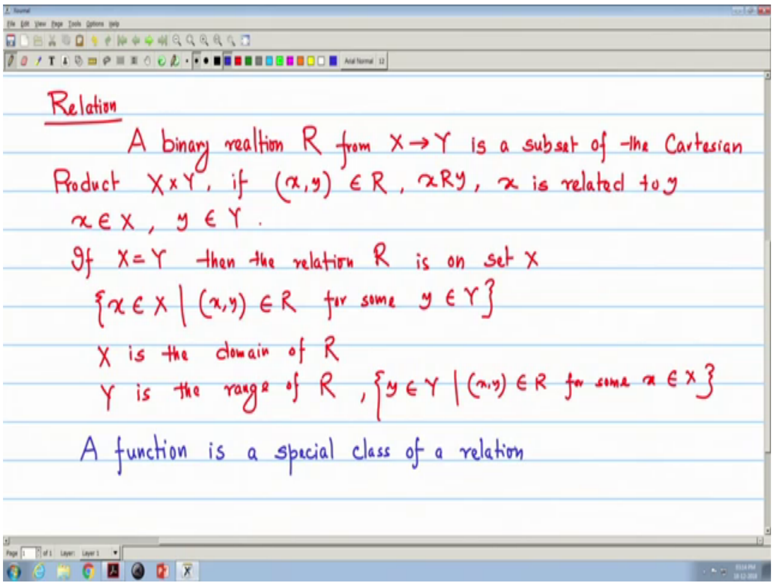
Discrete Structures

Prof. Dipanwita Roychoudhury
Department of Computer Science & Engineering
Indian Institute of Technology, Kharagpur

Lecture – 21
Relations and their Properties

Relations are very important role in computer science as well as mathematics and there are almost many braches of engineering. It is a mathematical structure and mainly relation is defined on a set. So, now, we have the concept of sets, we have read the functions and today we will see the relations and the properties.

(Refer Slide Time: 00:48)



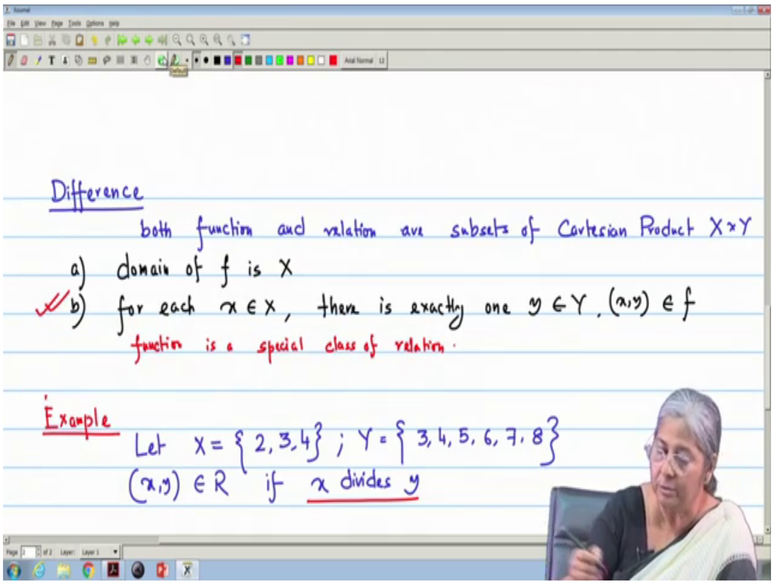
So, first we defined a relation mainly relation is defined on a set. Now, we defined taking two sets. And so, relation or better I can tell a binary relation first, normally we define it by R from a set X to Y; that means, the elements of set X how they are related to elements of Y and we define these relation as a subset is a subset of the Cartesian product X cross Y, if that the ordered pair x, y belongs to R.

And, normally we define this thing or we denote this thing as a x R y where here x is related to y, x R y; that means, it x is related to y and x as usual that my x belongs to the set X, y belongs to the set Y. And if X equal to Y then the relation R is on set X only relation R is on set X only; that means, it that time it denotes the relation of the elements of only one set X.

So, formally we can or in mathematical notation we can write that x belongs to X such that x, y belongs to R for some y belongs to capital Y, that is it and here also we called the X is the domain of R and Y is the range of relation R. Here y belongs to Y and such that x, y belongs to R for some x for some x belongs to X.

Now, the definition is very similar to the definition of a function we have read then what is the difference between a relation and a function. So, first I make one statement that a function is a special class of a relation and what is the difference?

(Refer Slide Time: 05:34)



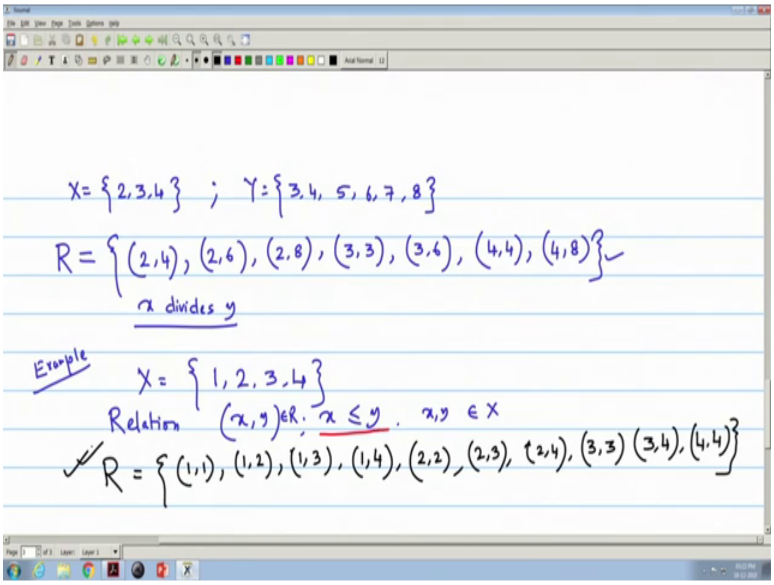
See, when we have defined function that we have the domain of f because both function and relation are subset of a Cartesian product x cross X. So, I write that both function and relation are subsets of Cartesian product X cross Y.

Now, the definition of function if we remember then it tells that domain of f is X here also the domain of f is X for is X for R now for each x belongs to X there is exactly one y belongs to the set Y such that x, y belongs to f. So, this is the difference is here and that is why we call that the function is a special class of function is a special class of relation of relation, because here only for each x there is exactly one y, ok. Now, it is not true for relation. So, relation is actually a general say situation.

Now, we take one example of relation. So, mainly we are considering that one elements of a set how it is related to the elements of other set. We take one example that let two set we consider one is X is say 2, 3 and 4 these three elements and Y has some elements say 3, 4, 5, 6, 7, 8 and the relation how X and Y are related that with that x, y belongs to R x y belongs to R if x divides y. So, that means, this is the relation between the elements of x and the elements of y.

So, all possible Cartesian product of x, y only we will be taking that subset where this property holds or only for those pairs relation consists of that that pairs where x divides y.

(Refer Slide Time: 09:52)



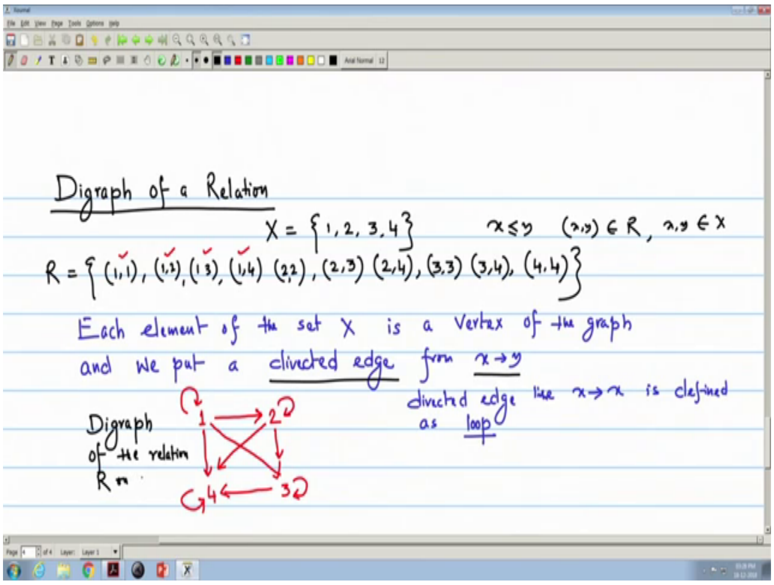
So, we see that which are the pairs it is. So, if we continued that with that thing. So, our X is 2, 3, 4 and Y is 3, 4, 5, 6, 7, 8. So, what will be the relation R? R as we have defined that subset of Cartesian products. So, 2, 3; 2 does not divide 3; 2 divides 4; so, 2, 4 is one pair then 2, 6 then 2, 8. Now, if I consider 3 then 3, 3 3, 6 if I consider 4 then 4, 4 and 4, 8.

So, this is a subset of the Cartesian products because Cartesian products are all possible ordered pairs and then this is the subset that holds therefore, these the relation holds in that x divides x divides y x divides y.

Now, if I consider another example if we consider say simply I am considering as if one relation is given say R equal to 1, 2, 3, 4 say, ok. I take the set X I take the set X is one only four elements and the relation is defined that relation is x, y where x less than equal to y; that means, x, y belongs to R if x less than equal to R and then all x, y belongs to x then the relation is that I consider that we will consider all such pairs where that x less than equal to y.

So, my relation I can tell that this is 1, 1 because equal to is there, so, 1, 1 then 1, 2 then 1, 3 1, 4 then 2, 2; 2, 3; 2, 4 similarly 3, 3; 3, 4 and 4, 4. So, this is the subset of the Cartesian products or these are the ordered pairs set of ordered pairs that for these it holds that x less than equal to y. So, this is a one relation.

(Refer Slide Time: 14:03)



Now, sometimes we denote the relation by using some figure or some picture and that is a very informative way to present one relation and it is called that digraph it is called the digraph of a relation. So, digraph is a general term, but here we are defining digraph only on the context of relation.

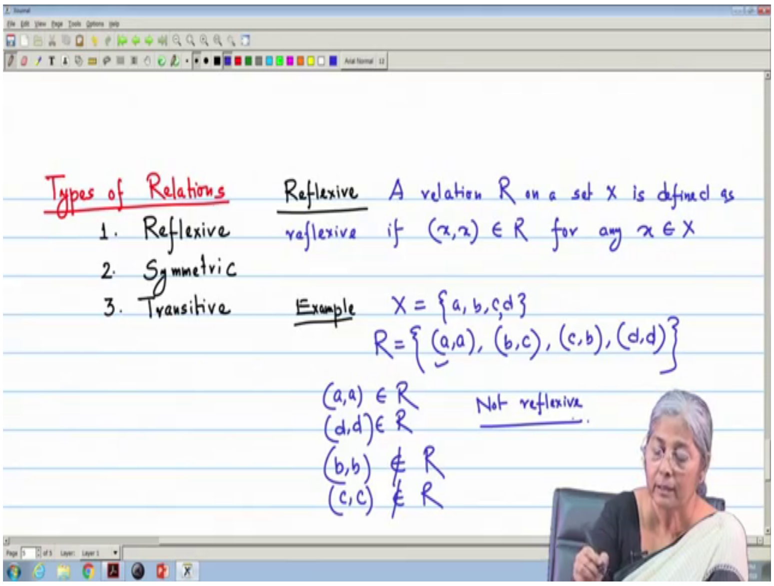
So, if I consider that set just now the example we have seen that X the set 1, 2, 3, 4 and the relation we got the like 1, 1; 1, 2; 1, 3; 1, 4; 2, 2; 2, 3; 2, 4 and 4, 4 and we have defined that x less than equal to y then x, y belongs to R and x, y belongs to X.

Now, digraph we how we define that the elements of X the vertices of this graph or just I that each element of the set X of the set X is a vertex of the graph and we put a directed edge from x to y; that means, for all pair the pair exists we give a directed edge and why it is directed edge because it is a ordered pair because relation is defined as the ordered pair. So, we give a edge.

So, for this particular relation if I draw the directed or digraph of this thing, so, I am first thing I have four elements. So, I give 1, 2, 3, 4 then I see that 1, 1 is there. So, normally if there is a directed edge from x to x we call that is a loop. So, loop is defined a directed edge like x to x is defined as loop. So, we give a loop here. So, 1, 1 then 1, 2 so, we give one directed edge 1, 2. So, 1, 1; 1, 2 then 1, 3 so, I give directed edge here then 1, 4 now 2, 2; 2, 3; 2, 2 again it is a loop 2, 3, then 2, 4, then 3, 3 again it is a loop, 3, 4 and then 4, 4. So, this is the digraph. This is my digraph of this particular digraph of the above relation.

Now, we have a different type of relation normally we use and these are the relations are of now we consider is of three types.

(Refer Slide Time: 19:47)

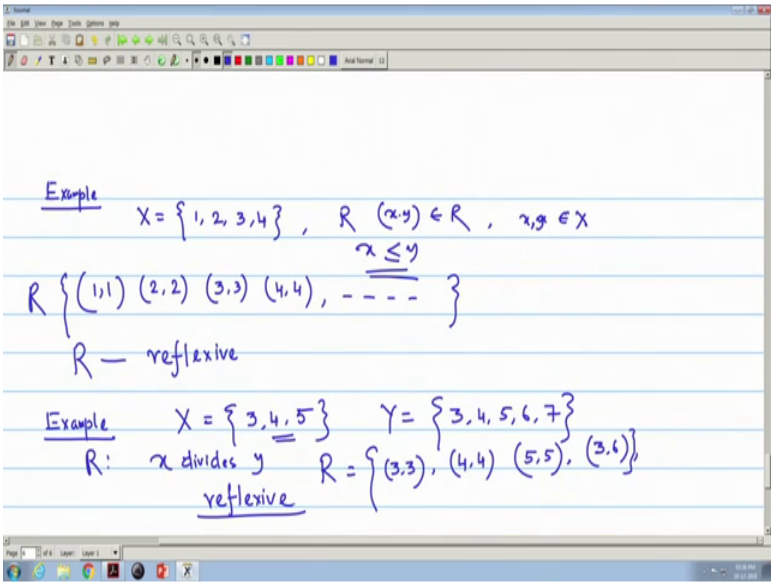


So, the types of relation it is reflexive, it is symmetric and in this context we will read also not symmetric or anti symmetric and one is called the transitive relation. So, now we define one by one that what are these type three different types and what are their differences. So, first we define the or reflexive. So, a relation R on relation R on a set X is defined as reflexive if the pair x, x belongs to R for any x belongs to x. So, the relation now we consider if x is x belongs to x, x is one element in the set x, now the relation if x is related to x. So, that x, x pair this is this belongs to R, then it is reflexive.

We take one example one very simple example if we see that. So, I consider a relation R which of elements R consider the a, a say b, c; c, b and d, d. So, R is defined on set Xs it has four elements a, b, c, d.

Now, we see here that for any x a that a, a is here a, a belongs to R d, d belongs to R, but we see that b, b; b, b does not belongs to R or c, c that pair is not also in R.

(Refer Slide Time: 24:21)

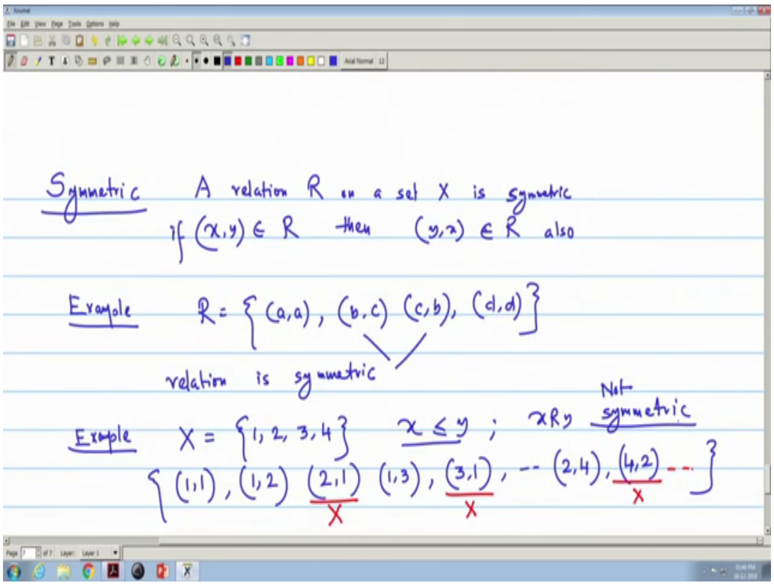


So, this is not reflexive we take another example that previous example. We have seen that less than equal to we have seen one very simple example that if I consider that again that previous example that say at x again 1, 2, 3, 4 and we have seen relation is we have taken the relation that x less than equal to y, x, x y belongs to R the relation we have taken that x, y belongs to R and then x, y.

Now, here we see that since it is less than equal to we have what we have seen that x less than equal to x less than equal to y. Since it is less equal to sign is there, so all such pairs 1, 1; 2, 2 because 2 equal to 3, 3; 4, 4 all along with other that it will be in the relation R. So, this relation must be a, this relation is a reflexive this is an example of a reflexive relation.

And, then another example, we have seen that x divides y. We have taken some X say 3, 4, 5 and Y some 3, 4, 5, 6, 7. Then if the relation is that R is that x x divides y, as we have considered earlier that x divides y then R that say 3, 3 will be there, then 4, 4 will be there because 4 divides 5, 5 is also there since 5 divides 5, but see here that 6, 6 or 6, 7 because here there is no element of 6 because x divides y that 3, 3; 4, 4; 5, 5 will be there and other along with the other elements that 3 3, 6 or no other will be there only 4, 4; 5, 5 only 3, 6 will be there. So, here this is for this will it is also a reflexive this is also a reflexive because the element divides itself. So, this is reflexive. Now, we see that symmetric.

(Refer Slide Time: 27:51)



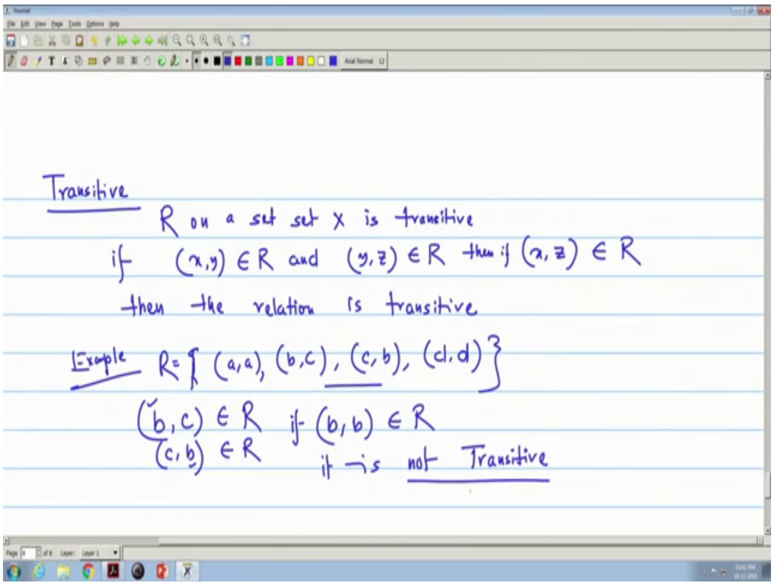
Symmetric that a relation R on a set X is symmetric if x, y belongs to R then y, x also belongs to R and if we it see that same example the example that R say a, a then b, c then c, b then d, d. Then this is symmetric because b, c is there then c, b is also there. So, this relation is symmetric.

See if we consider that our earlier example that X equal to 1, 2, 3, 4 and relation is x less than equal to y x less than equal to y that is we called x or y, that is x less. Now, see that if x less than equal to y then say one if I consider that 1, 1 is there 1, 2 is there, but 2, 1 never can be there here never we get 2, 1 because it is only some ordered and x less than equal to y. So, y never can be less than. So, this cannot be here.

Similarly, that if 1, 3 is there 3, 1 cannot be here or 2, 4 is there. So, 4, 2 cannot be there. So, that means, these are the pairs never it will come in this relation. So, here this relation is not less than equal to x this relation is not symmetric, it is not symmetric.

Similarly, our example of that x divides y that is also not symmetric because if x divides y then y divides x that is never true. So, if x, y is in the relation R then y, x cannot be in the relation R and our is it is transitive I have another transitive.

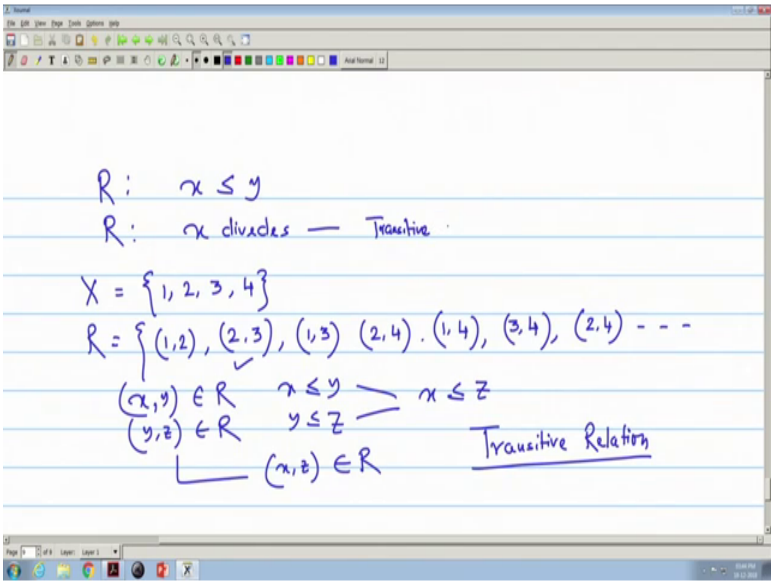
(Refer Slide Time: 31:26)



So, relation R on set R on a set X is transitive if x, y belongs to R and y, z belongs to R then x, z if x, z also belongs to R then if x, z also belongs to R then the relation is relation is transitive.

So, that same simple relation if we consider that R that a, a; b, c; c, b; d, d. So, b, c b, c and c, b, so, b, c belongs to R, c, b belongs to R then b, b if b, b because x, y y, z the next z; that means, here it is b, b. So, b b belongs to R then it will be transitive, but what do you observe that here b, b is not here in this relation. So, it is not a transitive relation.

(Refer Slide Time: 33:41)

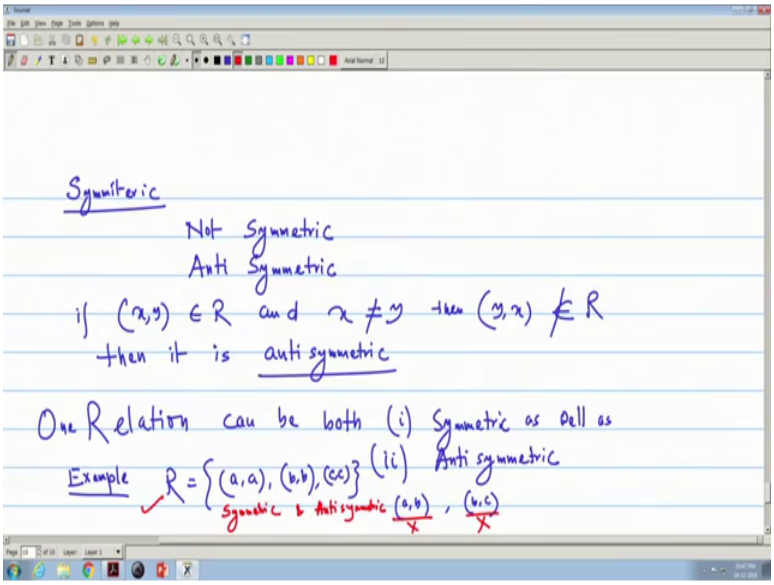


So, it is not transitive, but it is not transitive. But, our previous example that the relation x less than equal to y or the relation x divides y. See less than equal to y again if I consider the set x is 1, 2, 3, 4 then in the relation R we will get if I get 1, 2 we will get 2, 3 because 2 less than 3 we will get 1, 4; 1, 3; 1, 2; 2, 3 we will get 1, 3.

Similarly, if I get 2, 3, I get 2, 4. So, 2, 3 I get 1, 2; 2, 4 I get 1, 4. If I consider 2, 3 then 3, 4 so, it will be 2, 4 also it means that if x since x less than equal to y, so, some x, y belongs to R and if my y, z belongs to R then because x less than equal to y similarly here y less than equal to z. So, from there I am get that x less than equal to z. So, we can tell always that x z must be belongs to R. So, this is a transitive relation, this is a transitive relation.

And, similarly for if x divides y we know that if x divides y and the x, y is in the relation and then y divides z then x must divides z. So, that is also a this is also a transitive relation this is also a transitive relation.

(Refer Slide Time: 35:55)



Another relation is that it is a variant of symmetry symmetric relation when we have read that there is two other options one is not symmetric one is called the anti-symmetric. So, if it a relation is not symmetric if it is not symmetric we called anti symmetric we define that if x, y belongs to R x, y belongs to R and x not equal to y then y, x does not belongs to R. Then it is called then it is anti-symmetric. One very funny thing is that one relation can be both one relation can be both symmetric as well as anti-symmetric, symmetric as well as anti symmetric. And, the example is very simple example we give one relation say if it is a b, b; c, c this is the relation; obviously, this is symmetric because x, x belongs to R.

And, since there is x not equal to y; that means, a, b or b, c this type of pair is not here a, b or b, c this type of part is it is not in this relation. So, this is also so, it is vacuously true; that means, the statement is false that x not equal to y in this type of topple is not here. So, it is automatically it is anti symmetric. So, this particular relation is both symmetric and anti symmetric.

So, with this we give the idea of that basic relations on the three different type of the relations and then we will read the properties of relations.