Discrete Structures

Prof. Dipanwita Roychoudhury
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture – 24
Relations and Their Properties (Contd.)

So, last lecture, we have read the partial ordered set and their some practical applications. Now, these lecture we will see that how the partial order set can be represented pictorial. And finally that becomes a algebraic structure, and we call this is Hasse diagram. So, we read Hasse diagram opposite here.

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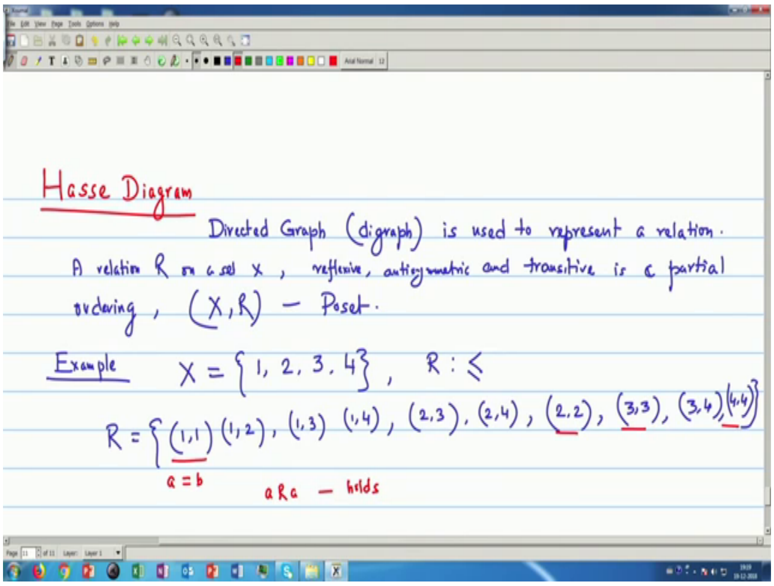

So, today we will read the Hasse diagram. So, explain the Hasse diagram with some example. If we remember that we have read the directed graph of a relation that means, a relation can be directed graph or not we call digraph is used to represent a relation.

Now, a relation on a relation R on a set x, with the property reflexive, anti-symmetric, and transitive is a partial is a partial ordering. And the set is X is X with R is called a partial ordered set or poset. Since, it is finally a relation on a set X, so we can represent this thing by a digraph.

Now, we take one example first that same example if I consider that, we have a set X having four elements say 1, 2, 3, 4, and relation we take is the less than equal to relation is less than equal to.

Now, the if we remember the diagraph, so relation we remember the relation will be the pairs 1, 1; 1, 2; 1, 3; 1, 4; then 2, 3; 2, 4; since it is less than 2, 2 less than equal to then 3, 3; 3, 4; 4, 4.

So, first we see that 1, 1 is there, since it is less than equal to since it is equal to so a equal to b. 2, 2 will be there, 3, 3 is there, 4, 4 is there, so that means or in other way I can tell that it is since it is reflexive it is reflexive. So, all 1, 1 a a R a it a R a it holds.

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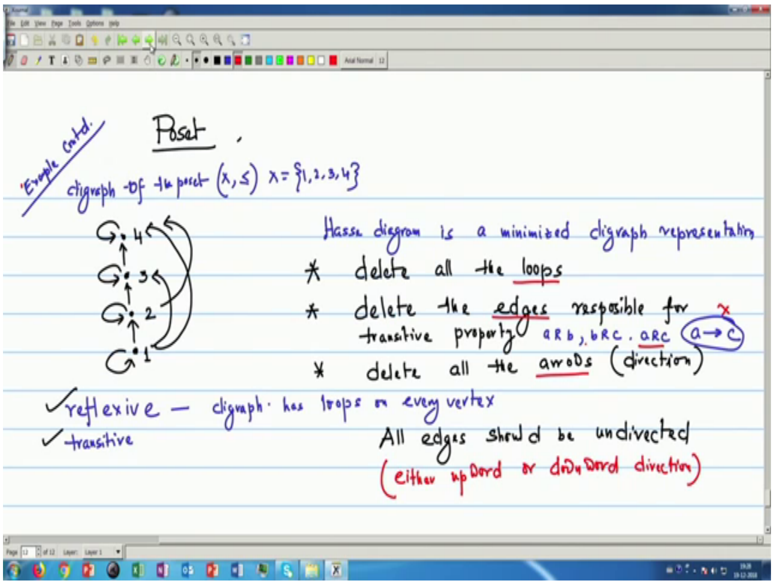

So, if we draw the diagraph, now if for this example if, the example continued, so if I draw the digraph. And this time we are giving that first thing is we know that for digraph that all elements will be a vertex, all elements will be a vertex.

So, this is my 1, say this is 2, this is 3, and this is 4. Since, it is reflexive. So, every time there will be a loop, since it is a R a holds. I can tell that since it is reflexive, it is a partial order, so it is reflexive. And it digraph has loops on every vertex.

Now, if I see that it is less than equal to, now I am considering some direction either upward or the downward direction see 1, 2; 1, 3; 1, 4. So, I can now put one directed edge that I can put 1, 2 1, 2 since 1 is less than 2, 2 is less than 3, 3 is less than 4, again 1 is less than 3, 1 is less than 4 also.

So, 1, 1; 2, 2; 3, 3; 4, 4; then 1, 2; 1, 3; 1, 4 what about 2, 2 less than 3, again 2 less than 4. So, this is my this is my digraph of this is my diagraph of the poset on the poset say X less than equal to and X equal to 1, 2, 3, 4[noise]; now, see since it is a poset, so always it is reflexive always it will be a transitive.

So, now we can minimize, so Hasse diagram is some minimized digraph. So, simply I can tell that Hasse diagram Hasse diagram is a minimized digraph representation, how we can minimize. See since it is poset, it is always reflexive. So, what I can do, I can I can delete all the loops, since it is a poset always it is true. So, we will not show separately.

Similarly, since it is a transitive. So, what I can do that if we only 1 to 2 or 1 to 2 to 3, then 1 to 3 always will delete. So, delete the edge or the edges responsible for all for transitive property. Finally, we will delete all the arrows. We will delete all the direction, all the arrow or the direction that means, all edges should be that means, since all edges should be undirected so all edges should be undirected.

Now, why we can do this thing, because we are considering we are considering only the poset we are considering the poset, and the digraph of a poset. So, poset is reflexive and transitive. So, we are not separately giving any representation for the property reflexive that is why, we delete we delete loops.

Since, it is transitive that is why, we delete the edges responsible for transitive property, what does it means that if it is a or c a R b, then b R c, then I will put a R c. So, a to c will delete these, we delete a to c. Since, we have already a to b and b to c. So, we do not give we do not give a R c a to c.

And now for this type of as diagram in the Hasse diagram always we give the arrow either in upward direction or downward direction since we give either upward direction or downward direction. So, I do not need any arrows for that. So, we delete all the arrows from here. So, in this way we can minimize; then how we can do that thing.

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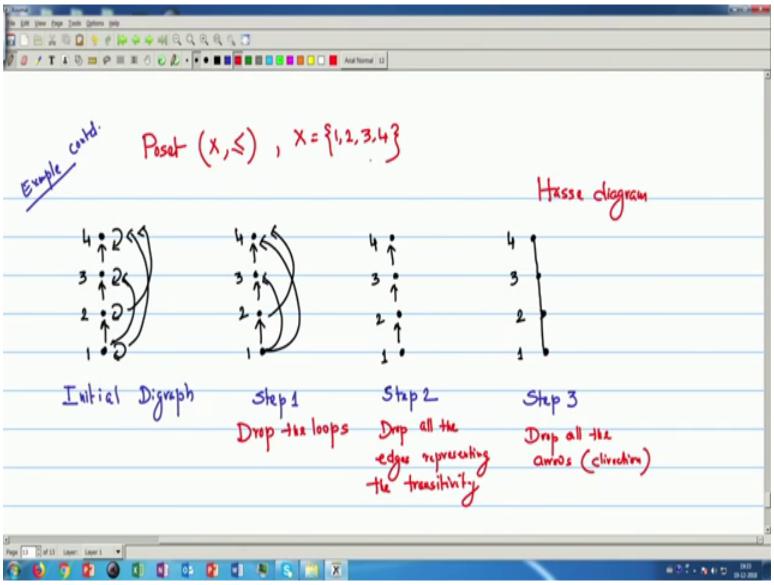

So, in the first step so Hasse diagram, so you we continue the same example. So, again if I quickly draw 1, 2, 3, 4, then I give 1, 2, 3, I give 1 to 4, then I give 2 to 4. So, what will be mine? So, this is my initial digraph. Then what will be mine step 1. We first drop the all the loops, so we drop all the loops. So, only I this is my step 1, and I can write that step 1 is drop the groups.

Now, what will be mine step 2? Then all arrows for transitive are all the edges for transitive property that means, we can drop 1 to 3, 1 to 4, and 2 to 4, so step 2 will be 1, 2, 3, 4. So, all the transitive thing we draw. So, this is my step 2, we drop all the edges representing the transitivity.

Now, my final step, we drop all the arrows. This is only connection. So, drop all the arrows or direction all the direction, and this is my final Hasse diagram, so this is my final step or this is my Hasse diagram. So, how we construct the Hasse diagram of the poset; poset X less than equal to, where X is 1, 2, 3, 4, because earlier we have shown that for this relation are less than equal to it is a poset. So, the in this way, we can draw the draw the Hasse diagram. And see that it is very minimized form and very simple form of the digraph ok.

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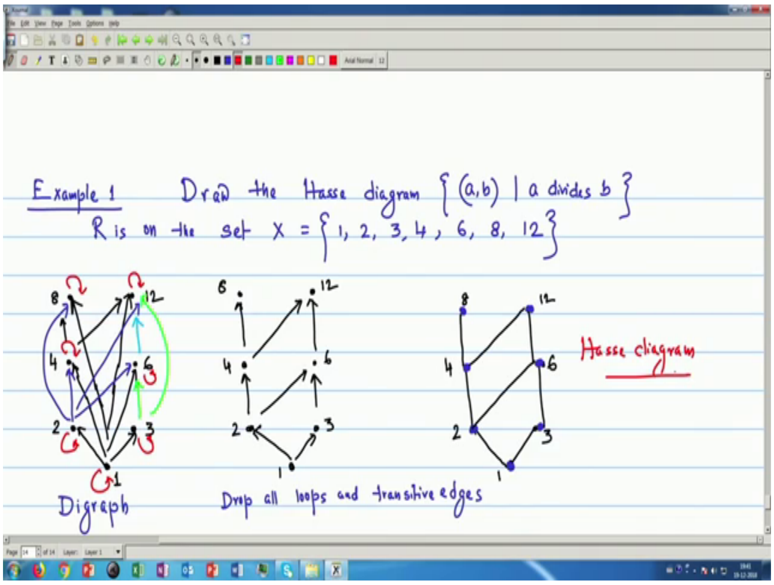

Now, we see for some other example. So, again that earlier we have seen that for division, so give the draw the Hasse diagram for the poset I write in this way a, b, and the relation is divisibility where a divides b. And on the set on the set the relational relation on the set R is on the set X equal to say 1, 2, 3, 4, 6, 8 12.

So why? Now, we know the procedure for drawing the Hasse diagram. So, first we draw the digraph. Since, I have a 6, 8, 12, 4 or 3, 7 nodes ok. We draw the digraph, I have 1, so 2, 3. So I, given this way 6; I draw this is 1 so it divides b.

So, 1 divides everyone, 2, 3, 4, 6, 8, 12. So, there will be some initially the directed edge to 1 to everyone, 1 to 2, 1 to 3, 1 to 4, 1 to 6, 1 to 8, 1 to 12. Then 2, 2 divides 4, 6, 8, 12; so 2 divides 4, 2 divides 6 ok, we can give a different color, so that it can I will give you a different color for 2 to 4; so that it will be clearly we can understand 2 to 6, then 2 to 8 and then 2 to 12.

Then, for 3, 3 divides only 6 and 12; 3 divides 6 and 12, I give a different color. 3 divides 6 and 3 divides 12, then 6 divides 12, then since every one divides a divides a. So, I have loops to every node or vertex. I have loops to every nodes, 4 I am not given 4 is 8. So, 4 divides 8, 4 divides 12, but 8 does not divide 12 or only 8 divides a 12 divides. So, this is my digraph.

Now, first thing I step 1 I draw, the all the loops. So, if I step 1 and step 2 together if I give, then the transitive edges also we can drop. So, I write if I giving in, I give 1. So, I have only, I will give only 1, 2, 3 then 2 to 4, 3 to 6, 4 to 8, 6 to 12; then I have I have 2 to 6, I have 4 to 12.

So, this time I have drop all loops and transitive edges and transitive edges. Now, finally we drop all the arrow diagrams. So, these becomes this becomes my finer that simply I draw 1, 2, 3, 4, 6, 8 and 12. Well, I can give the nodes blue; so that it will be clear and I have 4 to 12; I have 2 to 6. So, simply this is my Hasse diagram.

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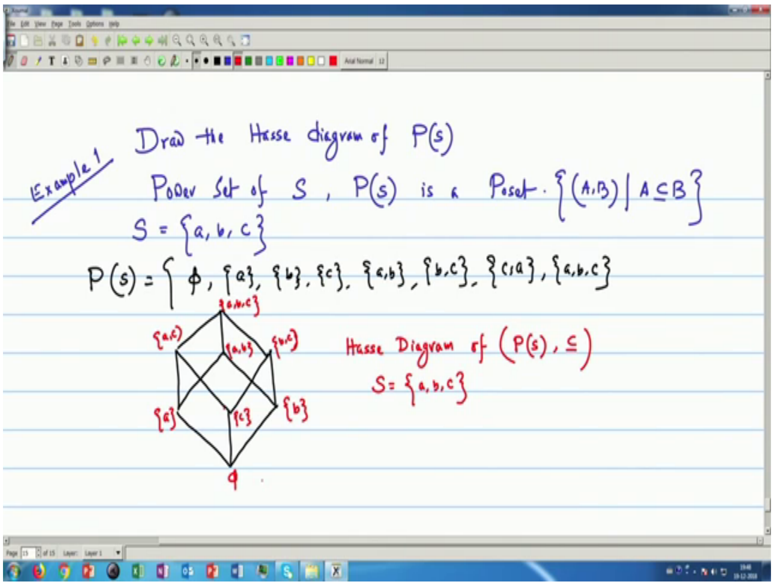

Now, we have if we see that in the last lecture, we have seen that our power set is a poset, some power set S is a poset is a poset.

Now, if we consider S the set S is 3 elements, then we can draw the Hasse diagram of p s. So, draw of power set s. So, if we remember; what are the elements of the power set see S is equal to a, b, c; we remember the elements will be 2 to the power 3, 8. And what are those elements, the power set, it will be the empty set that is now set the a, then b, c; a b, b c, c a and a b c, a b c.

And the relation here we have considered when it is a poset, the relation is the subset ok; that means, we have to show p s. And the poset is, poset is I can write A, B such that A, B or I can write A is a subset or inclusion, set inclusion ok.

So, how we can draw? We have 8 elements, and all the elements must be the vertex of the digraph. So, directly if I draw, then this will be always some upward direction, we can give so I can draw in this way. See I can give my the empty set as the bottom. So, I write my empty set or I write I write this is my null set. So, this is my a, this is my b, this is my c.

Now, a and c. So, this connection this must be my a,c. Similarly, c and b so this h this node must be b, c. Then a and b, so this will be a, b; and this will be the a, b, c. So, these are all my all subsets. And this is the simply the Hasse diagram if I these will be my Hasse diagram of the poset, Hasse diagram of p s. Relation is the set inclusion and S is equal to a, b, c. So, this is my poset of that thing.

Now, we can define some here see for this type of diagram always, since it is in upward or bottom downward direction that is why, we are omitting or we are deleting the arrows or the direction. So, we can define some maximal element or the minimal element. So, see which simply which is bottom element that is my minimum, and which is my top element that is my maximum element. So, how we can define some maximal or minimal? I can tell some maximal.

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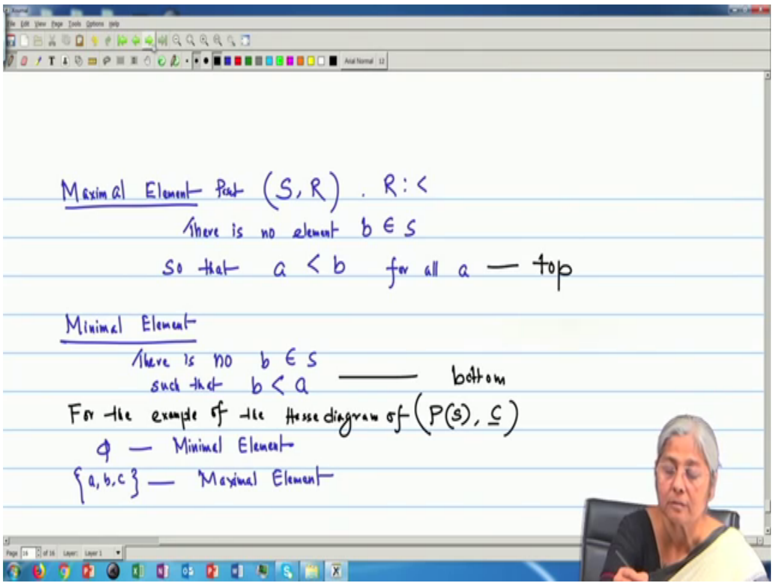

And this is for my Hasse diagram that simply I can tell, there is no element b belongs to S if you consider a element of the set s. I am telling the poset is the set is with a relation R, this is my poset. So, such that there is no element b belongs to a so that for so that a is less than b or that relation holds for all a, and this is for all a. So, obviously this is this gives me my top element. So, this is my top element. Similarly, the minimal element it will be that there is no b belongs to S such that b less than a, where r I am related say taking in less than. So, obviously this becomes my bottom element.

So, for the previous example that my phi, then I will set is my minimal element and that a, b, c the subset a, b, c is my maximal element. This is for my for the previous or for the example of the Hasse diagram of the power set S. So, if we just consider some Hasse diagram at random.

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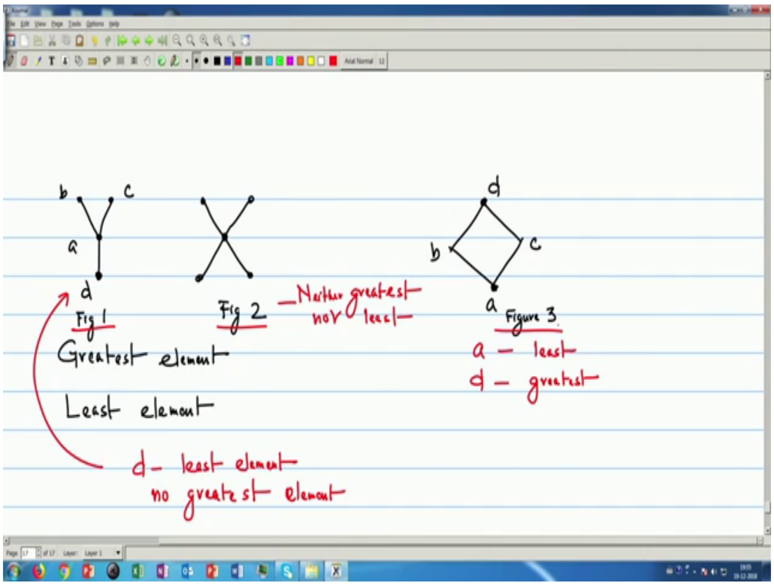

Say, I have I have some this type of say it is a, it is b, it is c or I can take some. So, here I can the previous example, we can tell that some it is called some greatest element or greatest element or the least element. Then I can tell that for this example say for so if say for figure 1, this is for figure 2. I can tell for figure 1, that for figure 1 that d is the least element, but there is no greatest element, but no greatest element.

Maximum elements are b and c, but no greatest element. Here for figure 2, so this is for figure 1. Then for figure 2, no greatest or neither greatest nor least. I can write neither greatest, nor least, but for our this type of diagram that we have seen in our power set Hasse diagram that we can tell that if it is a, b, c, d, then a is if a is least, and d is greatest element.

So, with this we finish this talk of Hasse diagram. And again next lecture we will continue this thing.