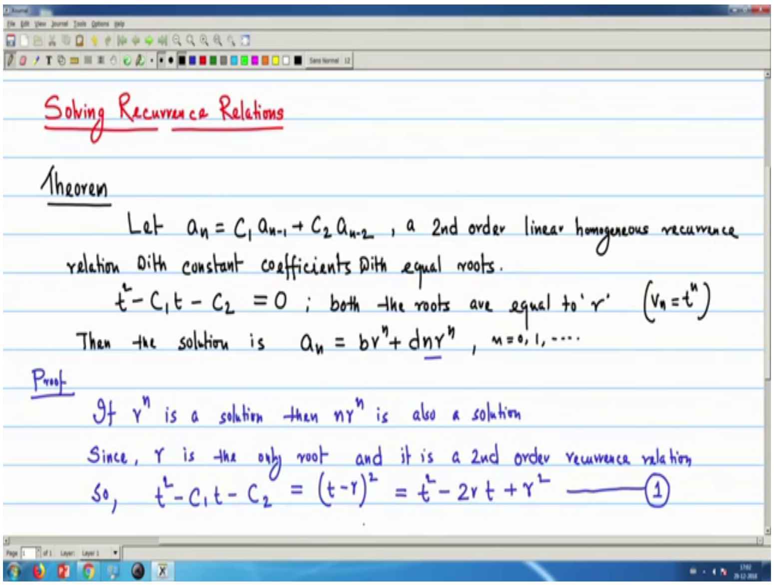
Discrete Structures
Prof. Dipanwita Roychoudhury
Department of Computer Science & Engineering
Indian Institute of Technology, Kharagpur

Lecture – 34
Recurrence Relations (Contd.)

We are discussing about the different techniques for solve solving Recurrence Relations. In the last lecture we have learnt that how we can solve the linear homogeneous recurrence relation with constant coefficients, and also the simple technique of applying iteration how we can get the explicit formula of the recurrence relation; that means, the solution of recurrence relation.

Today again we will see read the how to solve the recurrence relation that are linear homogeneous recurrence relation with constant coefficient, but when the roots are equal because last lecture we have seen when the roots are not equal. So, we will continue the solving recurrence relations.

(Refer Slide Time: 01:21)



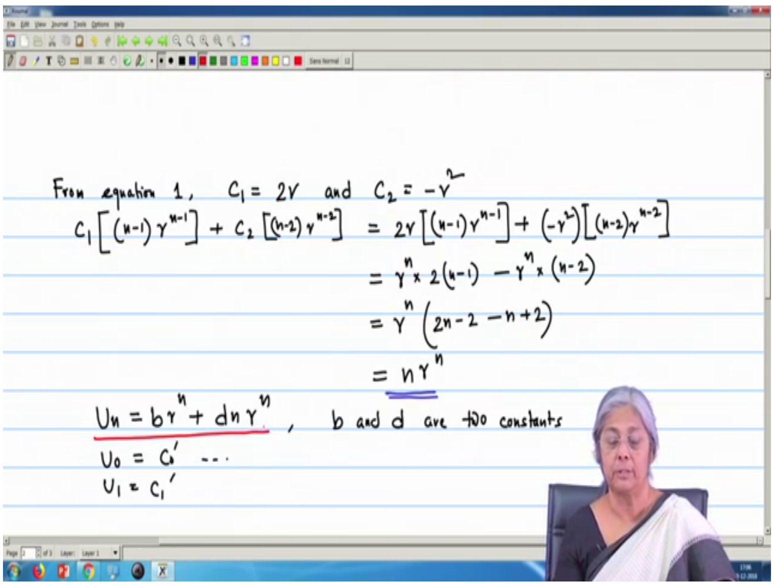
And we will see today, how we can solve the linear homogeneous recurrence relations with constant coefficients when the roots are equal. So, first we read the theorem. So, let a n equal to C 1 a n minus 1 plus C 2 a n minus 2 a second order linear homogeneous recurrence relation with constant coefficient. Yes C 1 and C 2 are the, but here the roots are with equal roots are equal. Last lecture what we have done that the roots are unequal.

So, we can consider that the solution as if we consider that solution is of type V n equal to t to the power n then we get the equation as t square minus C 1 t minus C 2 equal to 0 and the root is both the roots are equal to r since the roots are equal. So, both the roots are equal to r. Then the solution is, then the solution is of the form is a n equal to b r to the power n plus d n r to the power n, where n equal to 0 1 like that.

So, if we remember that when the roots are not equal unequal roots, the solution is of the form b r to the power n, d r to the power n it is of the form this. Now we will see that that here if the roots are equal it is a n b r to the power n plus d n r to the power n. So, how we can first we prove that theorem. So, here mainly we have to show that if r to the power n if r to the power n is a solution, then n r to the power n, n r to the power n is also is also a solution ok. Now, since r is the only root one root since r is the only root and the recurrence relation the equation is a second order recur recurrence relation and it is a second order recurrence relation.

So, we can write that t square minus C 1 t minus C 2 equal to t minus r whole square because both the roots are equal to t square minus 2 r t plus r square. So, this I give equation 1. One thing we write here since last theorem we have seen that this t is coming from the that as if the solution is V n equal to t to the power n is of the this form.

(Refer Slide Time: 08:17)

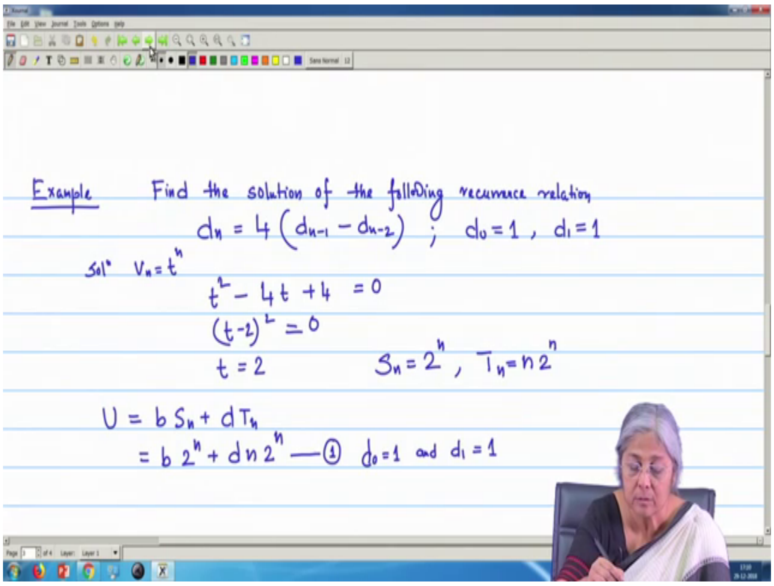


Now, from equation 1 what you see from equation 1 that C 1 equal to 2 r, C 1 equal to 2 r and C 2 equal to minus r square. If we equate the coefficients of the left hand side and the right hand side of the equation. Now we have to show that if r to the power n is a solution then n r to the power n is also a solution. So, we take that one solution with C 1 C 2 as the coefficient that C 1 n minus 1 r to the power n minus 1 plus C 2 n minus 2, r to the power n minus 2 this is equal to we put the values of C 1 and C 2.

So, this is equal to 2 r n minus 1 r to the power n minus 1 plus C 2 is minus r square; so, minus r square into n minus 2 into r to the power n minus 2. So, this becomes r to the power n into 2 into n minus 1 minus here also r to the power n into n minus 2. So, if I take r to the power n common, then 2 n minus 2 minus n plus 2. So, I get n r to the power n. So, my if I take the same convention that my solution u will equal to b r to the power n plus d n r to the power n these are b and d are two constants are two constants.

And we can take that U 0 equal to some C 0 dash and U 1 equal to some C 1 dash that type of constant. So, what we prove that that n r to the power n is a n r to the power n is a solution. So, if we see the theorem that this is a second order linear homogeneous recurrence relation with constant coefficients y with equal roots equal roots, then we get that n r to the power n is a or I give the solution as un equal to b r to the power n plus d n r the power n. Now, first we see one example we see one example.

(Refer Slide Time: 12:15)

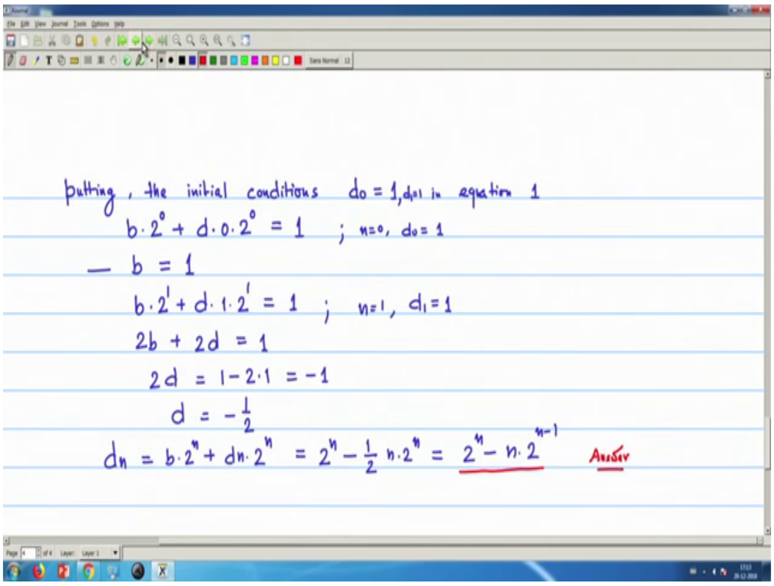


We take the find the solution of the following recurrence relation. Relation is d n equal to 4 into d n minus 1 minus d n minus 2 coefficients given d 0 equal to 1 and d 1 equal to 1. Now if we take since it is linear homogeneous recurrence relation of second order, we can consider the solution is of the form solution is of the form V n equal to t to the power n.

So, we get that t square minus 4 t plus 4 equal to 0. So, t minus 2 whole square equal to 0. So, I get t equal to 2. So, here the both the roots are equal. So, my solution is of the form. So, I get if it is we take the convention because t equal to 2. So, my con S n equal to I can take S n equal to 2 to the power n and T n equal to n n 2 to the power n n 2 to the power n from the previous theorem just now the theorem we discussed. So, my solution becomes U equal to U equal to b S n plus d T n equal to b 2 to the power n plus d n 2 to the power n.

Now, we put the d 0 equal to 1 the condition we put d 0 equal to 1 and d 1 equal to 1 in this equation 1. So, putting the initial conditions d 0 equal to 1 in equation 1 d 0 equal to 1 and d 1 equal to 1 in equation 1 we get, b 2 to the power n d n 2 to the power n.

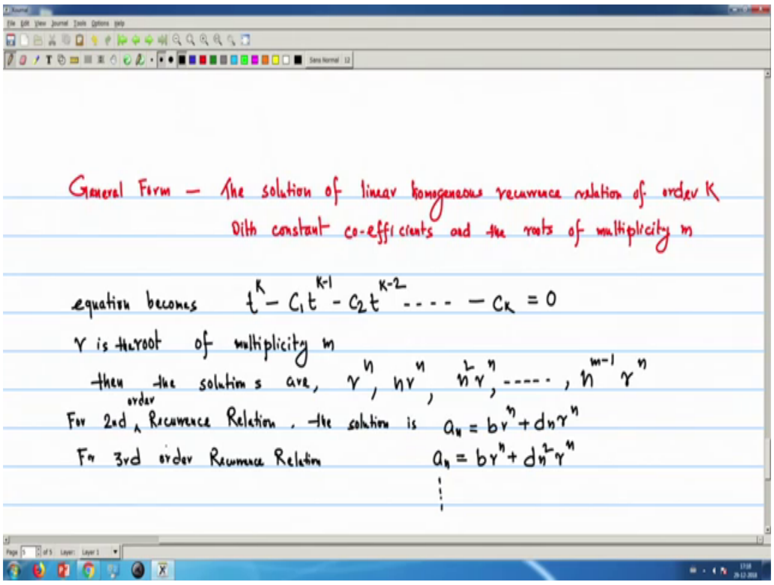
(Refer Slide Time: 16:01)



So, we get b into 2 to the power 0 plus d into 0 into 2 to the power 0 equal to 1 where n equal to 0 d 0 equal to 1. So, I get b this becomes 0. So, b equal to 1. Now, if I put d 1 equal to 1 then b 2 to the power 1 plus d 1 2 to the power 1 equal to 1; that means, I put n equal to 1 and d 1 equal to 1. So, this becomes 2 b plus 2 d equal to 1 already I we got b equal to 1 b equal to 1. So, 2 d equal to 1 minus 2 b equal to minus 1 so, d equal to minus half.

So, I get the solution that my solution is d n equal to, I get the solution dn equal to b 2 to the power n b 2 to the power n plus d n 2 to the power n equal to 2 to the power n minus half n 2 to the power n equal to 2 to the power n minus n 2 to the power n minus 1. So, this is my solution of the given recurrence relation this is my answer. Now, in the theorem we have considered the second order we have taken the second order equation. So, if we increase the order; that means, in general form if we take the solution of k order.

(Refer Slide Time: 18:47)



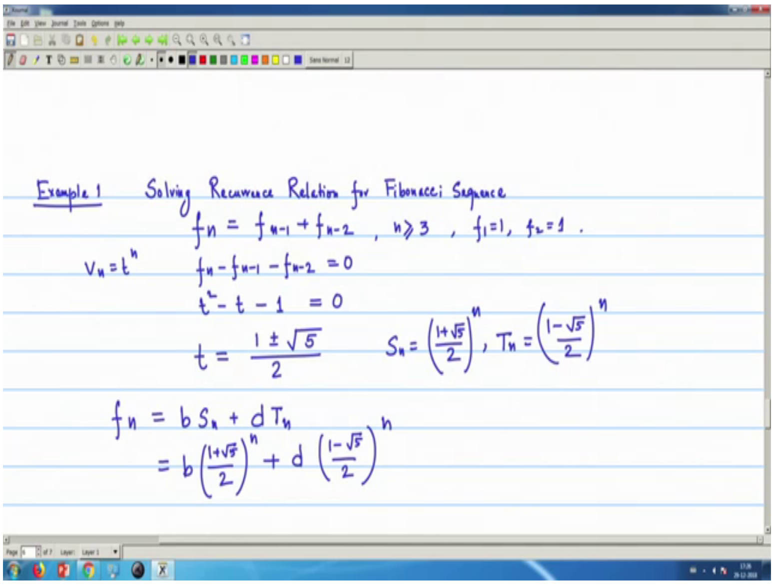
So, the general form that is the solution of linear homogeneous recurrence relation of order k with constant coefficient and when the roots are and the roots of multiplicity m; so, for second order equation we have seen that both the roots are r. So, I can trick the value like now I can write that my equation becomes t to the power k because it is a order k, minus C 1 t to the power k minus 1 minus C 2 t to the power k minus 2 minus C k equal to 0.

And we will be getting the roots are or the roots or roots of multiplicity m or r is the I write the r is the root roots of multiplicity m. Then the solutions are r to the power n n r to the power n, n square r to the power n in this way of multiplicity m. So, I get n to the power m minus 1 r to the power n.

So, for second order recurrence relation, we have already seen the solution is a n equal to b r to the power n plus d n r to the power n. Similarly for third order n, we will be getting that a n equal to b r to the power n plus d n square r to the power n like that and you can continue when the order will increase. So, this is the general form we get. Now so, we have read the two type of solution techniques that mainly for solving recurrence relation linear homogeneous recurrence relation, with constant coefficients when the roots are equal or the roots are not equal.

Now, we see that some linear homogeneous recurrence relation earlier we have seen that how to frame the recurrence relation or as a recursive function also, now we see that how we can solve those recurrence relation. So, one such equation is our Fibonacci sequence. So, we know that. So, I take the example.

(Refer Slide Time: 25:07)

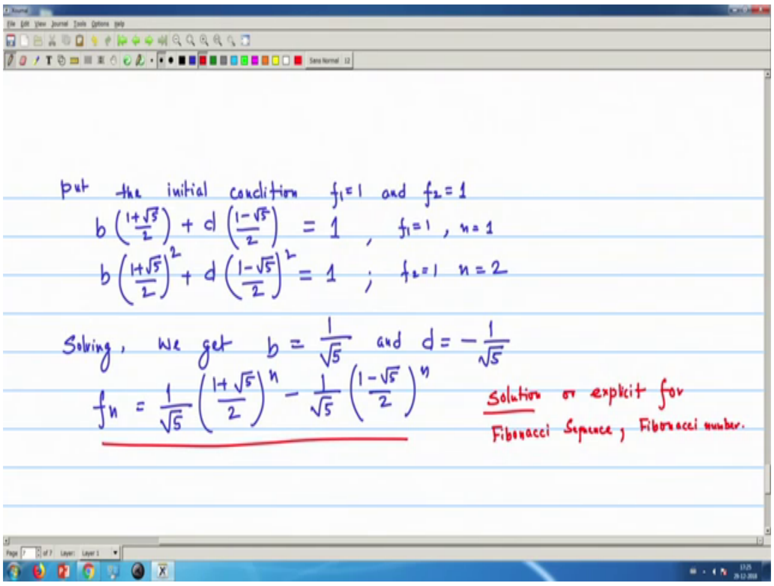


I take this example of solving linear homogeneous recurrence relation. So, this is relation is called; so, solving recurrence relation for Fibonacci sequence ok. We know the sequences f n equal to f n minus 1 plus fn minus 2 and for n greater than equal to 3, I write f one equal to 1, f 2 equal to 1. So, if I put that the equation or the solution is of form b n equal to 2 to the power n we get, first we take f n minus f n minus 1 minus fn minus 2 equal to 0. So, I get t square minus t minus 1 equal to 0.

So, if I put minus b plus minus root over b square minus 4 ac by 2 a. So, this t becomes 1 plus minus b square minus 4 ac becomes 5 divided by 2; 2 into 1. So, t equal to 1 plus 1 plus minus root 5 by 2. So, I can write the solution f n equal to I know that b S n plus d T n and here I can write S n equal to 1 plus root 5 by 2 and T n equal to 1 minus root 5 by

2. So, this becomes is equal to b 1 plus root 5 by 2 plus d 1 minus root 5 by 2. Now if I put the initial conditions f 1 equal to 1 and f 2 equal to 1.

(Refer Slide Time: 28:19)



So, first I put of the initial condition f 1 equal to 1 and f 2 equal to 1 we get we get b into 1 plus root 5 by 2 plus d into 1 minus root 5 by 2 equal to 1 when putting f 1 equal to 1; that means, n equal to 1 and b into 1 plus root 5 by 2 square plus n equal to 2. So, 1 minus root 5 by 2 whole square equal to 1; that means, I put f 2 equal to 1; that means, n equal to 2. So, if I solve these 2 equation; some solving we get we get b equal to 1 by root 5 and d equal to minus 1 by root 5 solving the log 2 equations.

So, I can write fn equal to 1 by root 5 1 plus root 5 by 2 to the power n minus 1 by root 5 1 minus root 5 by 2 to the power n. So, this is my solution. So, this is my solution of the explicit formula for solution or the explicit formula for Fibonacci numbers. You can write Fibonacci sequence of Fibonacci number that is Fibonacci numbers.

This becomes S n is 1 plus root 5 by 2 to the power this is n, 1 minus root by 2 to the power n ok. So, we have seen that theorems that how to solve the linear homogeneous recurrence relation when both roots are equal or the roots are different, when we have considered second order recurrence relation and we have seen some of the examples like the Fibonacci number, how to solve these sequences. So, next lecture we will see how we can solve the non-linear or linear non homogeneous recurrence relation.