Discrete Structures
Prof. Dipanwita Roychoudhury
Department of Computer Science & Engineering
Indian Institute of Technology, Kharagpur

Lecture – 15
Proof Techniques (Contd.)

So, we are discussing the mathematical induction; that means, the Proof Techniques that normally we use the mathematical induction. We have already read the normal form as well as the strong form of mathematical induction. Now today we will see different type of problem not only mathematical, but other problems how directly or indirectly we use the mathematical induction.

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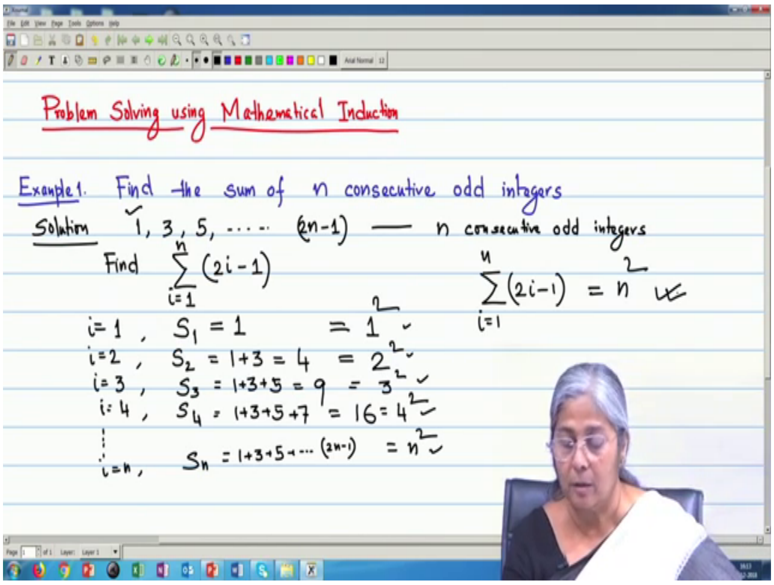

So, this session is the problem solving using mathematical induction techniques. So, first we see one very simple problem. So, find the sum of n consecutive odd numbers, find the sum of n number of consecutive odd integers. See normally the problem we see that while we use mathematical induction that some well known formula or the correct formula is given and then normally we validate it by induction or we proof that the formula is true.

So, for this particular example that formula is not given, we already discussed this type of problem. So, first we have to find out a correct formula and normally we do that thing for some, taking some primitive values and then to observing the sequences and we try to identify or try to frame one formula for that. So, this is sum of n consecutive odd numbers, if we so we know that n consecutive number or numbers are that 1, 3, 5. So, n is 2 n minus 1 these are the n consecutive odd integers and we have to find out the find the sum. So, i equal to 1 to 2 i minus 1 or 2 n minus 1 or I can give are better I write 2 equal to 1 to n, 2 i minus 1 I have to find this sum.

Now, if I consider i equal to 1 then the sum say S 1 is only 1, if i equal to 2 S 2 is 1 plus 3 equal to 4. 1 plus 3 plus 5 is 9. And if we observe the pattern or the sequences we see that S 1 equal to 1 I can write that as if 1 square, this is for S 2 that is i equal to 2 I can write this is 2 square, for 3 this becomes 3 square, for 4 this becomes 4 square.

So, I am getting a compact formula for this. So, i equal to n this S n is 1 plus 3 plus 5 to 2 n minus 1 is equal to n square. So, first we get a formula that it is n square. Now we have to proof by induction that this formula is indeed true; that means, if I write that my sum equal to i equal to 1 to n 2 i minus 1 equal to n square whether this formula is true or not. So, earlier we have seen when we have discussed the mathematical induction how to get a correct formula. So, normally this is one of the technique that we get that thing. Now, we have to proof and we remember that to prove this thing we have to do the two steps, we have to follow, one is the basis step.

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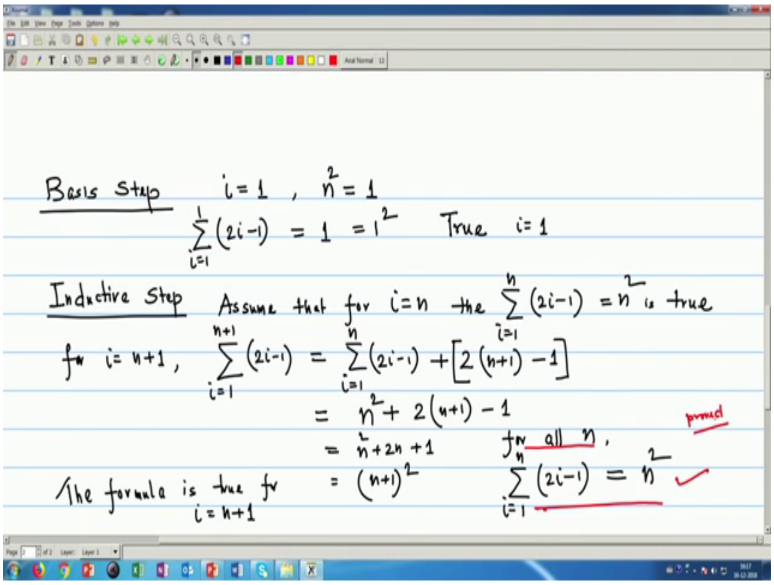

So, basis step where we consider for i equal to 1, we know the n square equal to 1 and the sum i equal to 1 to 1 2 i minus 1 equal to 1 which is equal to 1 square. So, my basis step is true, it is true for n i equal to.1

Now, we see the inductive step, inductive step we assumed that for i equal to n the proposition is true, in this case the formula for this sum the sum i equal to 1 to n 2 i minus 1 equal to n square that is true, sum is true. We have to proof that for i equal to n plus 1 or we have to show that the equation is true or the formula is true.

So, assuming that the, for i equal to 1 it is true and then for i equal to n it is true. So, for i equal to n plus 1 what is our expression? That our expression is i equal to n plus 1 2 i minus 1 and these I can write that i equal to 1 to n, 2 i minus 1 plus the value for the for i equal to n plus 1; that means, 2 into n plus 1 minus 1. So, this is my last element for I equal to or the last odd integer when I equal to n plus 1.

Now, according to that inductive state it is true for n so this part is the first term is n square and this becomes 2 into n plus 1 minus 1. So, this is n square plus 2 n plus 1 which is nothing, but n plus 1 whole square. So, when i equal to n plus 1 again it is giving n plus 1 whole square so; that means, the formula is true for; is true for i equal to n plus 1. So, we can write that for all n i equal to 1 to n 2 i minus 1; that means, sum of all consecutive integers up to n equal to n square. So, it is true; so, for all n the this formula is true. So, it is proved; so, normally the, how we use mathematical induction to proof the simple mathematical formula.

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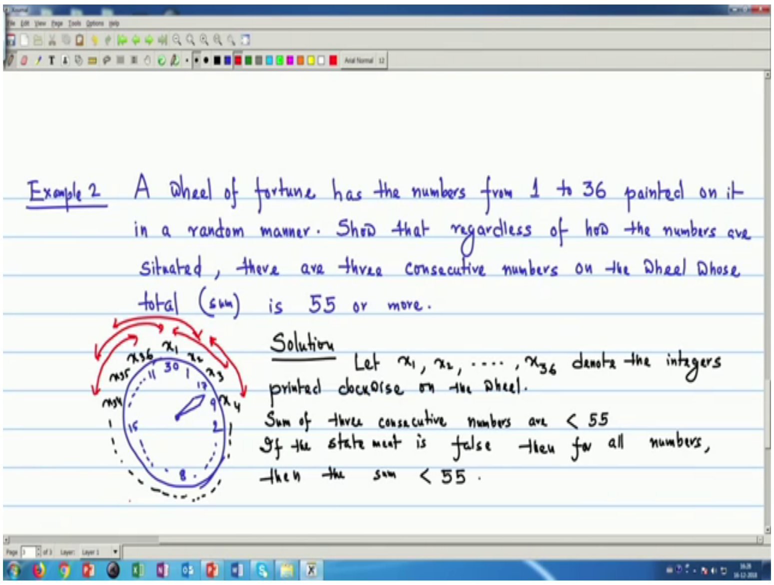

Now, we see a different type of example, but when indirectly we can apply the mathematical induction. We take one problem statement, let a wheel of fortune has the numbers from 1 to 36. So, wheel of fortune has the numbers from 1 to 36 painted or need in a random manner. Show that regardless of how the numbers are painted, always there are 3 consecutive numbers on the wheel whose total is or sum, it is the sum is 55 or more. So, this is one type of game, I think all of we have seen in sum fair somewhere that it is in a seat is a circular wheel and we have only one pointer type. See so, some randomly say I start from 30, then 1 then 17, 9, 2 in this way sum random way it is painted and this is this is we call that the wheel of fortune.

So, it is in this way it is. Now our and these numbers are in random manner it is printed on the wheel, now we have to show that whatever random way or whatever way the numbers are printed on this there are always there exist 3 consecutive numbers whose numbers are, whose total sum is greater than equal to 55. So, we have to show this thing.

Now, we use mathematical induction very indirect way to prove this or statement ok. So, the solution we do. So, we have 36 integer integers. So, let x 1, x 2 that x 36 are the or denote the integers printed on the wheel, printed clockwise; that means, now we since our numbers are not in order. So, we consider x 1, x 2 x 36 in order and they denotes the numbers printed clockwise on the wheel.

Since the sum of 3 numbers we have to consider. So, we first assume that if we consider that the result is false; that means, the sum of sum of 3 consecutive numbers or less than 55, why? Because the problem statement tells that there exists or there are 3 consecutives, there are 3 consecutive numbers on the wheel whose total sum is 55 or more. If the statement is false, if the statement is false of the result to be proved statement is false then for all numbers this sum of 3 consecutive numbers are less than 55. Then the sum of, then the sum less than sum of 3 numbers consecutive 3 numbers are less than 55.

Now, how many sums are there on a wheel? See we have 36 some numbers, now I can write that numbers are the way we have taken as if this is my x 1, this is x 2, x 3, x 4, in this way I have x 36, x 35, 34, in this way the 36 numbers are given.

So, from the result to be false all such 3 consecutive numbers 3 consecutive numbers say x 1, x 2, x 3 these 3 then x 2, x 3, x 4 these 3, x 3, x 4, x 5. Similarly, all these numbers this is x 34, x 35, 36 x 35, 36 x 1 then x 36, x 1, x 2 all such all these consecutive numbers must be less than some of these 3 numbers must be less than 55.

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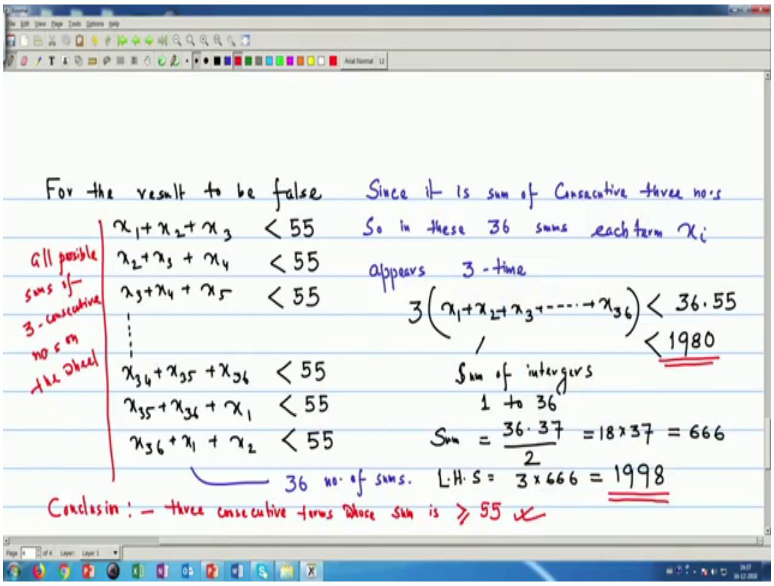

So we can write that for the result to be false since for all sums, for all sums of 3 consecutive numbers we have to show that this is false. So, we can write that x 1 plus x 2 plus x 3 less than 55. Say x 2 plus x 3 plus x 4 less than 50 55 x 2 x 3 plus x 4 plus x 5 we can continue in this way. And in this way x 34, x 35, x 36 less than 55 then since it is a will; that means, circular in fashion. So x 35, x 36 plus x 1 that is again 3 consecutive numbers, then x 36 plus x 1 plus x 2 this is also less than 5. So, these are my, these are the all possible all possible sums of 3 consecutive numbers on the wheel numbers on the wheel.

So, how many sums are there? You can see that it starts with variable x 1 or it starts with integer x 2, x 3, x 36. So, there are all together we have 36 sums. So, we have 36 number of sums. Now see if we just observe that thing; that means, in these 36 sums each term each x i appears 3 times since it is a some of consecutives 3 numbers since, it is sum of consecutive 3 numbers. So, in these 36 sums each term or each integer x i appears 3 times.

So, we can write that 36 into x 1 plus sorry 3 into sorry this is since each one is appearing 3 times 3 into x 1 plus x 2 plus x 3 or 36 that will be less than 36 into 55, which is less than 36 into 55 is if we multiply this is 1980.

Now, what is this x 1 to x 36 because this is the integers, sum of integers 1 to 36 only they are print printed on a different order. But when we are taking the sum these are whatever way we take this is actually sum of integers and that should be 1 to 36 that num sums are n into n plus 1. So, this sum is, sum of these number is 36 into 37 divided by 2 is 18 into 37 and this is equal to 666.

So, my LHS is 3 into 666 is this becomes 1998. Now what do you see? That these number these 30 sum of 36 sums this becomes the 1998, but if it would be, if this is the statement, the early the first statement we have assumed that the for the result to be false; that means, for each consecutive number must be less than 55 then it must be less than the all sum of 36 terms must be less than 1980. But what we see that it should be 1998 so; that means, our assumption is false; that means, that for this result to be false.

So, we the conclusion is that we have at least 3 consecutive terms. So, 3 consecutive terms whose sum is or total is greater than equal to 55 say this is a indirect way that we have actually proved and this is that these sums the 36 sums we have written that is actually by induction and this is a indirect way of applying the induction.

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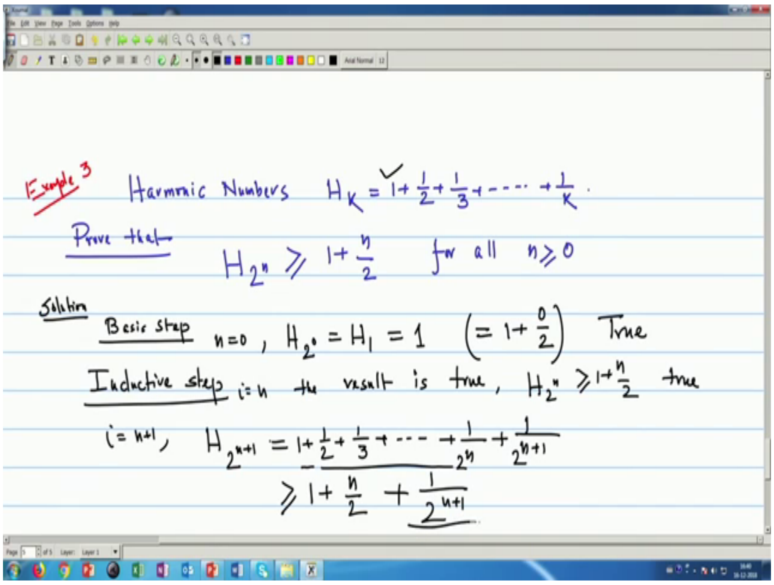

And if we just quickly see one another example that some mathematical example say we know the harmonic numbers. How we define that thing? We define that say H k equal to 1 plus 1 by 2 plus 1 by 3 is 1 by k.

Now, there are a number of problems, the number of inequalities equalities that we can proof on this harmonic numbers. Now, one is we can write that proof that H 2 to the power n is greater than equal to 1 plus n by 2 for all n greater than equal to 0, here we directly apply our mathematical induction. So, what is our basis steps? So, solution for our basis step you see for n equal to 0. So, H 2 to the power 0 equal to H 1 equal to 1 so I can write this is equal to 1 plus 0 by 2 so, this is true. So, our basis step is true.

For inductive step we assume for i equal to n it is true for i equal to n the result is true ; that means, our result is true is that H 2 to the power n greater than equal to 1 plus n by 2 that is true. So, we have to show for i equal to n plus 1. So, i equal to 1 n plus 1, 2 to the power n plus 1 greater than equal to 1 plus n plus 1 by 2 that we have to show.

Now, what is h is H 2 to the power n plus 1 by 2? This is 1 plus 1 by 2 plus 1 by 3 plus 1 by n plus 1 by 2 to the power n plus 1 by 2 to the power n plus 1 like that. So, up to this is my H 2 to the power n, this is H 2 to the power n is 1 plus I can write greater than equal to this is greater than equal to 1 plus n by 2 plus 1 by 2 to the power n plus 1. Now how this term I can show?

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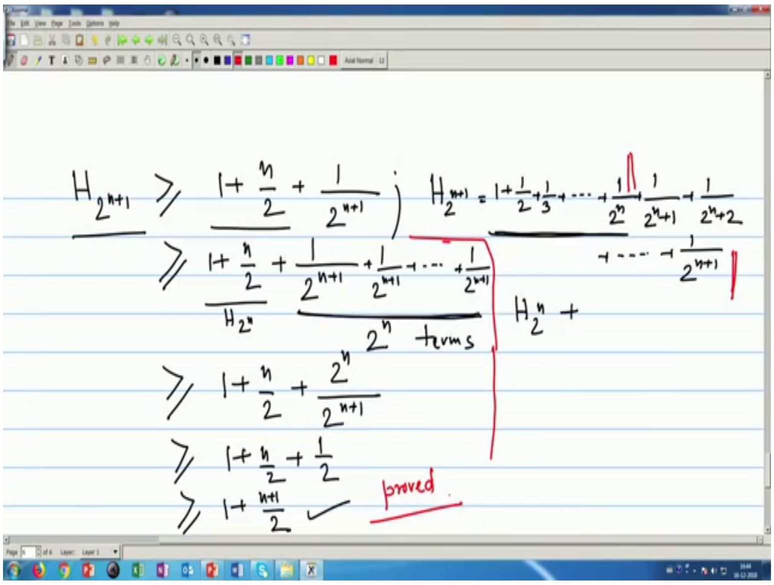

So, our H 2 to the power n plus 1 I can show that this is greater than equal to 1 plus n by 2 plus 1 by 2 to the power n plus 1. So, I can write this thing that because this is up to 2 to the power n. So, these term is if I know that what is our H n ok, if I consider my H 2 to the power n plus 1 this will be 1 plus 1 by 2 plus 1 by 3 plus 1 by 2 to the power n plus 1 by 2 to the power n plus 1, 1 by 2 to the power n plus 2 plus 1 by 2 to the power n plus 1.

So, my H n is only up to this term and I have these are the terms that I have to add now what are those these terms. See I can write so this is my H 2 to the power n plus see this is the lowest term. So, when I am giving the greater than equal to then what I can write each term I can replace by 2 to the power n plus 1 since, this is my lowest term. So, this is 2 to the power n plus 1 plus 2 to the power n plus 1 plus to the power n plus 1.

So, how many terms are there? Since this is H 2 to the power n this is my H 2 to the power n and the whole term is H to the power n plus 1; that means, here there are 2 to the power n terms are there. So, I can write that 1 plus n by 2 plus 2 to the power n by 2 to the power n plus 1, which is greater than equal to 1 plus n by 2 plus 1 by 2 which is greater than equal to 1 plus n plus 1 by 2. So, for H 2 to the power n plus 1 we show that this is greater than equal to 1 plus n plus 1 by 2. So, this is proved.

So, what we have seen that mathematical induction is very the most important proof techniques that we have read and different type of problems not only the mathematical things or just to verify the formula, we can also prove or verify the other type of problems where directly or indirectly we can apply the mathematical induction. And in computer science as well as in different fields of other science and engineering streams mathematical induction is very much required true proof the formulas or to verify the formulas.