Discrete Structures
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Lecture – 22
Relations and Their Properties (Contd.)

So, we are discussing about the Relations and Their Properties. We have read the different basic definition of relation. We have seen some basic three different types of a relation; mainly the reflexive, the symmetric and the transitive. And, today we will read that some other type of relation and which mainly we consider as if the properties of the relation what type of properties it holds.

So, we will first consider the equivalence relation.

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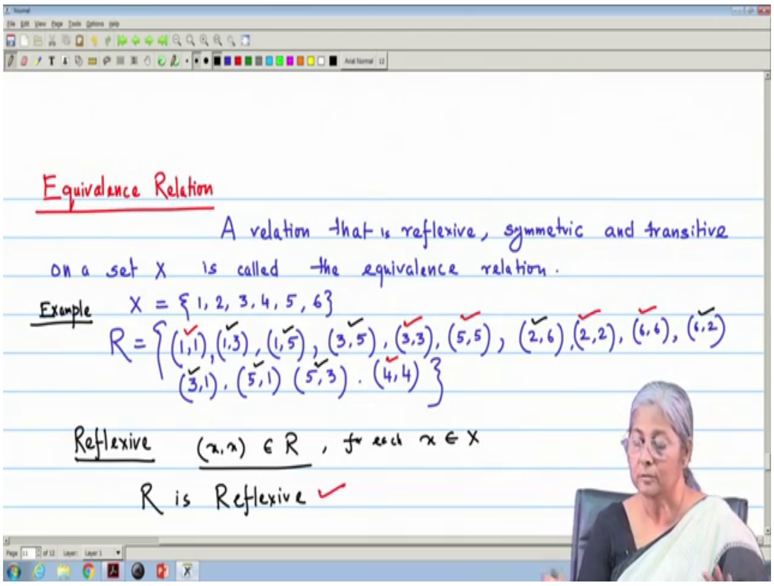

We will consider the equivalence relation. Now, since we know already the three basic type the reflexive, symmetric and transitive. So, with respect to that we define that a relation that is reflexive that reflexive, symmetric and transitive on a set X is called the equivalence relation ok; that means, a relation R which is a reflexive symmetric and transitive.

We see one simple example we consider a relation R where the relation is given and say the relation R on a set x and that is set having elements 1, 2, 3, 4, 5, 6 and the relations are given 1, 1; 1, 2; 1, 3. So, I take 1, 3; 1, 5 then 3, 5 I take 3, 3 I take 5, 5 then I take 2, 6; 2, 2; 6, 6; 3, 6 2 I also take 3, 1 also take 5, 1; 5, 3 and then say 4, 4. So, I have taken this relation is given.

Then first we see whether it is a check for its whether this is reflexive or not. If we remember that the definition of reflexive or x, x belongs to R. So, for this one we see that 1, 1 is there, 2, 2 is there, 3, 3 is there, 4, 4 is there, 6, 6 is there, 5, 5 is there. So, for all; that means, for every x for every x belongs to X for each x belongs to X, x, x belongs to R, so, it is reflexive. So, it is reflexive relation is so, R is reflexive.

Now, we check for symmetric we check for symmetric.

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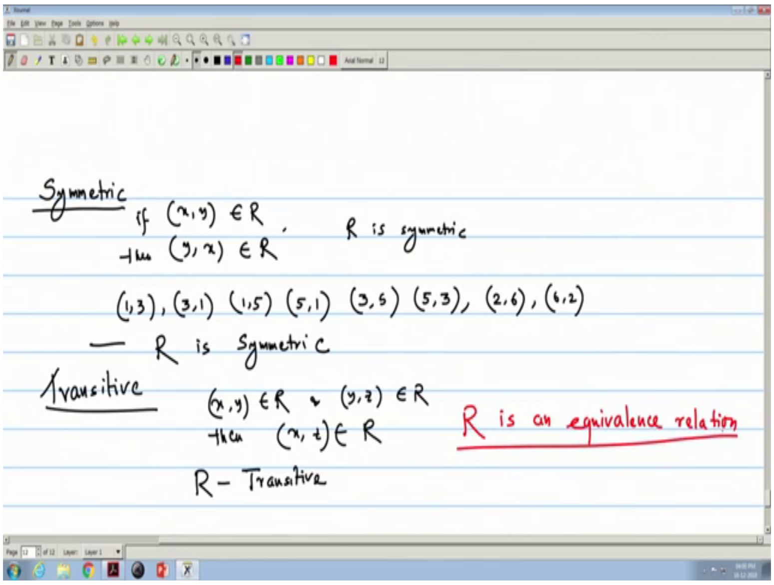

So, symmetric if x, y belongs to R then y, x if y, x also belongs to R then it is R is symmetric. Now, we check for that thing. So, we see that 1, 3; 3, 1; 1, 5; 5, 1. So, we see that 1, 3 here 1, 3; 3, 1; 1, 5; 5, 1; 3, 5; 5, 3 then 2, 6; 6, 2.

So, if we see this is 1, 3 in different color 1, 3 then I see that 3, 1 is there, 1, 5; 5, 1 is there, 3, 5 then 5, 3 is there, 2, 6 then 6, 2 is there. So, I can tell that this is symmetric relation. So, R is R is symmetric. Now, we have to see whether it is transitive or not.

So, in the relation if I see that x, y belongs to R and y, z belongs to R then if x, z is also belongs to R then it is transitive. So, we see the see that 1, 1; 1, 3. So, 1, 3 is there 1, 3 is there then 1, 1; 3, 5. So, we should check that 1, 5 is there similarly if I check that 1, 3 we do not have any other such cases, 2, 6 6, 2. So, 2, 2 is there 2, 6; 6, 2 and 2, 2 is there 2, 6; 6, 2. So, 2, 2 is there.

So, we can tell that this property also holds. So, R is also transitive. So, this particular relation R is a equivalence relation since all three a reflexive symmetric and transitive three properties it holds. So, R is an equivalence relation.

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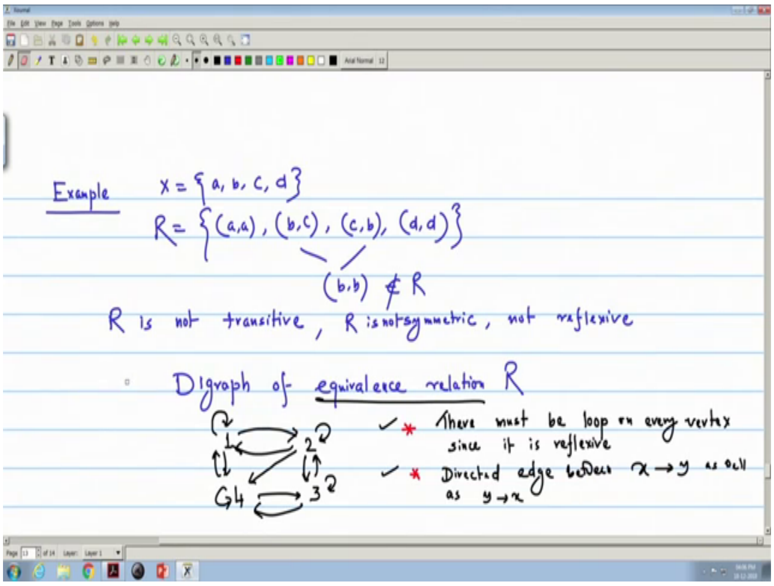

Now, we see that some previous examples that we have seen that say we have taken some x equal to a, b, c, d and the relation we have taken a, a; b, c; c, b and d, d ok.

So, b, c and c, b is there, but b, b is not in relation R. So, obviously, it is not R is not transitive, R is not transitive. R is b, c is there c, b is there. So, R is symmetric R is symmetric a, a and d, d is there, but b, b and c, c is not there, so, not reflexive. R is not symmetric, not reflexive. So, now, we see some other properties of relation, ah.

Before we go to that other properties 1 property of that equivalence relation when we see the digraph we must see that thing say if it is a for R is an equivalence relation and if I draw the digraph of R. So, for if R is equivalence or I can write the digraph of equivalence relation R.

So, first thing is. So, I have say three elements any I am considering. So, since it is a equivalence relation. So, first thing it must be reflexive. So, there must be loop for every vertex. So, there first thing we identify that there must be loop must be loop on every vertex because it is since it is reflexive since it is reflexive. So, this is 1. Now, if it is symmetric; that means, always there must be if it is 1 to 2, x to y, x, y then y, x. So, always there will be edge between an edge there will be edge between x to y and y to x, it will be a directed edge directed edge between x to y as well as y to x.

So, here if it is always this will be nothing another is that 2 to 3 and 3 to 4 is there. So, there must be must be 2 to 4, this means it is transitive and same thing will happen. So, for equivalence relation that we from the digraph this property, we can always check very easily. So, this is one property of the relation R.

Now, we see another important property of equivalence relation it is called the partition.

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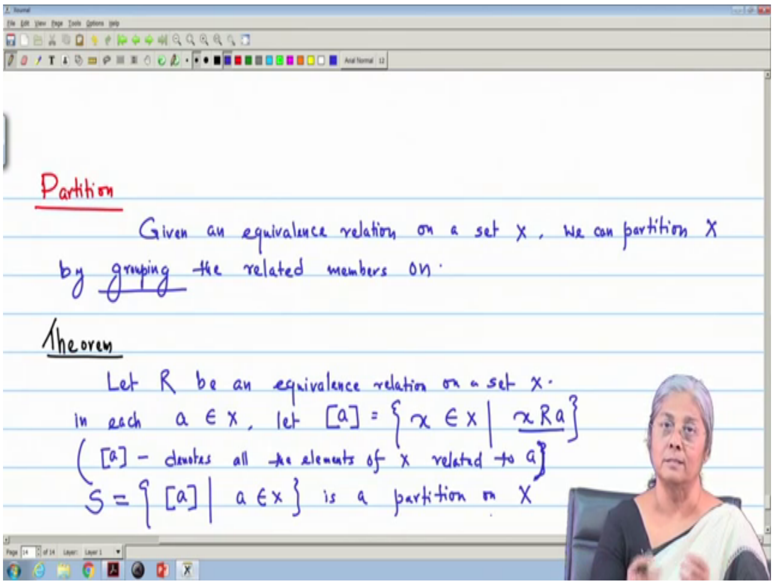

So, given an equivalence relation, so, define partition first you define that given an equivalence relation on a set X, we can partition X by grouping the related members on X. So, main thing is that grouping and here elements that are related may be thought of that are as if equivalent. So, equivalency. Now, first we give a theorem on this definition of partition we give a theorem. We write let R be an equivalence relation on a set X. Now, in each a element belongs to X that element of X let a is the set of all elements x belongs to X such that x R a; that means, x is related to a and by this notation a that this notation a this a, it is it denotes all the elements al l the elements of x that are related to a we denote by this.

Then, we define S; S is a partition and these are these box such that a belongs to X and is a is a partition is a partition on X. So, mainly very simple way if I tell that R is an equivalence relation on a set X and we are identifying all the elements which are related and we denote that thing by this box a. Say a is one such elements and all for all x which are related to a that is why we have written x R a. So, these all elements which are related to a we are denoting that thing by this box a and this S for these boxes this is a partition on X; that means, all elements that are related to a, we are grouped together and we are telling that this is a partition on X.

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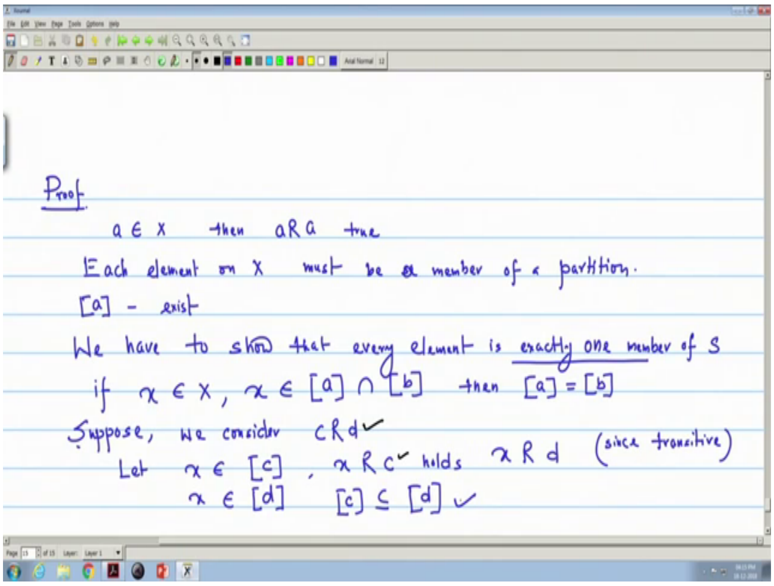

Now, how we can proof that thing that if we want to prove that a belongs to X then a R a, it is true. So, first thing we can tell that for each element each element on X must be a must be some member must be a member of a partition because a R a; that means, the this a exist. So, it is a member. Now, we have to show exactly one member of X. Now, show now we have to show that every element is. So, that every element is exactly member of exactly a one member of S exactly a one member of S.

So, if we can write in this way if x belongs to X and x belongs to some part a and intersection b; that means, all elements related to a and then all elements related to b then this partition must be the same partition since x must be in a and b. So, suppose we consider some two elements c and d, ok; suppose we consider a relation c R d; that means, that c is related to d. So, let x belongs to partition c so; that means, x R c holds.

Now, since c R d also holds and x R c. So, since it is an equivalence relation. So, I can tell. So, it is x R d holds since it is transitive since transitive since x R d; that means, x belongs to d also. So, I can tell that this set; that means, all elements which are belonging to c, it is a subset of d I can tell this thing.

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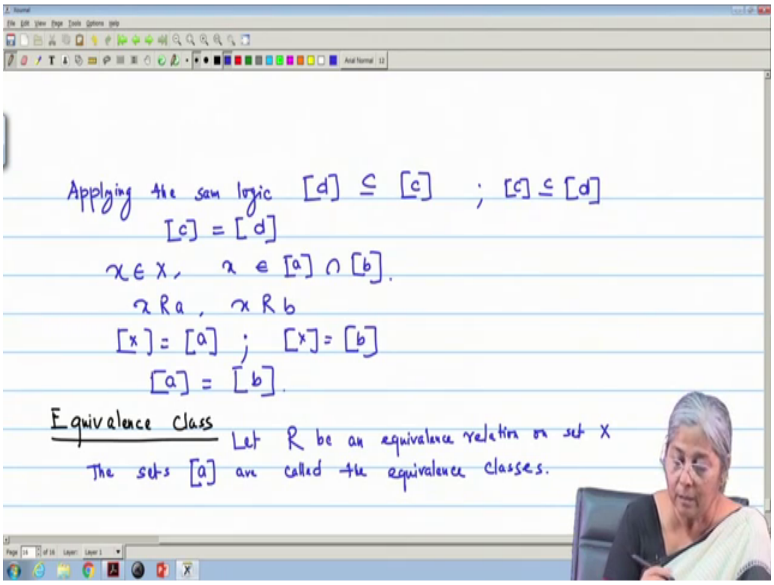

Now, using the same logic that applying the same logic I can tell that all elements that are related to d, again they are subset of c and earlier we got, I we got that c is a subset of d. So, from the property of subsets of a set we see we must take that c equal to d.

So, we have assumed that x belongs to c x R c so; that means, here we tell that x belongs to X that x belongs to. So, we can tell that x R a; x R a then since it is intersection so, x R b also it is true and I can tell that this is equal to a, I can tell that this is equal to b and so, from here I can tell that box a; that means, the partitioning is the same as that of partition b.

Now, I give another definition of equivalence class, from this is very much related of this equivalence class. So, let R b and equivalence relation on set X. So, let R be an equivalence relation on set X then the all the elements related to a and that we have defined as the set the sets as the box a are called the are called the equivalence class equivalence classes.

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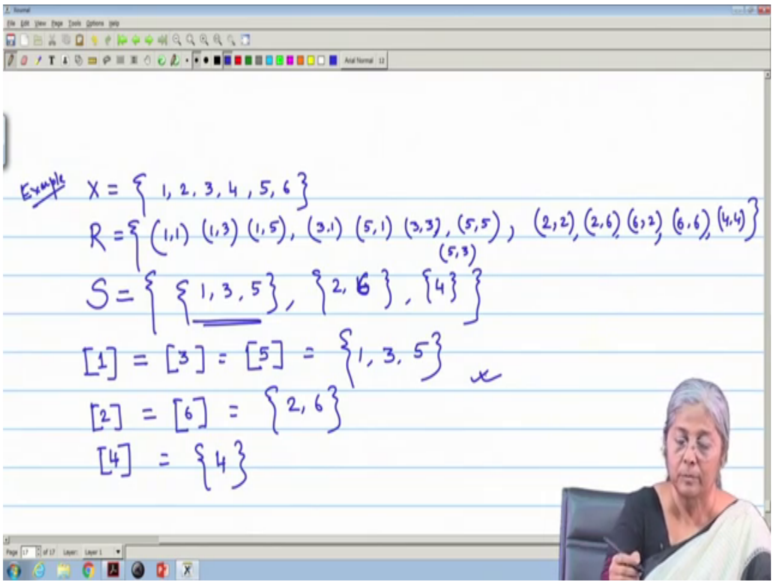

So, if we take the example of say I have I have a one set X equal to say 1, 2, 3, 4, 5, 6 and say on some relation some relation I get that is using some that the previous relation if I write 1, 1 then 1, 3; 1, 5 then 3, 1 and 5, 1; 3, 3 say 5, 5 and then I get say 2, 2; 2, 6; 6, 2; 6, 6 and say 4, 4. So, this is some relation given.

Then, we can tell that s I can tell the partition that where 1, 3, 5 then 2, 4, 2, 6 I get because 1, 1; 1, 3; 1, 5 then 3, 1; 3, 5; 5, 3 I should add 5, 3 here and 5, 5 then similarly 2, 6 and 4, 4, so, I can get this type of three partition. Then I can write that box since this is my related elements 1, 3. 5, so, I can tell 1 and this can be related as the box 5 equivalence class and they are actually 1, 3, 5.

Similarly, my 2 equivalence class or 6 this can be only two elements 2 and 6, and 4 is only one, only 4 is there. So, this is one example of equivalence class.

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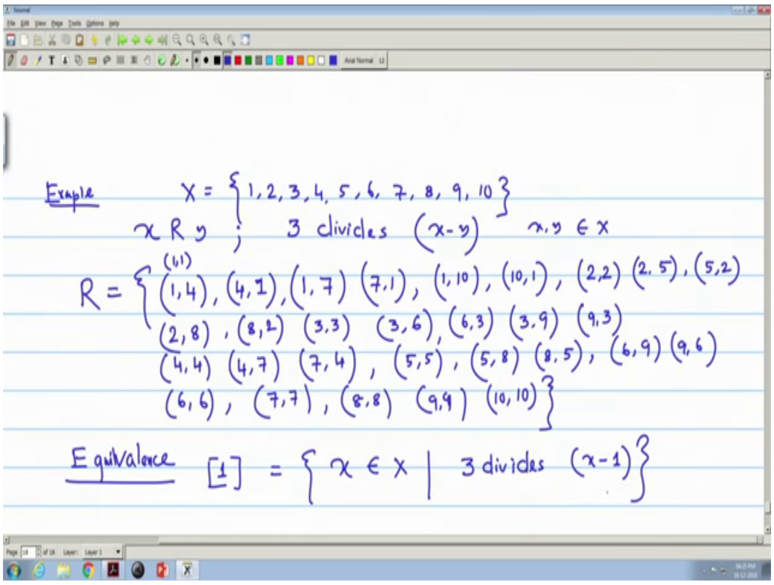

I take one example quickly say I have a set X equal to. So, I am taking all integers 3, 4, 5, 6, 7, 8, 9, 10 and I am taking the relation x R y where if 3 divides x minus y, and x, y belongs to X. So, if I write quickly my relation R will be all such topple say 1, 4; 4, 1 because 3 divides then 1, 7; 7, 1; 1, 10; 10, 1 these are all divided.

Then I should take 2 I should take 1, 1 also, first one then I should take 2, 2 then 2, 5; 5, 2 then 2, 8; 8, 2 then I have 3, 3 then I have 3, 6; 6, 3 I have 3, 9; 9, 3 I have 4, 4 then I have 4, 7; 7, 4 because x minus y divides by 3 and I have 5, 5. So, all such 5, 5 then 5, 8; 8, 5 I have 6, 9; 9, 6 then 6, 6 6, 6, then I have 7, 7 or 8, 8; 9, 9; 10, 10 all these tuples will be my relation.

Now, what will be my equivalence class from here? I can consider equivalence class as that say I am writing one box 1, I define that x belongs to X that if 3 divides x minus 1 as if I am divide writing like that.

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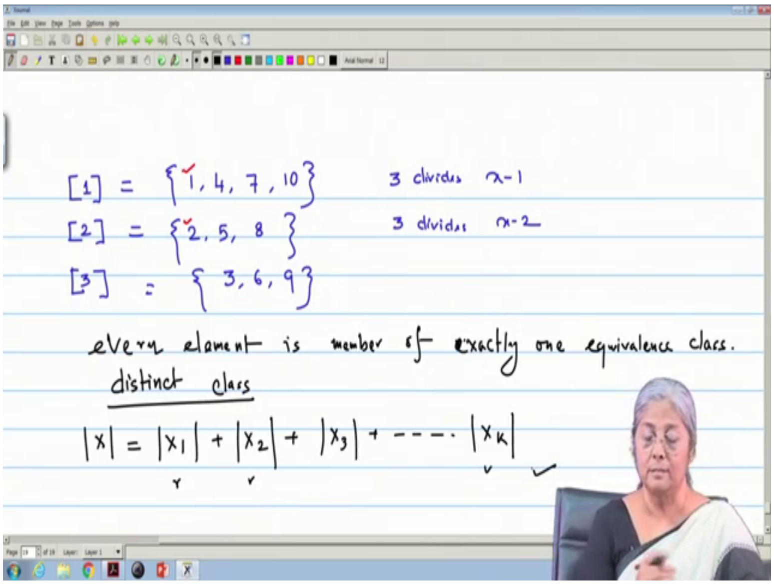

So, what will be my then 1? Then 1 will be that elements 1, 4, 7 and 10 because all 4 minus 1, 1 minus 4, 7 minus 1, 1 minus 7, 10 minus 1, 1 minus 10 this all will be divided by 3.

Similarly, what will be my 2? Because x 3 divides so, here it is if I quickly write 3 divides x minus 1, here 3 divides x minus 2. So, it will be 2, 5, 8. Similarly, my box 3 will be 3, 6, 9 and see the way we have defined that every element belongs to only one partition we if we consider say here 1 is the here, 2 is here. So, of every element so, if I every elements is member of only exactly one equivalence class. So, and there these classes are all in that way it is are all distinct class in that sense is a distinct class.

So, I can write actually that if X 1 if I set 1 X and these are my classes, so, if there are k such equivalence classes then this will be X k; that means, all that cardinality will be same and if all are they consider r elements then actually these are full k n number of elements will be there. So, this is that concept of equivalence class on the equivalence relation R and we get the partition from here.