Discrete Structures
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Lecture – 25
Relations and Their Properties (Contd.)

So, in the last lecture we have learnt the partially ordered set and today we will see the some properties of partially ordered set or poset we called.

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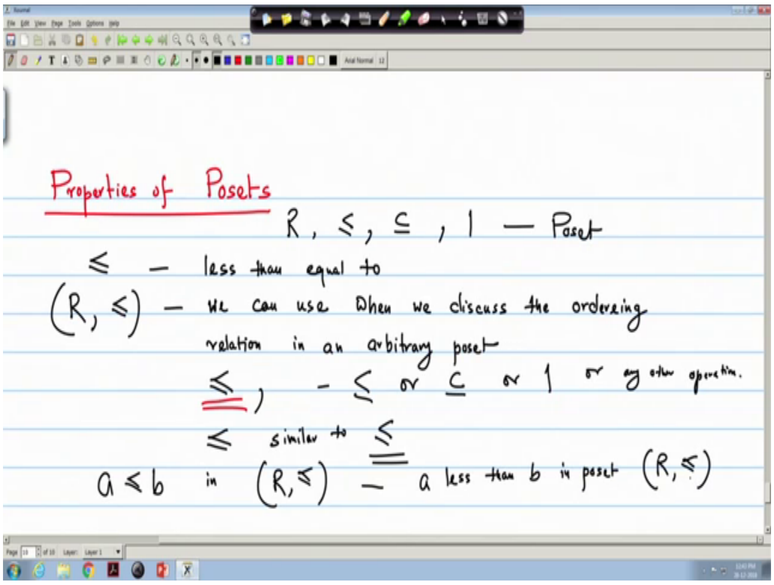

So, the and we will see that how it forms a algebraic structure which is very important in practical applications; like the information flow or in Boolean algebra called lattices. So, before we introduce latticed first we see the properties of posets, the partially ordered set. Before these we introduce a notation normally we used for posets. So, if we remember that when we have learned the posets the partially ordered set we have seen that normally the relation with the operation a relation R with operation less than equal to or the if we consider set inclusion the subset or the divisibility with these operations they form a poset R is a poset ok. So, I can write that this become this is a poset.

Now, this is different operations. Now, if we use a notation like this it is not always it represents that it is less than equal to less than equal to, but we define that R with this notation is a symbol, this is a symbol that we can use when we discuss the ordering relation in an arbitrary poset. Now; that means, this relation this can be less than equal to or this can be set inclusion or this can be divisibility or any other operation. This does not, why we have used the similar type of notation that less than equal to? So, if we write this thing that we use this notation which is similar to less than equal to because, that normally the posets are very easy to understand with the operation less than or the relation; that means, that order set and that is why it is a similar type of thing.

So, now onwards we will be using and this notation to represent an arbitrary put it posets and we will also use the term less than equal to, but we remember we must note that this is not always less than equal to; this can be any other operation also regarding one arbitrary poset ok. So that means, if I write a b; that means, we will be telling that in a relation in a poset R we will be using the term that a less that b in relation or in poset R this ok. Now, we see some of the properties of poset. So, this is a convention that we will be using, now first we see some greatest and least element of a poset.

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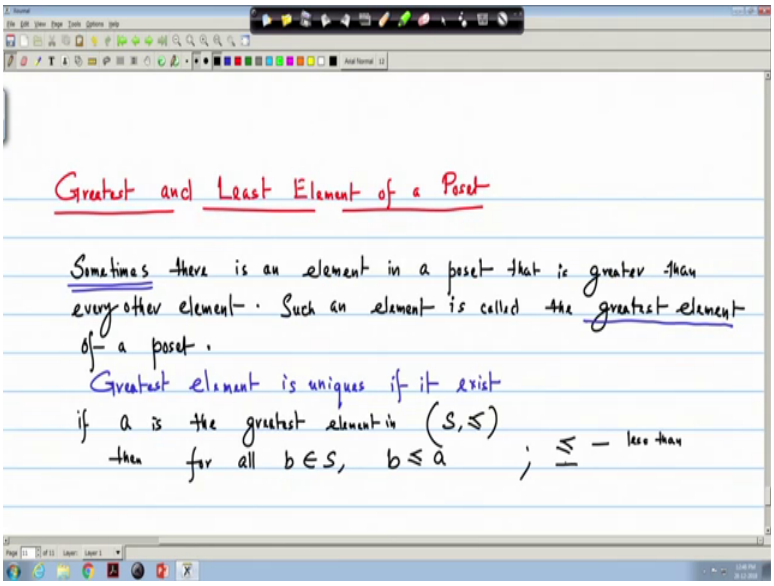

So, it define greatest and least element of a poset. Now sometimes if we or define that sometimes there is an element in a poset that is greater than every other element of the set. So, such an element is call the greatest element of a poset. Now, we must note that sometimes we get sometimes not always it exist we must note this term.

So, we can write that this element is unique and this is called the greatest element. So, I can write that greatest element is unique if it exist if it exists. So, if we call that is the greatest element if a is the greatest element in the poset S, then I can write for all elements b belongs to S that b is less than a because, just now we have defined the notation this as the as the as if we always tell this is less than as we have defined or we have introduced this notation.

So, this is my greatest element. Now, similar way I can define my least element for my for my list element that again sometimes we get an element an element that is less than all the elements of the posets.

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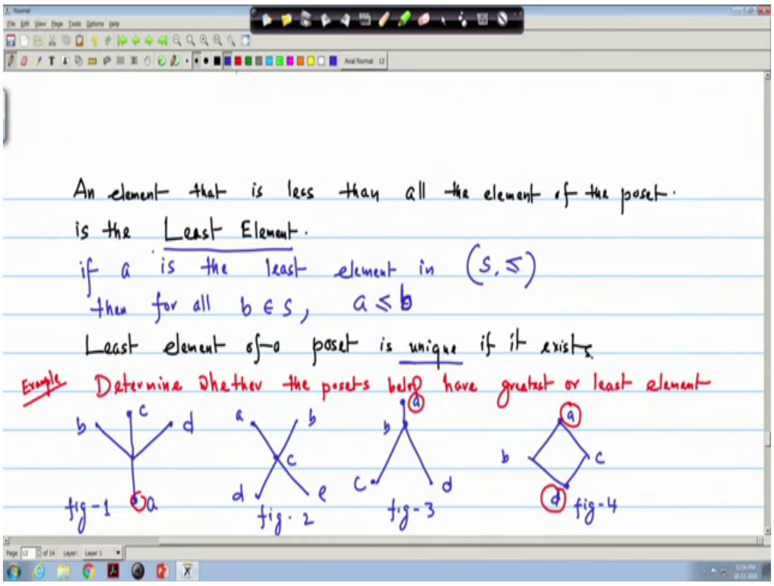

So, we can write that if so, an element that is less than element of the poset is called the is the least element. So, if is the least element in the poset S then I can write then for all element b belongs to S a is less than b. And again for least element also that least element is also unique of a poset is unique if it exist ok. So, it is unique if we take some example let us see some example that we determine whether the posets below I have greatest element have greatest or least element.

Now, we know that has a diagram is a representation of poset which is that minimize diagram. So, I give the Hasse diagram so, I write a Hasse diagram like I give an odd a b c d. So, this is my figure 1 this is my figure 1 or I give say a b c d e this is one Hasse a diagram of a poset figure 2 or reverse of the figure 1 a b c d figure 3 I get this type of a b c d figure 4. Now, we see that for figure 1 it has some least element it has some least element it has some least element a because, a is less than all elements of bcd. So, it has a least element least element I will write.

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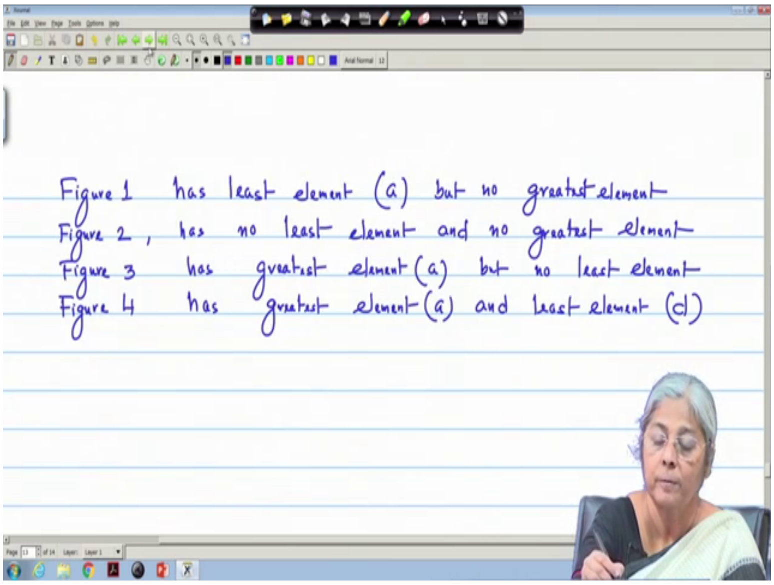

So, figure 1 has least element a, but no greatest element. If we see the figure 2 no least element no greatest element, here greatest element is a, but there is no least element. So, for figure 2; for figure 2 has no least element and also no greatest element. Figure 3 just now we have seen figure 3 has greatest element see, but no least element see if you got 4 figure 4 has figure 4 has greatest element as well as least element here it is it is.

So, figure 4 as greatest element a and least element d this is greatest and least element, but if it exist not always the greatest and least element exist as we have seen for figure 2 has no least or greatest element. Now, we can define the some greater some lower bound and upper bound; that means, sometimes.

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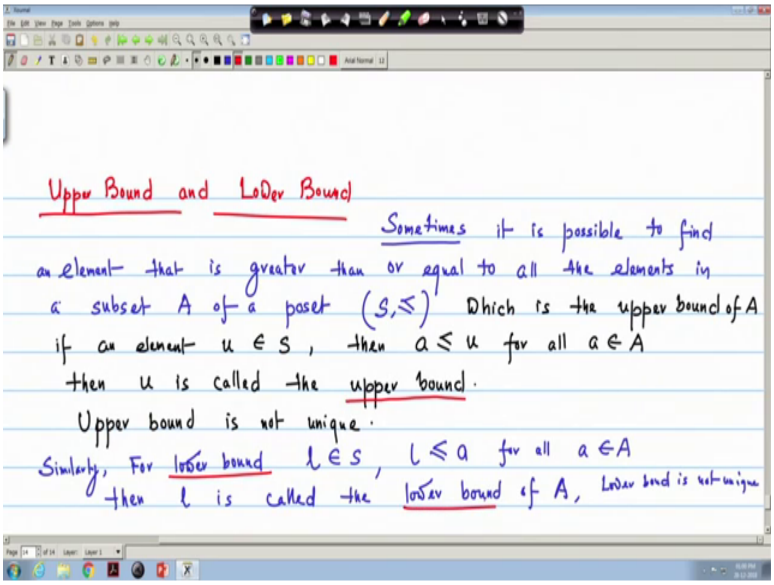

So, we can tell the upper bound, you can define the upper bound and lower bound. Since, we have just now we have seen the not always the greatest element or the least element exist. So, sometimes it is possible we can define like that again here sometimes we remember sometimes it is possible to find elements that is greater than or equal to all the elements in a subset A of a poset S of a poset S.

Now, if and this element is called the upper bound greater than all the limits and this is this element which is the upper bound of a. So, if an element u say u belongs to S then a is less than u for all a belongs to A because, if we consider a is the subset of the poset and we see that a is less than equal to u for all u belongs to S then u is called the then u is called the upper bound. And we will see upper bound is not unique another is upper bound is not unique there may be more than 1 element in the poset S where these property holds.

Similarly, we can define the lower bound similarly for lower bound we can write for lower bound we can write an element l belongs to s l is less than a for all a belongs to a the subset a capital A. Then l is the, then l is called the l is called the lower bound of a and again lower bound is not unique lower bound is not unique. So, we get some upper bound of a sub set a we get some lower bound of a get the lower bound of a subsidy we take some example. So, that it will be clear how to get the upper bound and lower bound from the Hasse diagram of a poset.

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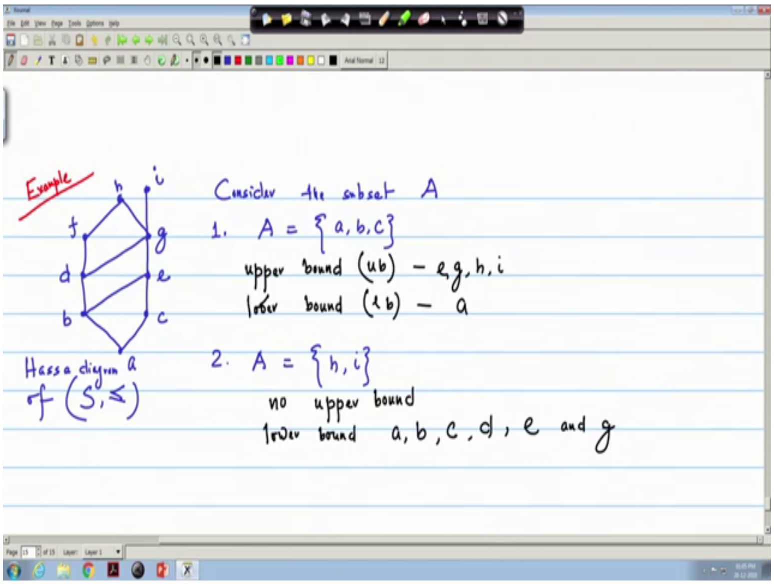

So, we take an example ok. So, poset it draw the some Hasse diagram give some note name. So, this is a b c d e f g h i we connect this is also some diagram is there. Now, we consider the. So, this is one poset calculation S and this is the Hasse diagram of this as a diagram of the poset S. Now, consider the subset a we will consider different subsets a say for the 1st case I consider A equal to sum we consider abc and we see that what will be the upper bound. So, upper bound we called u b is e g h i and lower bound lb is a.

Similarly, if I consider the subset as h i then we see that there is no upper bound because h and i so, there is no upper bound. That means, we do not get any such element in the subset a which satisfies the property, but there is lower bound that in this case that lower bound is lower bound is since it is a h i so, lower bounds are a b c d e g. So, all will be the lower bounds so, lower bounds are lower bound a b c d e and g all are one.

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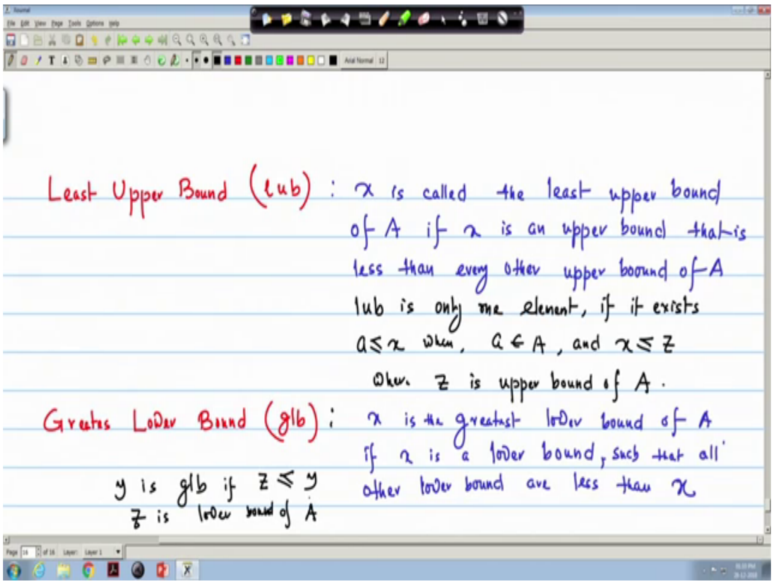

Now, we define the least upper bound l u b that x is called the least upper bound of A if x is an if x is an upper bound that less than every other upper bound of A. Since just now we have seen that there can be more than one elements in the upper bound. So, now, we are finding what is what is the least element with respect to that relate poset that is less than then that can be defined as the least upper bound.

So, it is one element only one element. So, l u b is I can write so, that is less than every other bound of a. So, I can write that l u b is only one element; that means, if it exist only one element if it exist because, we have seen that even upper bound may not exist then least upper bound; obviously, will not exist. And in notation we can write is less than x when a belongs to A and x belongs to or x is less than z where z is upper bound of A z is upper bound of A.

Similarly, we can define the greatest lower bound we call normally g l b of a and notation by notation. So, x is the x is the greatest lower bound of A if x is an x is a lower bound that is greater among or we can write all other greater bounds are I can write such that all other lower bound are less than a less than x. And I can write in this way simply that y is the y is glb greatest lower bound if z less than y and z is lower bound of a of A.

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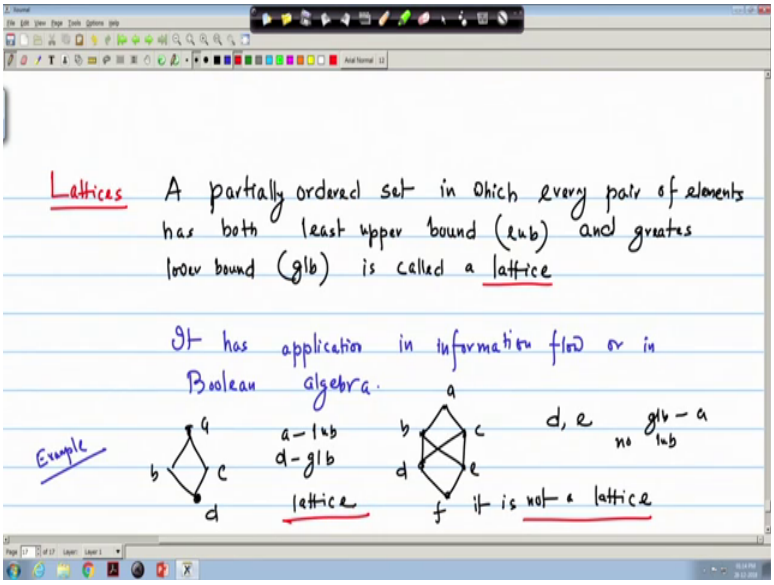

Now, with these properties we can define the lattice as the. Now, we can define the lattices that a partial order set in which every pair or every pair of elements has both least upper bound we called l u b and greatest lower bound called g l b is called a lattice. And lattice is a very important it is algebraic structure and which is application both in as it has application and information flow from in communication or in Boolean algebra.

So, if I draw a Hasse diagram some examples if we see if we see some examples we draw some Hasse diagram see it is just now we have seen that both it has both a as l u b d is so, it is a it is a lattice it is a lattice. If I consider I give it is not a lattice since if I consider b c or d e for d e it has g l b is a, but no l u b. So, it is not a lattice, but it is it is a lattice.

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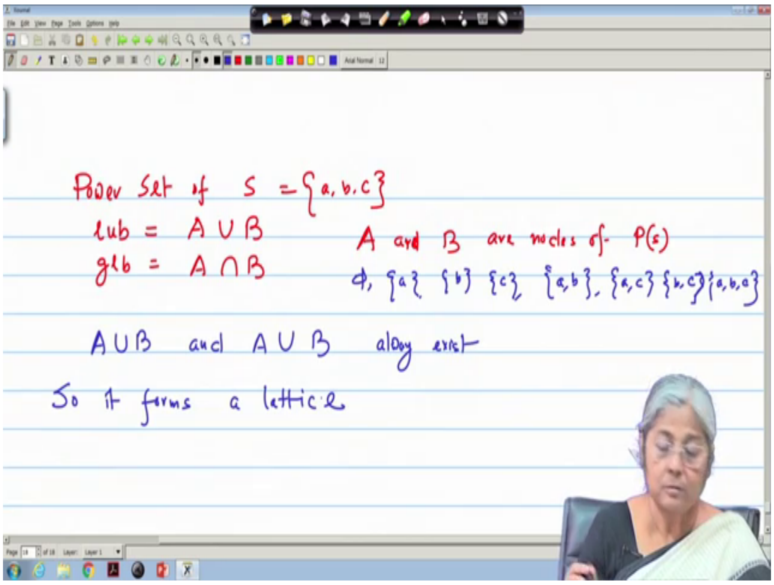

So, if we consider that earlier we have considered some power set if we remember the examples of power set of a S a b c. Then it has always it has some least upper bound and which is actually if I consider 2 nodes if I consider the power sets. So, l u b is the A union B and the g l b is the A intersection B where A B A and B are the nodes of the P s power set or the poset because, we have seen that it is poset and if we remember the nodes are that empty set then a then b c a b a c b c and a b c and a b c.

So, always this exist the A B and A union B; that means, g l b and l u b always exist. So, it forms a so, it forms a lattice it forms a lattice this is somehow other examples that we have seen that say divisibility.

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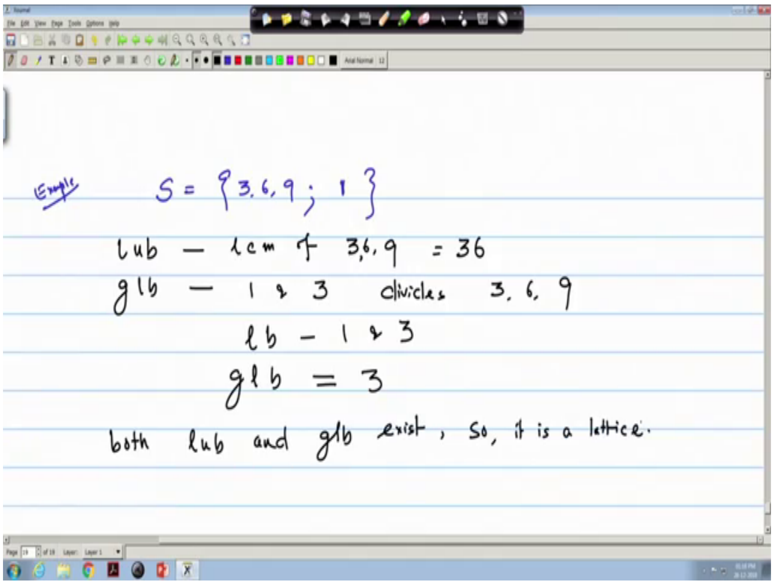

If we consider examples of some set S and say 3 6 9 is a set and the relation is the divisibility or the relation we can see is the divisibility this is a poset a. And then we can see that l u b the least upper bound is nothing, but the lcm of 3 6 9; that means, there which is a multiple of 3 6 9 and is 36. So, it exist and greatest lower bound is in this case 1 and 3 divides all the elements divides 3 6 9.

So, lower bounds are 1 and 3 so, greatest lower bound g l b is 3. So, both lub and glb exists exist and it forms a lattice so, it forms a. So, we see that earlier the examples we have seen that posets and that some of the posets with some operations like here we have seen the division or the power set they forms a lattice. And lattice is a very important algebraic structure that has uses in information flow in communication and the Boolean algebra etcetera. So, with this we finish this lecture of the posets and its property and that how it forms a algebraic structure like lattice.