainders and if we remember and m numbers of sum.

So, if we apply pigeonhole principle what I mentioned that first we have to identify which objects I am considering as pigeons and which objects I am considering as the pigeonhole. So, here if we apply the pigeonhole here m numbers of sum; that means, m sums are the pigeons and m minus 1 remainders are the pigeonholes then according to the pigeonhole principle that there will be two sums which have the same remainder.

So, if we apply pigeonhole principle there will be two sums which have the same remainder when divisible by m. Since there are m sums and m minus 1 remainders so, m sums each sum is divided by m. So, there must be two sums which have the same remainder when divisible by m.

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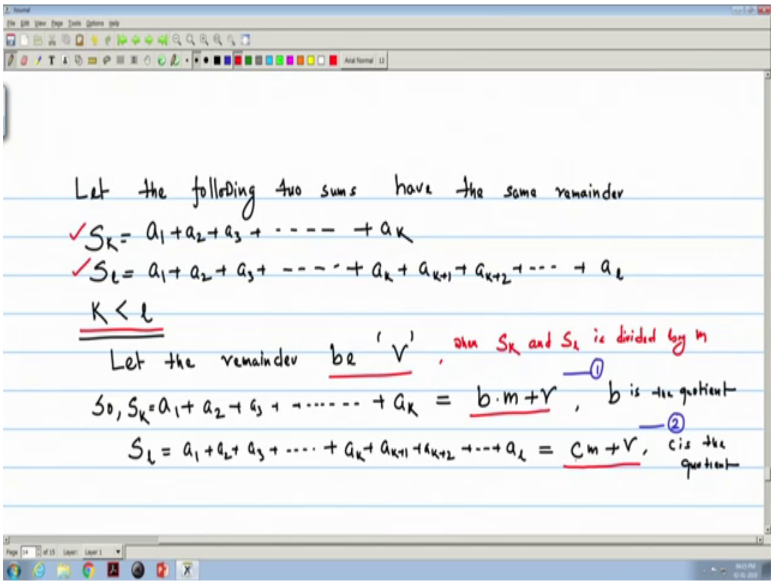


So, let the two sums of the same remainder when it is divided by m. One is the a 1 plus a 2 plus a 3 up to a k this is one sum, another is a 1 plus a 2 plus a 3 plus a k plus a k plus 1 plus a k plus 2 plus a l where we assume that l k is less than l as it is given in the problem statement that case if we remember that it is given that k less than l.

So, now if we subtract because they have the same remainders let the let the remainder be r it is the same remainder. So, I can write the first sum a 1 plus a 2 plus a 3 plus up to a k as if these equal to I can write b into m plus r or b is my b is the quotient when the first sum is divided by m let this sum I can write as if S k and as I can write this sum is as the S l. So, I can write my S k is a 1 to a k is b m plus r.

Similarly, I can write S l is a 1 plus a 2 plus a 3 plus a k plus a k plus 1 a k plus 2 plus a l is C m plus r s C is the quotient and in both the cases the r is the same remainder be r when S k and S l is divided by m. So, if I now subtract these equation 1 and equation 2.

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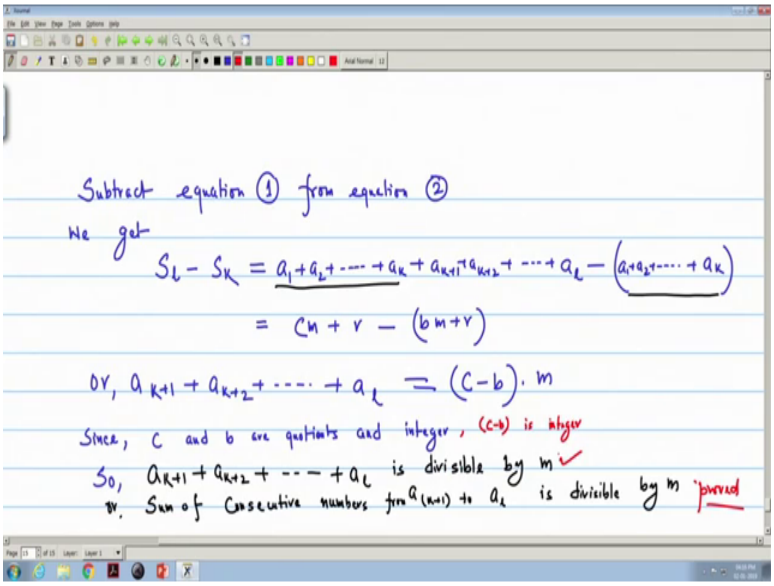


So, subtracting so, subtract equation 1 from equation 2 we get S l minus S k this is my S l minus S k is a 1 plus a 2 plus a k and these equal to these equal to I can write that b m plus r of l minus a k. So, this is my C m minus r C m plus r minus b m plus r or if I can write that a 1 up to a 1 plus a k and these a 1 plus a k these will cancel.

So, I can get a k plus 1 plus a k plus 2 plus a l equal to here r will cancel so, these becomes C minus b into m. So, since C and b where C and b are quotients so, they are integers. So, since C and b are quotients and integer. So, a k plus 1 a k plus. So, I can write a k plus 1 a k plus 2 plus a l is divisible by m or in other words or we can tell that the sum of consecutive numbers from k to l k plus 1 to l I can write from k plus 1 to l; that means, from a k plus 1 to a l is divisible by m.

So, it is proved. So, quickly if I see that my problem was that we had m numbers in the sequence and we have to prove that a k plus 1 to a l; that means, some consecutive a s in the sequence whose sum is divisible by m. And what we have done we have taken some consecutive sums we have given the name as sum S 1 to m consecutive numbers of the sum that S 1 to S k ; that means, S 1 equal to I can tell S 1 equal to as if I can write S 1 equal to a 1 S 2 equal to a 1 plus a 2 S 3 equal to a 1 plus a 2 plus a 3 similarly I have S k equal to a 1 plus a 2 plus a 3 plus up to a k and since I have m numbers so, I have m such sums.

So, we have m sums and sums are S 1 S 2 S 3 up to S k S l I have sums S m so, m number of sums. Then we have divided each sum by m, if any one of the sum is gives a remainder 0 when it is divided by m then it proves a statement. So, here this is one thing that if the remainder of one of the sums is equal to 0 then it proves. Now we assume this is where our general proof starts that we assume we consider that none of the sums are divisible by m. We have taken then m number of remainders; since we have considered that none of the sum is divisible by m.

So, I have m number of m minus 1 number of remainder that is from 1 to these are the these are my these are my remainders these are my remainders now I have m minus 1 number of remainders and m number of sums. So, identify the sums as the pigeons and treat remainders as the pigeonhole. So, there are m sums and m minus 1 pigeon m number of remainders like pigeonhole. So, two sums must have same remainder when it is divided by m and these two sums we consider as if S k and S l.

So, S k and S l is a 1 to a k sum of a 1 plus a 2 a k S l is a 1 plus a 2 a k plus 1 up to a l since it is written k less than l. So, we will consider since it is the same remainder. So, I consider when S k is divided by m the quotient is b is the quotient when Sk is divided by m and C is the quotient when S l is divided by m.

So, we can write that this sum equal to b m plus r and this is C m plus r, then we subtract equation 1 from 2. So, S l minus S k I get that C m plus r minus b m plus r and these will cancel. So, these becomes a k plus 1 plus a k plus 2 up to a l is C minus b into m since C and b are quotients. So, there they must be integer. So, C minus b is integer. So, C minus b is integer and I can write a k plus 1 plus a k plus 2 up to a l is divisible by m. So, it is proved. So, now similar type of numerical problems we can write another numerical problems.

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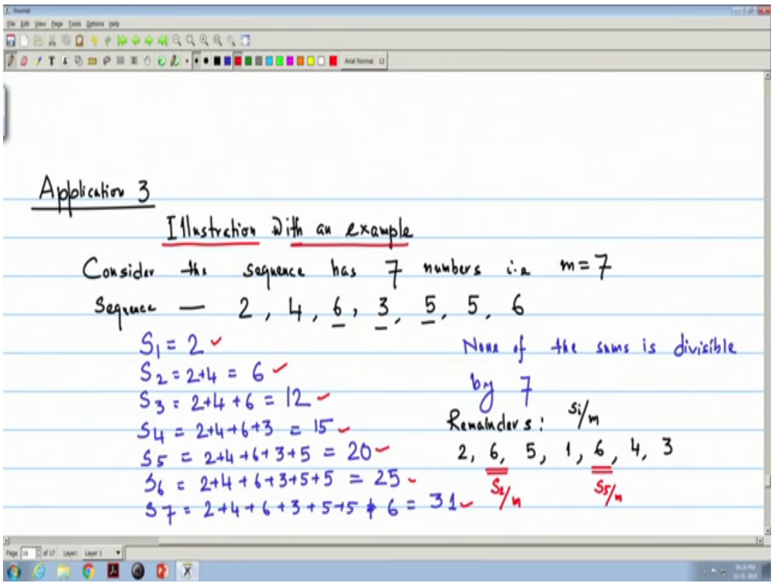


That I give some application 4 or we can just illustrate these thing with an example application 3 we illustrate with an example. So, consider the sequences has say 7 numbers; that means, on the sequence is; that means, m equal to 7 consider the sequence as the sequence is say 2 4 6 3 5 say 6.

So, if I take the consecutive sums the way we have done I have since I have m equal to 7. So, 7 sums are I take S 1 equal to 2 S 2 equal to 2 plus 4 equal to 6 S 3 equal to 2 plus 4 plus 6 equal to 12 S 4 equal to 2 plus 4 plus 6 plus 3 15 S 5 equal to 2 plus 4 plus 6 plus 3 plus 5 is 20 S 6 is 2 plus 4 plus 6 plus 3 plus 5 plus 5 equal to 25 and S 7 is 2 plus 4 plus 6 plus 3 plus 5 plus 5 equal to plus 6 equal to 31.

So, we have these are my 7 sums. Now if I and none of the sums are divisible by 7. So, I write that none of the sums is divisible is divisible by 7. So, if I take the remainder so, our sums are these if I take the remainder remainders are when it is remainders when sums are divided by 7.

So, I take S I by m remainders are 2 6 5 1 6 4 and 3. Now we see that two cases the remainders are same why because I have I have remainders can be only from 1 to 6.

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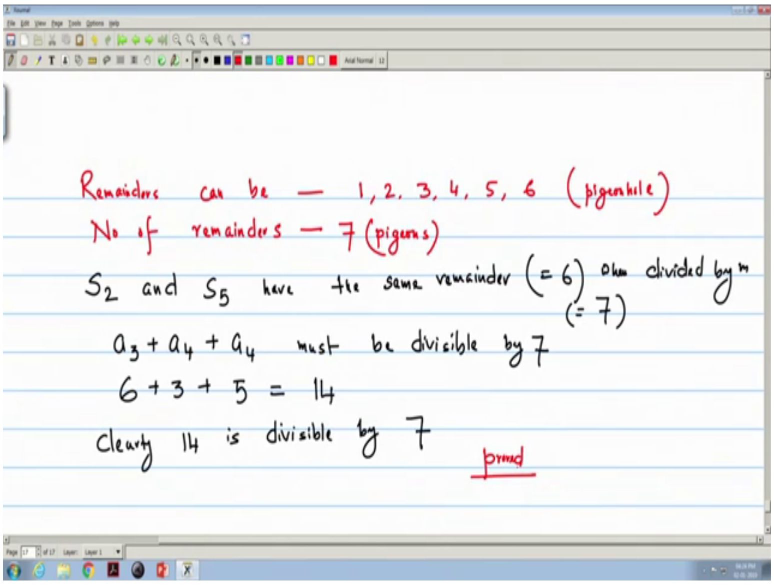


Since here the remainders can be remainders can be 1 2 3 4 5 6 since I am dividing by 7 and number of remainders are number of remainders since I have number of 7 sums a 7.

So, these are my pigeons these are my pigeons and these are my holes. So, what I get the two sums are the same reminder and they are S 2 by m they are S 2 by m and this is S 5 by m. So, just now there, but we have proved that S 2 by S 2 and S 5 have the same remainder same remainder that is equal to 6 when divided by 7 divided by 7 that is equal to m ok.

So, S 2 and S 5 so; that means, a 3 plus a 4 plus a 5 must be divisible by 7, now we see what is a 3 a 4 a 5. So, my a 3 a 4 a 5 is 6 3 5. So, my this 6 plus 3 plus 5 equal to 14 and clearly 14 is divisible by 7. So, with this example we see that what just now we have proved with pigeonhole principle that problem we have proved and we give some illustration some examples these it is an illustration with an example and it gives a correct result.

So, we see that numerical problems and counting problems or more minutely I can tell that counting problems related with arrangements that can be solved using pigeonhole principle. And we see one class of problems that can be solved using pigeonhole principle we in this next class. We will see that other classes of problems that can be solved using pigeonhole.