Discrete Structures
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Lecture – 23
Relations and Their Properties (Contd.)

So, we are discussing the relations. And last day we have seen a special type of relation, mainly it comes under the properties of relation is the equivalence relation. Now, we often use relations to order some or all of the elements of a set; and for this, we use some properties of relations. The three basic properties that we have read the reflexive, symmetric, we will see today some variant of symmetric, and the transitive property that are mainly used. But, here the relation is called the something called partial ordering.

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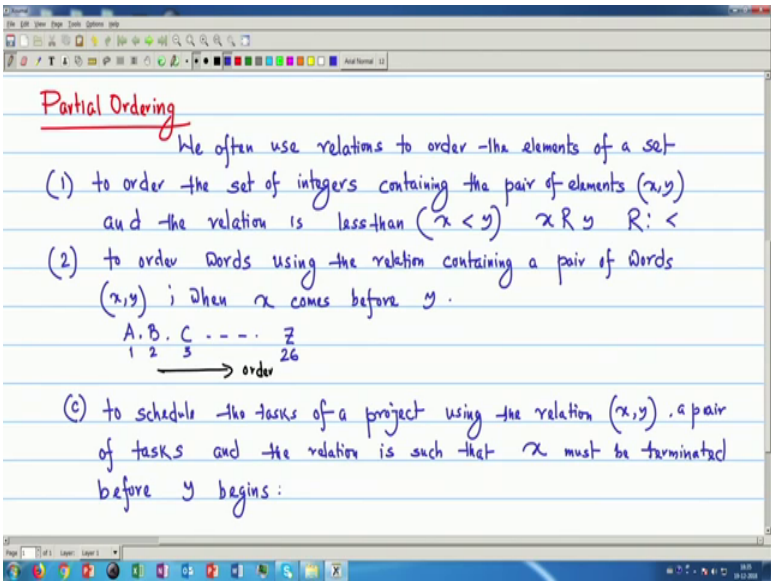

So, today we will see how relations are used to order the elements of a set. So, we will read the partial ordering or sometimes we call partial order ok. Now, when we use the relations to order the elements of a set, say if we see some example that we use relations to order the elements of a set, this is a general thing.

Now, as an example I can tell that to order the set of integers containing the pair of elements say x, y, and the relation is less than or greater than, relation is less than say x less than y. Normally, we called x R y, where R is the relation is the less than or we use the to order the words.

So, we write other example that to order words using the relation containing a pair of words say x, y when I think; when x comes before y. So, here if we consider our alphabet set say A to Z in order that means as if A is in position 1, B-2, C-3, and this is Z in 26, so this is in some in some order. And this ordering will govern that which letter comes before Y. Say X comes before Y, if X position is less than the position of Y, and here positions means that 1, 2, 3. And normally this is the ordering we use in dictionary, and that we will be again in details we will be describing later.

Now, another example we can think, say on more practical example that we can schedule to schedule the tasks of a project, again using the relation say a relation a pair x, y pair of tasks x, y a pair of tasks and the relation is and the relation is such that x must be terminated before y begins. So, for this type of examples, we can use relation. And this is some particular properties that mainly the reflexive, the anti-symmetric, and the transitive properties are used for that.

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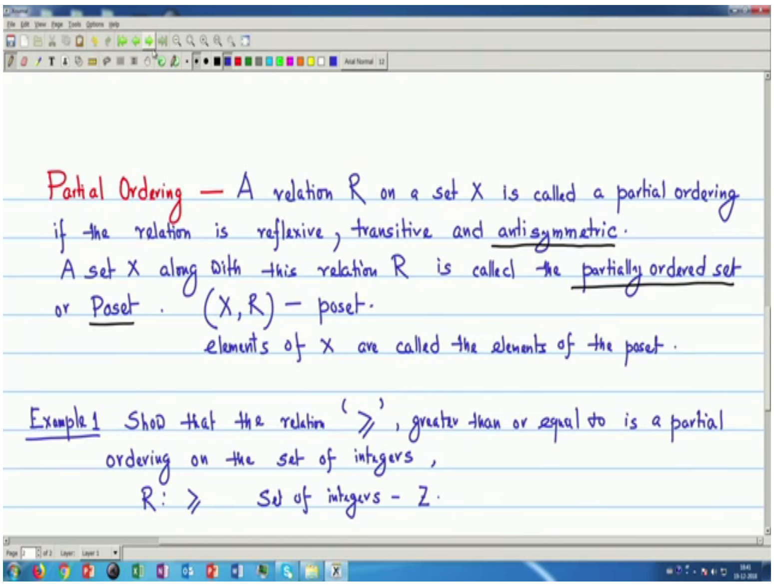

So, we define the partial ordering first. So, relation R on a set X is called a partial ordering, if the relation is reflexive, transitive, and anti-symmetric. Now, if we remember that instead of instead of anti-symmetric, if it is symmetric, then last day we have defined that as a equivalence relation.

Now, a set X along with this relation R is called the R is called the partially ordered set or more commonly the term is used called the poset correct. Normally, we denote that as the term here it is X, along with the relation R, we called this is my poset ok. And the elements of X the elements of X or members of X elements of X are called the elements of poset.

Now, we see first one simple example we see some example. So, show that greater than operation that the relation greater than equal to right is a partial order or partial ordering on the set of integers. Normally, the relation is relation is R; we is greater than and the set of integers the set of integers normally we denote as Z.

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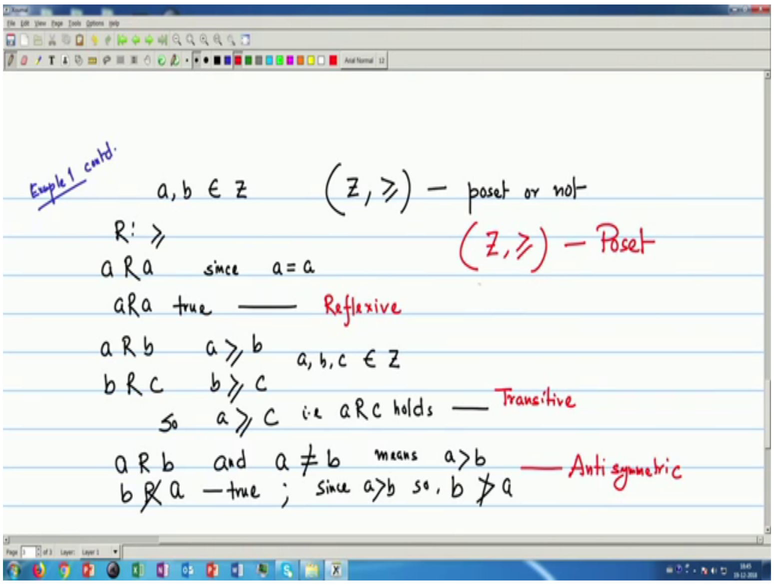

Now, see what are the properties, the relation holds? Previous example-1 continued. Now, if I consider a pair say a, b, so if or a, b belongs to z. So, my I am I am considering the set z, and the relation is greater than equal to, and I am checking, whether it is a poset or not ok. Now, a, b belongs to z. So, my relation is R is R is greater than equal to, so if I write a R a. Since, a equal to a, so a R a holds, a R a is true that means, my relation R is so a R a is true, so the relation is reflexive.

Now, if I consider that some relation a R b that means, a greater than equal to b. I consider b R c that means b greater than equal to c, where a, b, c a, b, c all belongs to z. Then since it is positive integer or a set of integers, so I can write that a so a greater than equal to c that means, a R c holds that is a R c holds, so it is transitive.

Now, if a R b if I consider a R b, and a not equal to b, since my relation is greater than equal to so that means, here I have to consider only greater than that means, a R b and a not equal to b means only a greater than b. So, now if I consider a greater than b that means, if we remember the definition of anti-symmetric that means, a R b and if a not equal to b, then b for this relation greater than b R a, this is it holds, because never b R a is true this is true. That means, if a greater than b, then b cannot be greater than since a greater than b, so b cannot be greater than a. So, this relation is anti-symmetric. So, my z, greater than equal to this is a these set of integers with the relation greater than equal to is a poset is a partially ordered set.

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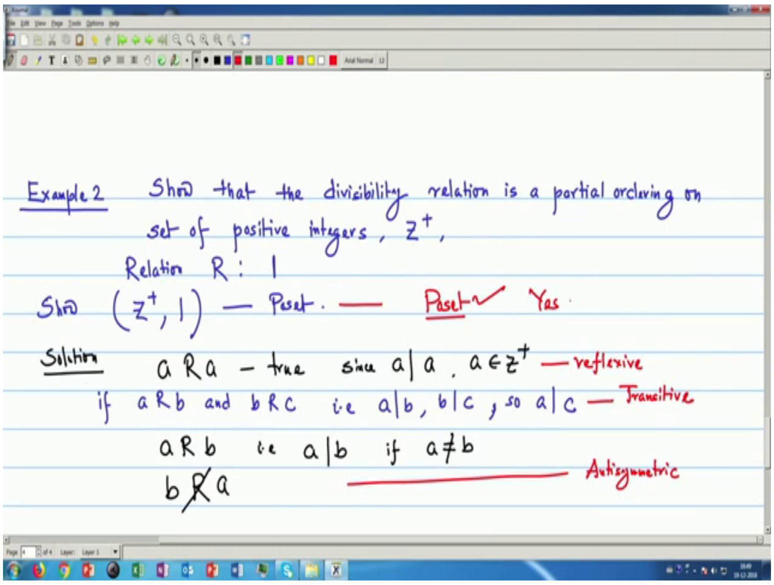

Now, we see another example. We give example-2 as the instead of the greater than equal to we take one relation the divisibility. So, you show that the divisibility relation is a partial ordering on set of positive integers, normally we denote as z plus. And the divisible the relation, we denote relation R, we denote as the by division. So, we have to show whether we have to show that the z plus with the relation divisibility is a show this is a poset.

Now, first we see the reflexive. See always a R a is true, since a divides a divides a is positive integer that a belongs to z plus, so it is reflexive, so it is reflexive. It is also transitive, because if a R b and b R c that means, that is a divides b, b divides c, so I can conclude that, so a divides c. So, it is transitive it is transitive.

Now, what about the anti-symmetric say if a R b holds that means, a divides b that is a divides b. So, if a not equal to b if a not equal to b, then b R a is never true that means, b is not in this relation a, so this is anti-symmetric so this is anti-symmetric. So, z plus is a so this is a proof, this is a poset yes it is a poset.

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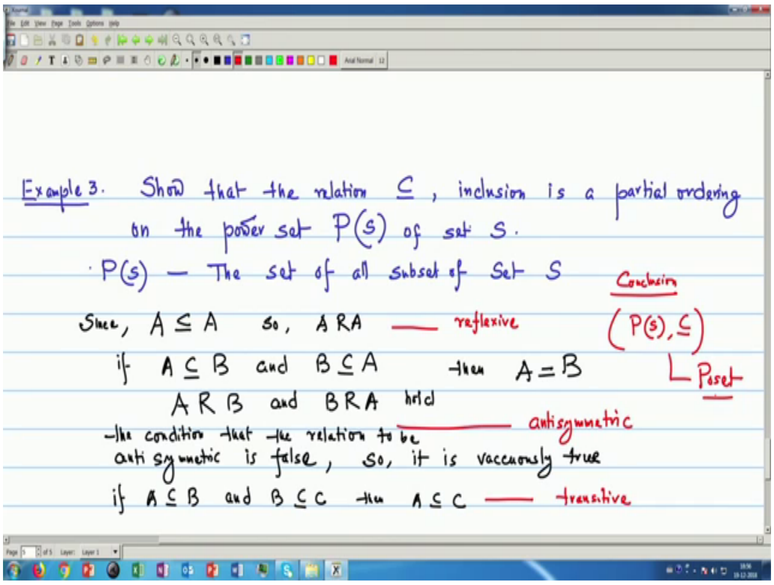

Now, we show a different type of example. In the last lecture, we have defined the power set which is nothing but the all the subsets of a set. Now, we will show that the relation inclusion set inclusion. So, show that the relation that means, if it is a subset, that is a inclusion is a partial order or partial ordering on the power set P S, where S is the or power set P S of set S. So, power set if I remember that power set is the set of all subsets of S of set S.

Now, we consider a subset A ok. So, if A is a subset of A or we know that it is always true that A is A subset of A, so a R a holds, which tells that it is reflexive. Now, if A is a subset of B and B is also a subset of A that means, a R a, a R b, and b R a if both hold, then we know that A equal to B.

So, if A not equal to B, then it does not hold; so that means, it is anti-symmetric, because if it is A equal to B, then actually the condition is false for anti-symmetric property, and so it is (Refer Time: 26:12) that means, for condition to be the relation one to be anti-symmetric is A not equal to B. So, then it is anti-symmetric. So, a R b and b R a hold, then A equal to B. So, it is then it is vacuously true for the property of or that the relation to be anti-symmetric or I can write the condition that the relation to be anti-symmetric is false, so anti-symmetric property, so it is vacuously true.

Now, we have if A is a subset of B, and B is a subset of C, so from the definition of subset, we know that a is A subset of C, so which directly gives that it is transitive. So, we can tell that my relation the partial order set. So, the conclusion is the partial order set on P S, with thus subset this is a poset, this is a partial ordering.

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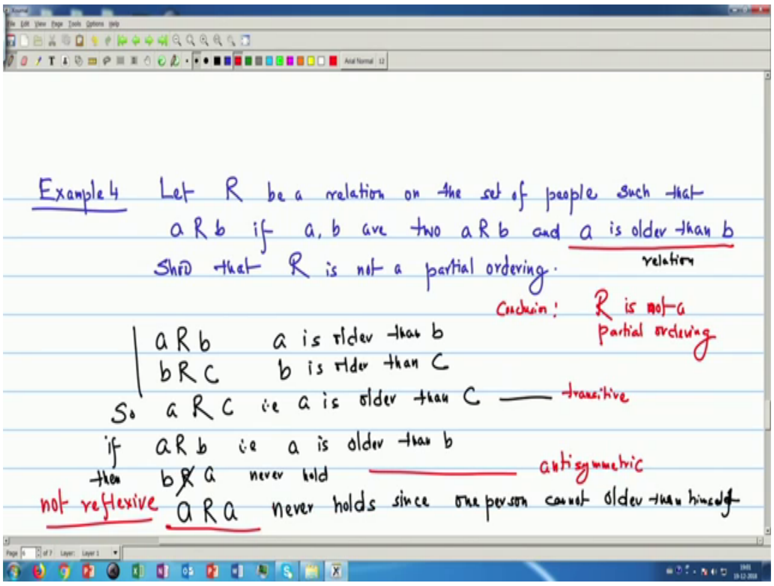

Now, we see one example, when the relation is not a poset. Let R be a relation on the set of people such that the relation holds that a R b if a, b are two people, and a is older than b. This is the relation a is older than b. So, this is my relation. So, we have to show that R is not a poset; R is not a partial ordering. Now, we see that the relation is older than.

So, first we see that we take three people a, b, c such that a R b that means, a is older than b. b R c that means, b is older than c, and which means that so a is a R c holds that that is a is older than c. So, it is transitive so it is transitive. What about anti-symmetric, because a is older than b, so b cannot be older than a. If a R b that is a is older than b, then b R a never holds, so that means, this is anti-symmetric this is anti-symmetric.

But, see that a is older than b, so a R a that one person can never be older than himself or herself. So, a R a it never holds a R a never holds. Since, one person cannot be older than himself or herself, so it is not reflexive so the relation is not reflexive not reflexive. So, it is not a the conclusion is conclusion is that relation it is not a R is not a partial ordering.

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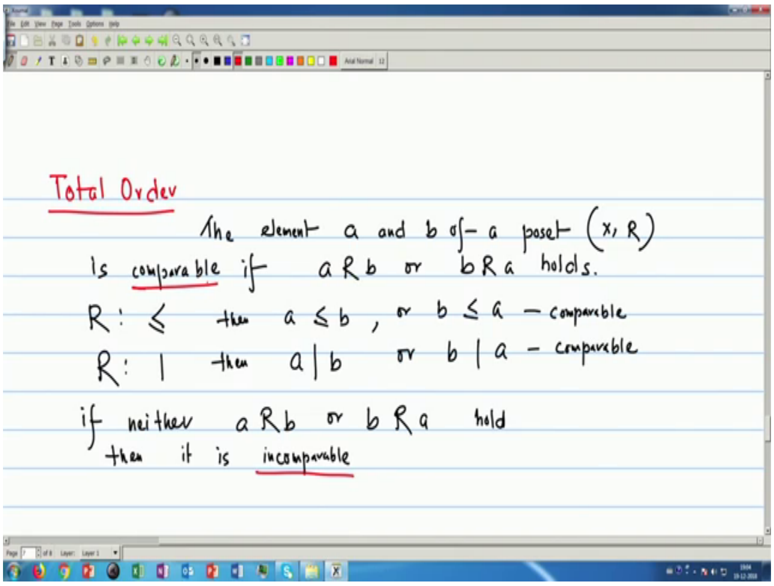

Now, we can define that in this relation one thing is called the total order, because we read the term partial order. So, what is total order for that we define a comparable or incomparable set. The element a and b of a poset; if it is a partial order set, say in X, R is comparable if a R b or b R a holds. That means, if I consider the relation R as the say less than equal to, then either then either a less than equal to b or b less than equal to a holds, then it is a comparable then this comparable.

But, say if I consider say relation is a division divisibility property, then a divides b or b divides a that is comparable, but not for all sets it may not be true. Then it is called then if it is not if that means, if neither a R b or b R a hold, then they then it is incomparable.

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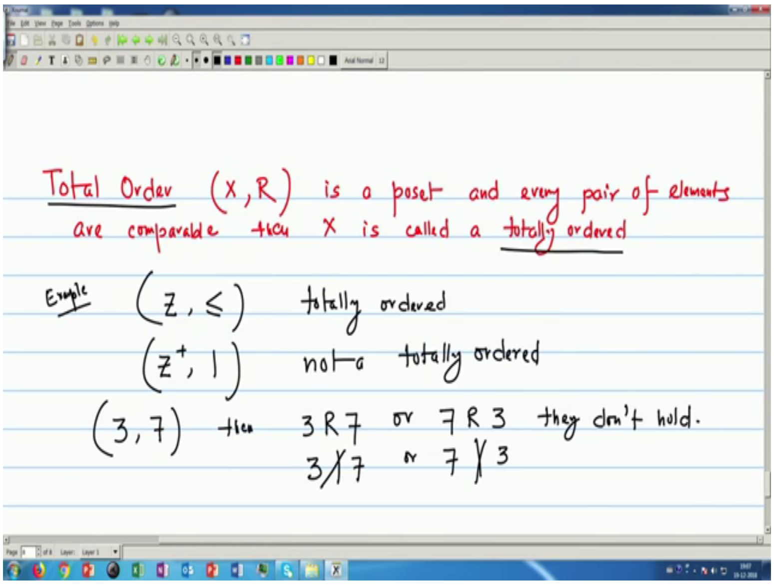

Now, now if I know the comparable and incomparable, so I can define the total order. See total order of set X, and I take a relation like this is a or I can give the simply I can give the relation R is a poset, and every pair or every pair of elements are comparable, then x is called a totally ordered.

So, just now we have seen that the set of integers set of integers z, and less than equal to this is a totally ordered set, because, if I consider any two elements in the set, they are actually comparable, but if I consider positive integer z plus, and division then not a totally ordered set. Since, if I consider say 3, 7 3 and 7 this pair, then 3 R 7 or 7 r 3 that means, 3 does not divide 7 3 does not divide 7 or 7 does not divide 7 does not divide 3. So, 7 R 3 in it they do not hold. So, it is not a totally order set.

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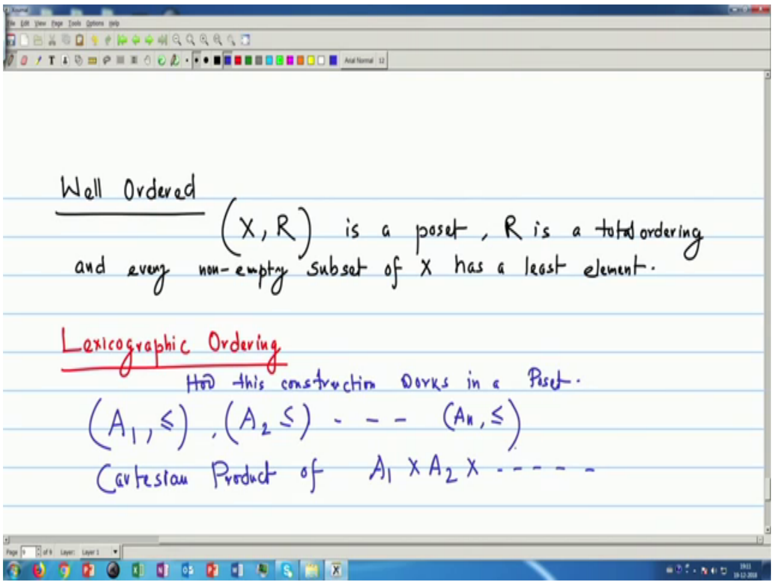

So, well ordered we can define as the so well ordered earlier we have defined. Now we give a different definition that if it is the set X with a relation R is a poset, where the relation R is a totally order, where R is a totally order or total ordering better I write R is a; R is a total ordering, and every non-empty set name subset of X has a least element ok.

Now, the most important application in poset is the lexicographic ordering in dictionary, we know this is lexicographic ordering. All of we know that in dictionary that normally the words are appeared according to the ordering of their letters. And we can show that how this a poset or how this constructions work in a poset so how this constructions work in a poset.

So, if we define say I have a poset A 1, say less than equal to; I have a another poset A 2, then I can take some Cartesian product of Cartesian product of these posets, because finally these are all sets with some relation. So, we can take Cartesian product of say A 1 cross A 2 cross say if I have some A n; now how I define this Cartesian product?

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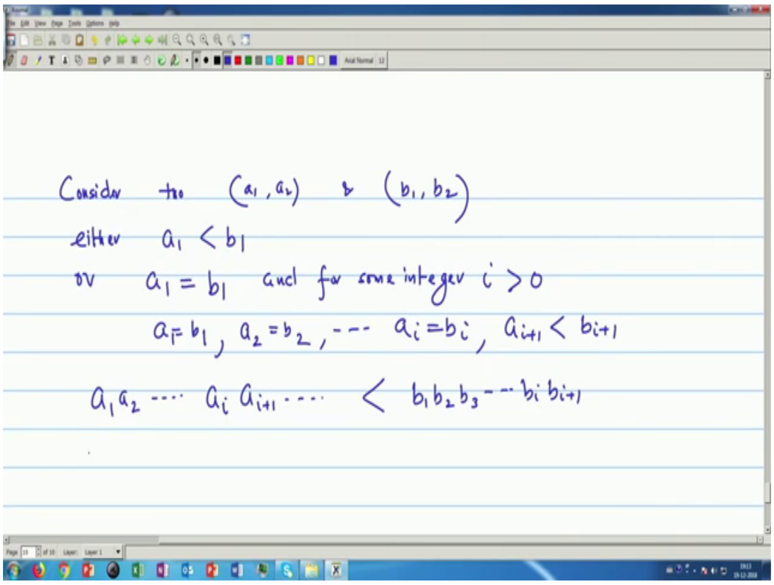

Say if so, I consider some two pairs consider two pairs a 1, a 2 and b 1, b 2. Now, either a 1 less than a 2 or a 1 equal to a 2, and for some integer I that I can consider that or some I greater than 0 that a 1 equal to a 1 equal to this is a 1 equal to a 2 this is wrong, a 1 equal to b 1 this is ok.

Since, we are considering a 1, a 2 and b 1, b 2. So, this is a 1 less than b 1 or if a 1 equal to b 1, and for some integer i greater than a 1 equal to b 1, a 2 equal to b 2 like a i equal to b i, then the next a i plus 1 less than b i plus 1. Then the lexicographic ordering is that the word say a 1, a 2 that a i a i plus 1 that is ordering is like that b 1 b 2 b 3, b i b i plus 1.

Normally, this is mainly used in our arrangement of dictionary. So, lexicographic ordering, which is nothing but the application of poset that is mainly used in our dictionary. So, how this relation and the property are used for practical purposes, this is one of the examples.