Problem Solving through Programming In C
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Lecture - 52
Trapezoidal Rule and Runge-Kutta Method

In this lecture, we will be looking at another numerical method technique; that is will actually look at two. First, we look at integration, how we can integrate a function, there are several methods for doing that. We will look at only one method and you can after that; you can look up at for other methods. Next will proceed to see, how ordinary differential equation can be solved using numerical techniques using program. So, first of all let us start with integration.

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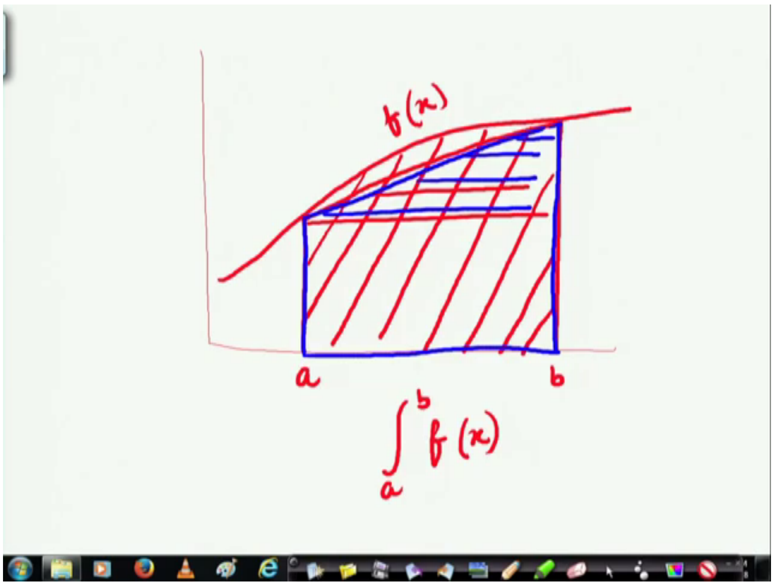

Now, all of you know; what integration means given a particular function given a particular function integration is. So, I have got a function like this and I want to integrate suppose this function is f x and I want to integrate it within the range a and b. So, we write that as integral a to b of f x, right and you also know that this integration actually means the area that is under this curve, all right, this value is the integration. So, we will look at how we can solved this problem. The simplest a very simple method is a trapezoidal method which will discuss here. So, you can see that ok.

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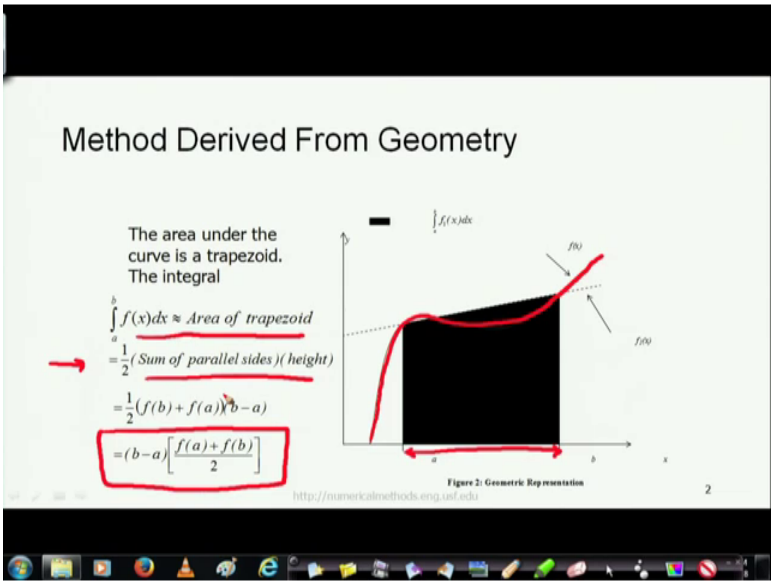

Let me go back to this this case. Now, you can see that I can approximate this curve in a very simplestic case using a trapezoid, right. So, I have got a trapezoid here, let me draw in blue. So, the area under this trapezoid, this is wade itself. So, the area under this trapezoid is approximating under this trapezoid is approximating the area under the curve, but that is becoming possible here except for this error that is becoming possible here here because the curve was very much not very scude in that sense, but if the curve where something like this.

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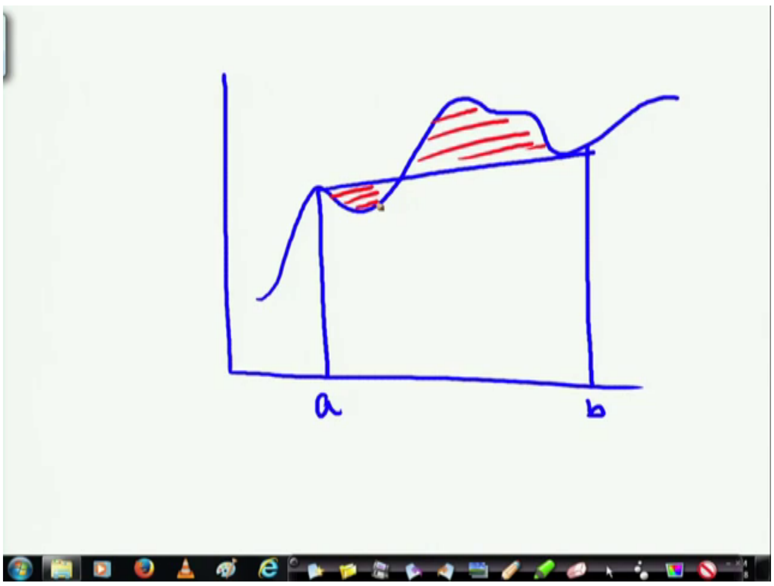

That save; it was something like this and I was trying to integrate it between these rangers then if I had fitted a trapezoid here, then the amount of error would be much more because I will be committing errors at this points ok, I will I am not considering this I am over considering this points, etcetera.

So, the error will be more. So, it is not always the case that a simple trapezoid one single trapezoid will solve, but let us start with that and try to understand how we can go ahead with the problem. So, the area under the curve in this particular curve as you can see here this particular curve that has gone through this is a trapezoid under the curve is a trapezoid this part if I assumed to be a trapezoid in that case the integral of f x d x is the area of the trapezoid and we know that the area of a trapezoid is not nothing, but half the sum of the parallel sides; that means, f a plus f b divided by the height.

Now, if I look at it this is a parallel sides then the height is this b minus a this amount, right. So, this is a known result therefore, I can see if I can approximate a curve by a single trapezoid in this way b minus a times f a, f b plus f a by 2 or f a plus f b by 2, but of course, we will see that there can be errors due to this approximation, but this is a simple formula which we can quickly compute is very easy to write a program for that you have got a function that will compute the curve. So, you call it for a and call it for b and compute this expression, you will get the integrate integral all right.

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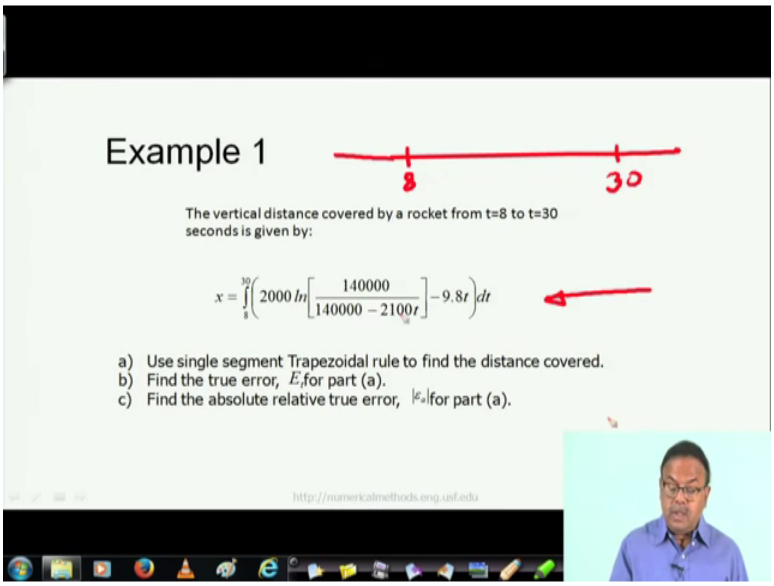

So, here is an example why is it important suppose the vertical distance covered by a rocket from time 8 to 30. So, my timeline is from 8 seconds to 30 seconds is given by this formula, all right. So, the vertical distance overall total vertical distances is this is a complicated formula now using single segment trapezoidal rule, let us try to find the distance covered, let us try to find out the distance covered for this.

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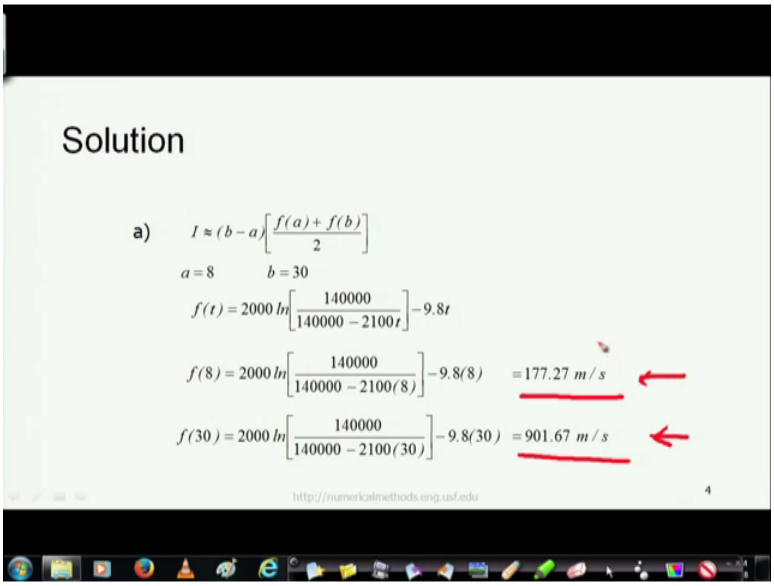

So, we know for 8 and for 30, if I compute this function this was my function.

So, for f 8, my function is yielding this value and for f 30 the function is yielding, this value, we can compute using you can use your calculator and find it out that you need not do right now. So, you can compute the values at these 2 points because this is the overall function and next. So, this is the f 8 and f 30. So, what will be my integral my integral will therefore, be 30 minus 8 b minus a f a plus ab by 2.

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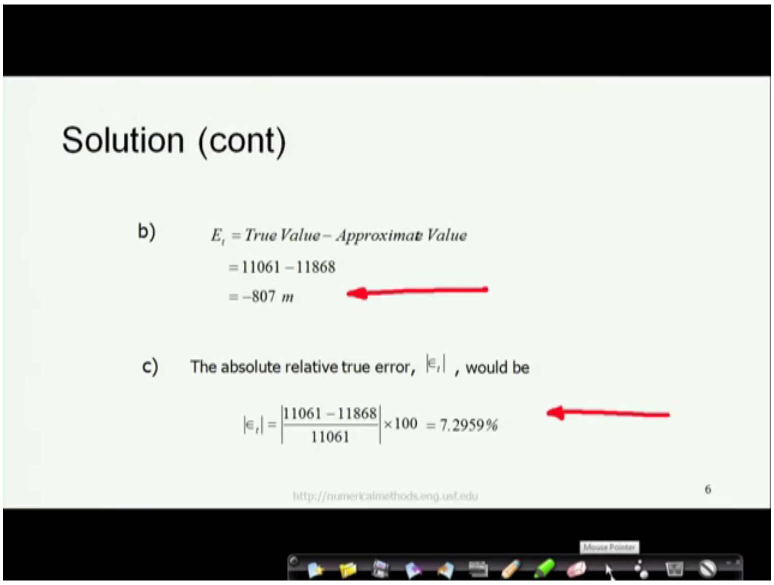

So, that is coming to this value sorry that is coming to 11868 meter, the distance covered is this, but; however, the exact value if you computed the exact value if you do a detailed computation will be 11061 meter. So, there is an error; obviously, there is an error because I have approximated the curve using a trapezoid. Now, let us see how great is the error how big is the error.

So, we can see that the true error is minus 807 meter, right; that is quite significant 800 meters and the absolute relative error which is the actual error true value and the computed value and divided by the true value you find that I am getting 7 percent more than 7 percent error, how can you better it how can you better with.

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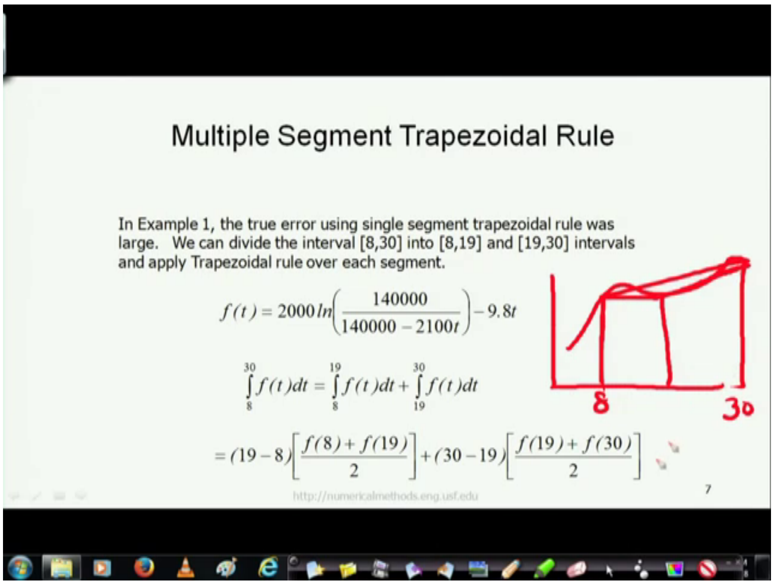

Therefore, so, our answer will be will instead of applying a single fitting in a single trapezoid, I can try to fit in more than one trapezoid here like something like this that these trapezoid here I fit in one trapezoid here I fit in another trapezoid here and I hereby I can approximate I can minimize my error to some extent.

So, here; what we are trying to do instead of taking 8 and 30 and fitting in one trapezoid for the whole thing install fitting and one trapezoid for the whole thing what we are trying to do is we are fitting in one trip trapezoid for 8 to 19 and other for 19 to 30 ok. So, now, again using the same formula we find them here is one 19 to 8 to 19, this is the integral plus the other trapezoid is giving me this. So, I am fitting in 2 trapezoids now clear.

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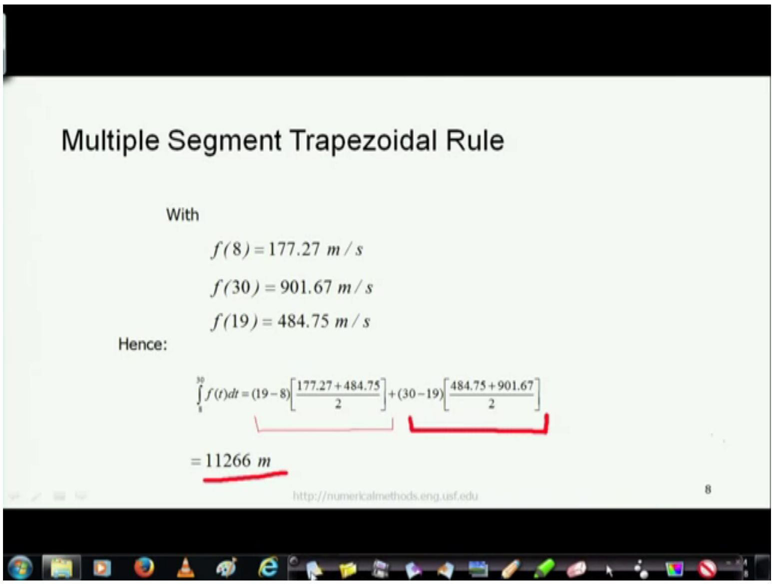

So, if I do that; then let us see whether the result is being bettered. So, I compute f 8 at f 30 and f 19 and compute both of these values the areas under the curve here and here and the result is 11266 meters. Now how far is it from the actual now you can see.

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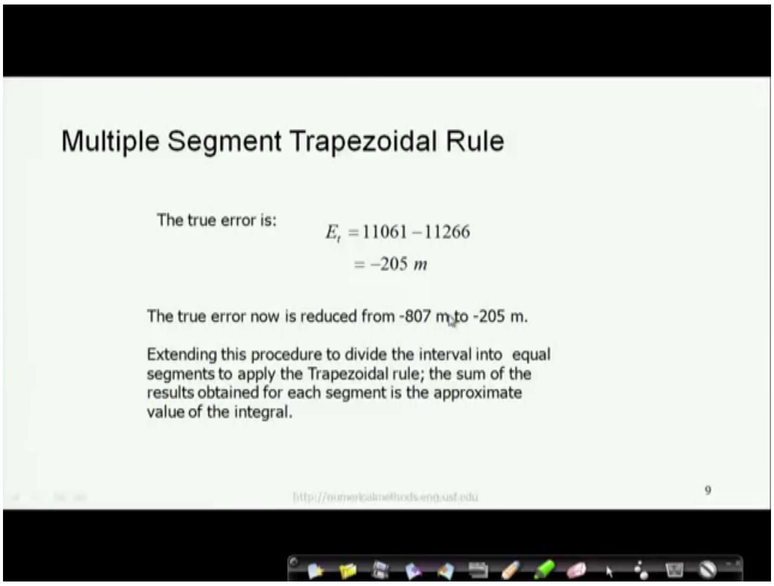

That the true error is now has reduced from 807 meters to 205 meters. So, that tells us that if we can extend this procedure and fitting more and smaller that more number of trapezoids my error will come down for the still taking q from this idea.

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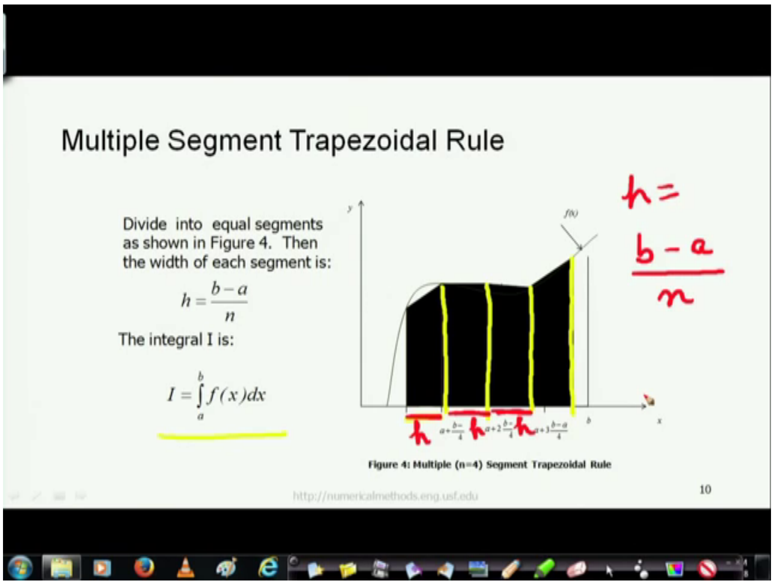

What is being done is the multiple segment trapezoidal rule; therefore, is that we divide into equal segments. So, here is one trapezoid here is one is its not coming here, let us see. So, here is one trapezoid here is another trapezoid here is another trapezoid and here is another trapezoid, I am getting 4 trapezoids here and trying to formulate this I can do that in this way. So, the integral is this whole thing which will be a sum of these trapezoids remember each these lines these lines this distance is the same. So, I am dividing b minus a by some particular value n and that is my h.

So, this is h this is h this is h like that I am going for equidistant points and drawing the trapezoid from there.

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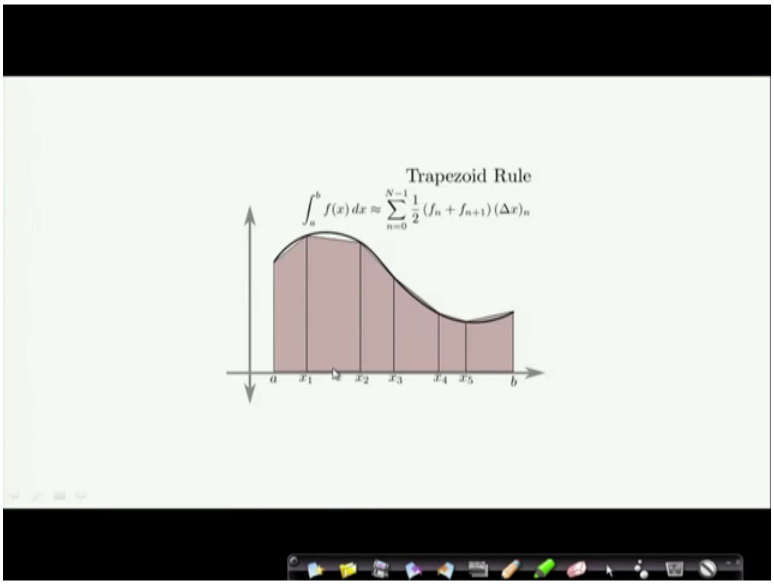

So, therefore, if I follow this as is shown in this way, the trapezoid will rule the trapezoid will rule can be written as integral of a from a to b is sum of half fn plus fm plus 1; that is f a plus f b by 2 times the particular distance that is there x o x 1 minus a x minus x 1 usually do it in the equidistant watt.

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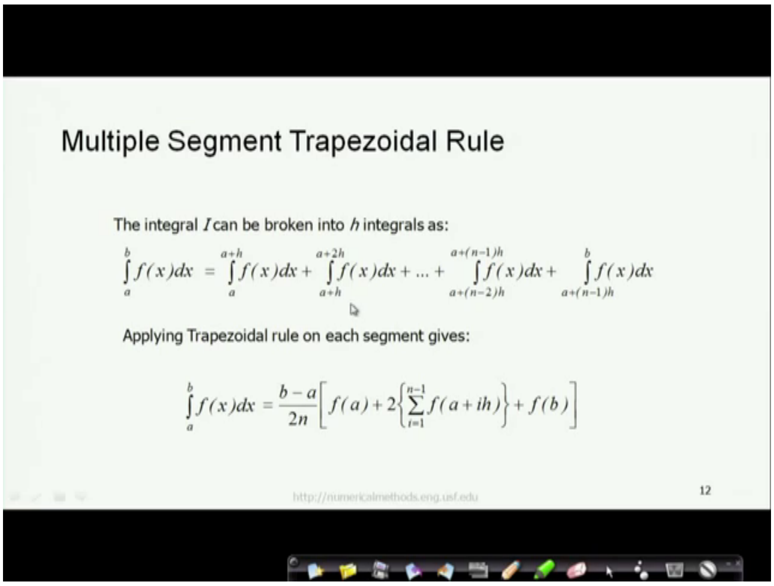

Ok. So, now, multi segment trapezoidal rule is therefore, an integral I you can be broken down this symbol from a to a plus h, I can have f x d x a plus h 2 a plus 2 h f x d x plus 8 plus n 2 h 2 a plus n 1 h f x d x a plus n minus 1 is to b.

So, all the segment serving as it together applying this I get this formula b by a divided by 2 and because if n is coming 2 is coming n times. So, f a plus f b plus 2 into f a plus size because that is coming twice here once and here, once you see here f a plus h will come here f a plus h will come right ok.

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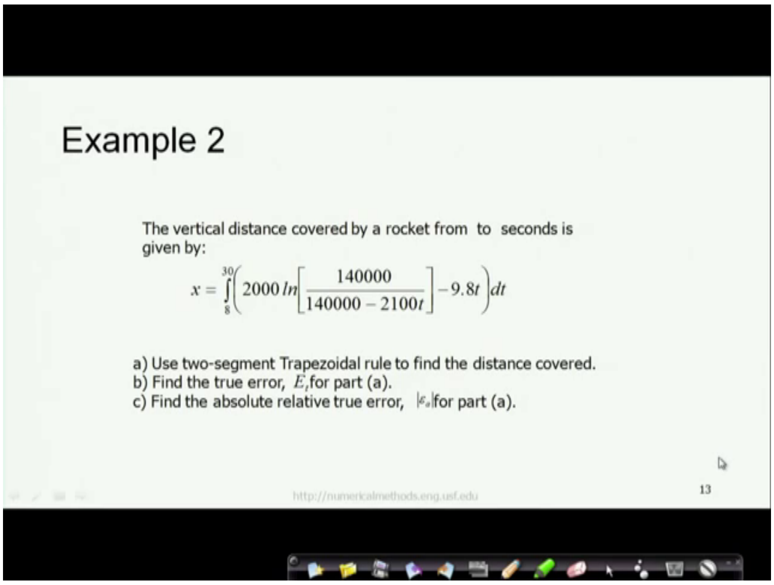

So, using this let us do the example again the same thing using 2 segment Trapezoidal rule, we could find that the error is coming down.

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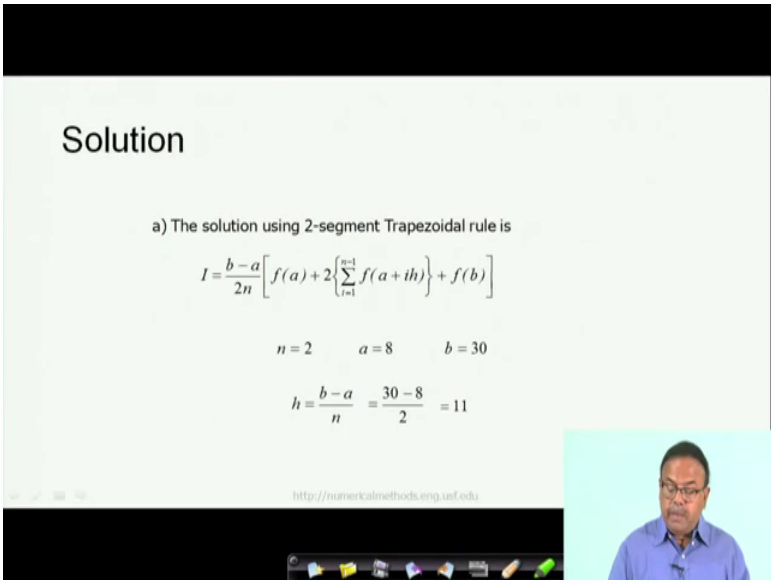

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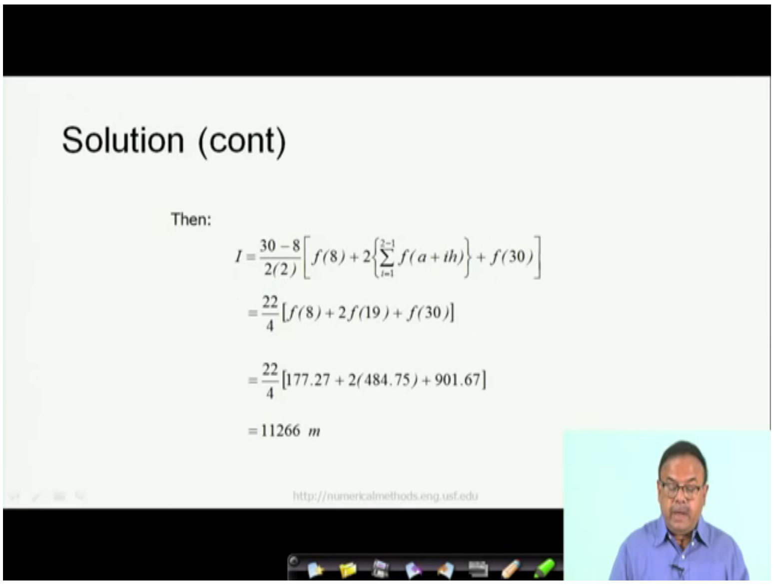

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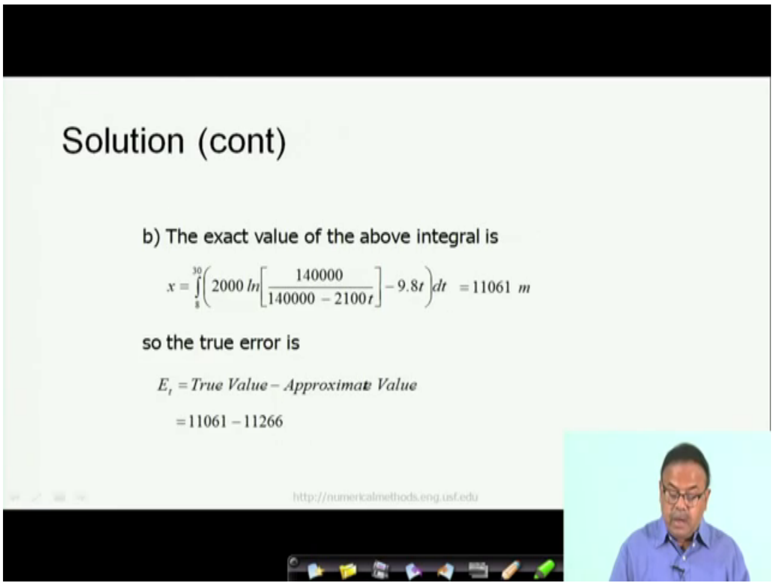

We have already seen that the true value of error is coming down.

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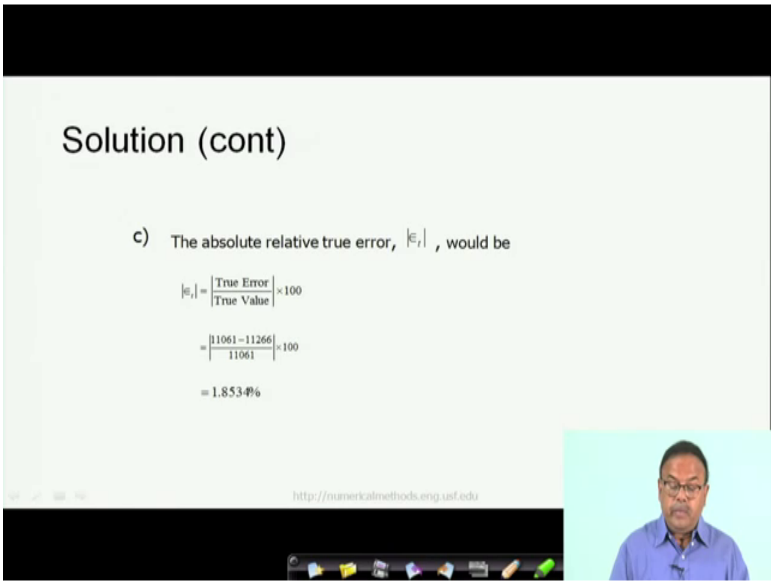

And the absolute relative error has come to 1.853.

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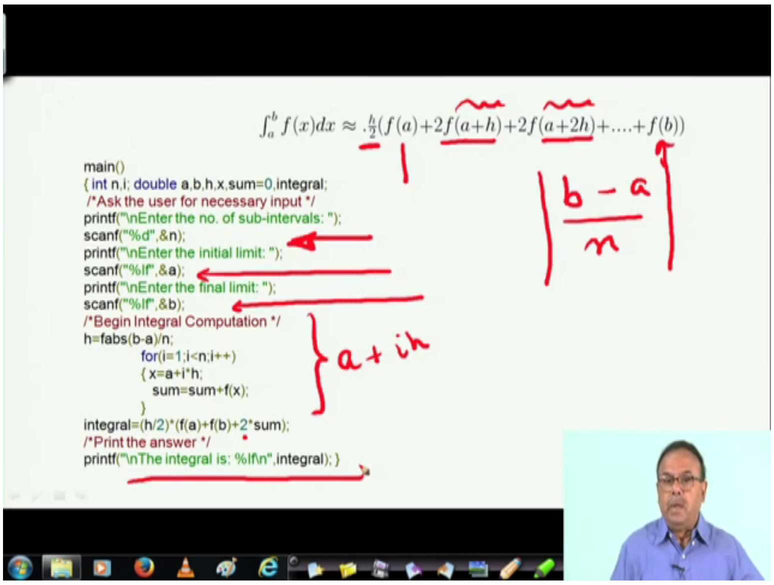

So, now let us come to c programming, straightway; how can we encode it using a c program you see here. So, I am reproducing the result here again.

So, integral of f x d x between a and b will be h by 2 f a plus 2 f a plus h plus 2 f a plus 2 h why this 2 is coming because in the first zone f a and f f f a plus h second zone f a plus h plus f a plus 2 h. So, each of these intermediate points are coming twice that is why these 2 and I have got this formula. Now, as a c programmer, your task is very simple you see here that I have defined n i; whatever I am asking the user for the necessary inputs how many number of subintervals that you want to have the initial limit a you are reading the initial limit b all those things you are reading.

Now, the integral computation is done here what is being done h is I am finding out the absolute value of b because b and a b minus a divided by n and I am taking the absolute value of that all right because it could have gone on the other side. So, now, here I am just computing the sum, what is sum this part? What initially x is a plus i h sum plus f x. So, each of them next time, it is becoming to all those things I am adding here. So, here I am in a loop I am doing a plus i h initially i is 1. So, 1 h 2 h like that I am going on doing that.

And ultimately and I am completing the sum here some plus f x and note that f x x is a separate function that is being kept somewhere here and then ultimately I find the sum is these things a plus h this points. Now integral is h by 2 this part f a plus f b plus f a plus f b plus twice the sum a plus a h a plus 2 h a plus k h like that that, but has been a plus i h has been completed inside this loop and that I had with 2 here and here is my integral. So, that is the trapezoidal rule it is the program is.

So simple, if you understand the concept ok, next we will move to another very important engineering computation that is needed is solving ordinary differential equations quickly lets all of you know what a differential equation is.

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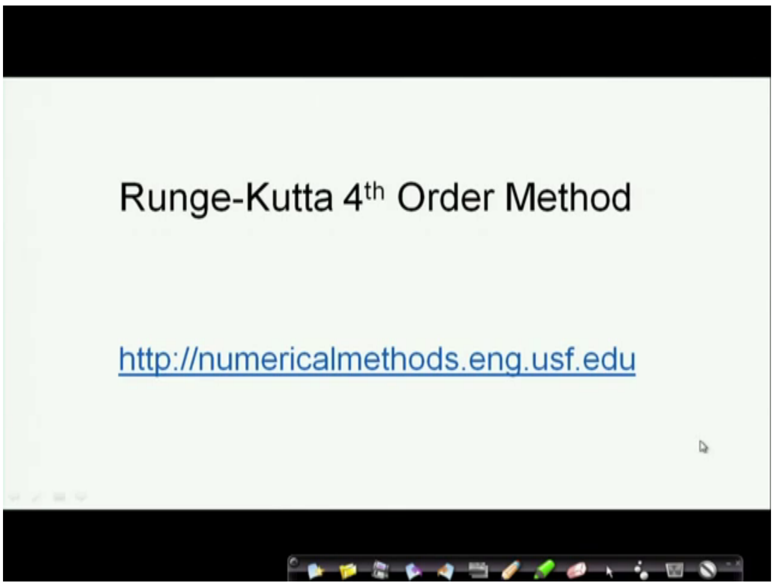

And we will in particular look at Runge Kutta fourth order method and here I will show examples that you can also find in this site of University of South Florida numerical methods and I have taken the slides from them with the permission. Now you see how to write an ordinary differential equation now an ordinary differential equation you know is d y d x f x y ok.

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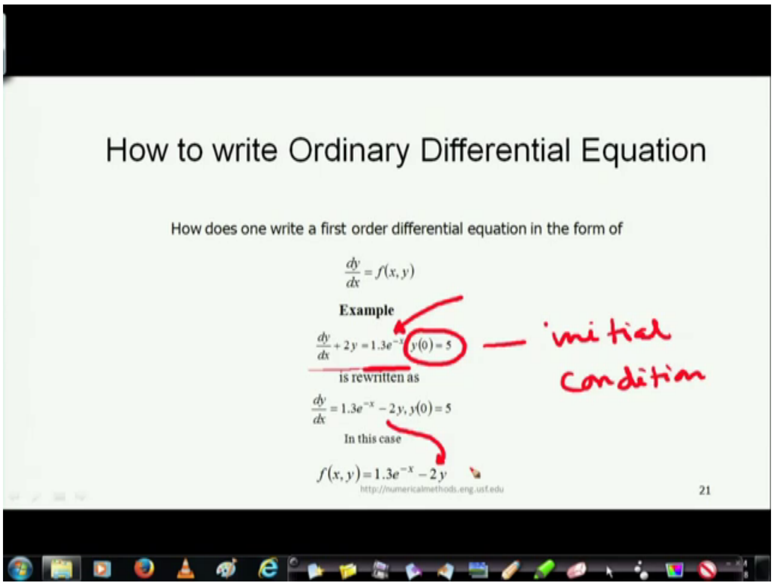

Now, d y d x; so, up how do I write it I write it as suppose this is something given d y d x plus 2 y is 1.3 into e power minus x and what is this part? This is the initial condition ok. Now this can be really nice by just changing the directions because I have to bring it to this form d y d x is 1.3 e minus x minus 2 y. So, in this case we will assume that our f x y f x y, there is a divide x is 1.3 this thing then e to the power minus x minus 2 y this is our function differential equation that we will have to solve.

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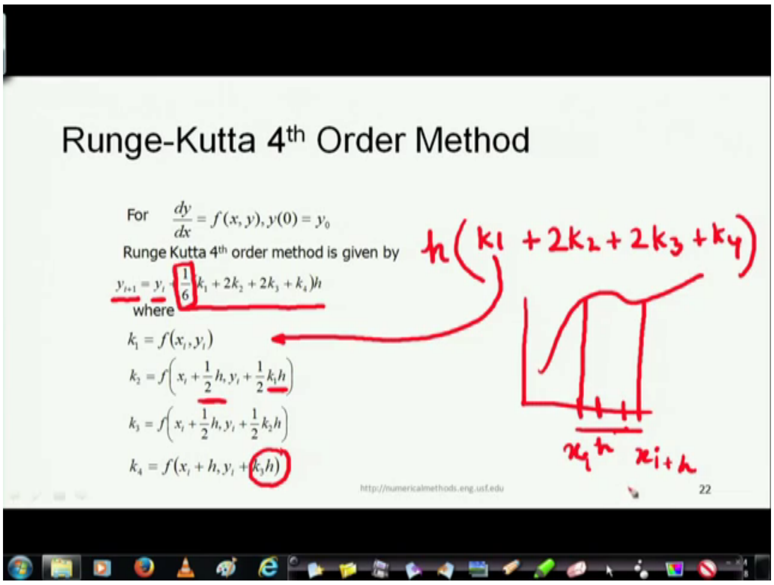

So, for d y d x, the Runge Kutta fourth order method, I am not going into the derivation of it for positive of time and you can always look at look into this at any website or you can look at any numerical method textbook. So, the Runge Kutta fourth order method takes 4 terms and if this is the expression given this d y d x my task of solving a differential equation is to find a particular y, right. So, what we are trying to do y i plus 1 is y i; some particular w i plus 1 by 6 followed by a term and what is that term 1 plus 2 k 2 plus 2 k 3 plus k 4, this whole thing multiplied by h. H is the again the sampling that that is a the distance between the individual points that we looked at; now what is k 1.

So, k 1 is nothing, but when I am taking for y i f x i y i that particular function is k 1 that is k 2 k 2 is f of. So, suppose there is a curve. Now I have been given the slope, I do not know the curve. So, I have to if I know the curve within the same problem, if I know the curve, then I can find out the value of any particular y i plus 1 given any y i. Now given any y i, I am trying to guess the curve right, I am trying to solve the curve. So, x i was here x i plus half h, I am taking whatever was my h, I am taking half of h and what is the white part of this function y i plus half of k 1 h ok.

So, because k 1 was f i k 1 was the function that was giving y given an x that was giving a y for an x. So, I am taking this what is k 3 k 3 is again this party same x i plus half h, but this part is now becoming much more predictive. So, it is half of k 2 h half of k 2; that means, o whatever has been completed here times h and k 4 is x i plus h, the last one is x i plus h because I am trying to find solve the equation within this zone h. So, x i plus h i start with x i and this is x i plus h all right k 3 h here there is no half. Now this derivation you can look at that ultimately I multiply with 1 by 6.

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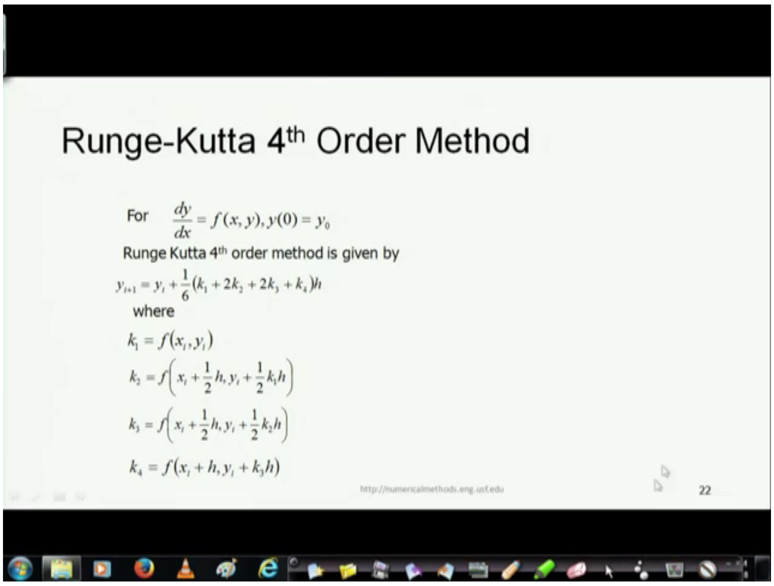

So, given this Runge Kutta formula let us quickly look at an example that we have a nice thing to look at.

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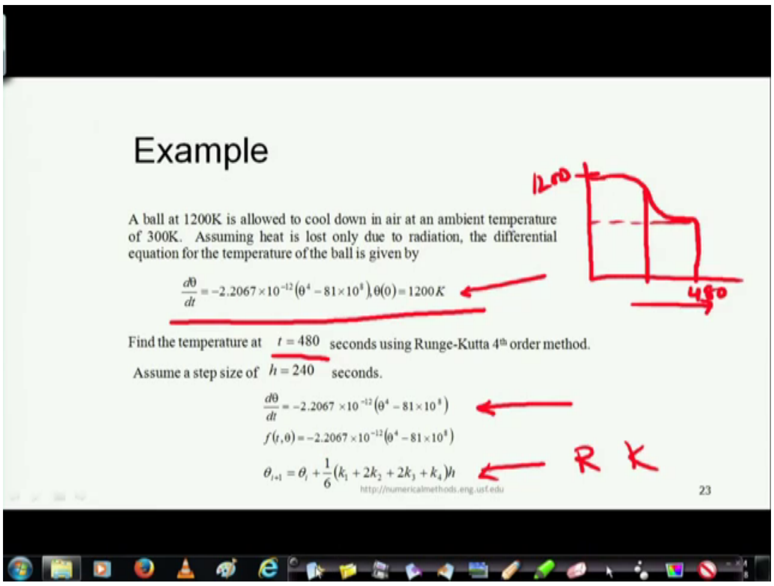

Suppose a ball is at 1200 Kelvin and is allowed to cool down in air in an ambient temperature of 300 Kelvin. So, the here the ball was heated, hence cooling down assuming that the heat is lost only due to radiation the differential equation for the temperature is given here. This one, all right where theta 0; we know initial condition is 1200 Kelvin; find the temperature at t 480. So, what is my x i and so, now, if I assume a step size of 240, I want to find out the temperature at 480. So, suppose it was up at a particular temperature after 480 seconds.

So, here is time, it was at 1200 and I have got some radiation formula using which it is coming down. So, I want to find out what would the temperature be at 480 seconds; 480 seconds from the starting point where it was 1200 degree Kelvin, I want to find out this temperature that is the y i, I want to find ait find out given the slope of this differential equation.

So, assuming a step size of each to be 240 half of this if I take half of this 240, then d theta d t, you can compute that here if you compute the d 3 d 3 that it is here. So, my formula will be theta i and theta i; k 1 plus 2 k 2 plus 2 k 3 plus k 4 divided by 6 times h that is the Runge Kutta formula of a formula. So, that is what I want to find out so right.

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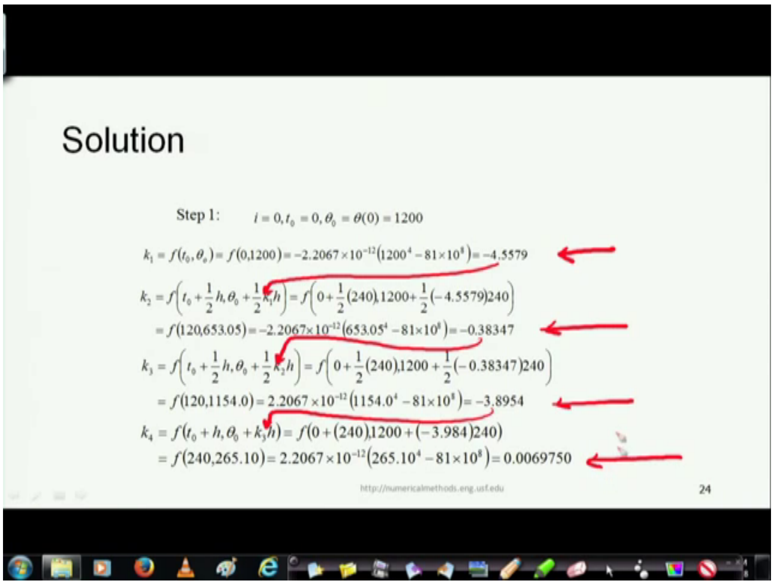

So, now stay pond; you see, I am not going into all these calculations, but I am finding out the value of k 1, all right, I am finding out the value of k 2 using this value of k 1, k 1 is being used here and I find out the value of k 2 again the value of k 2 is being used here, I find out the value of k 3 and k 3 is being used here I find out the value of k 4, I find out those values all right manually I am doing that now.

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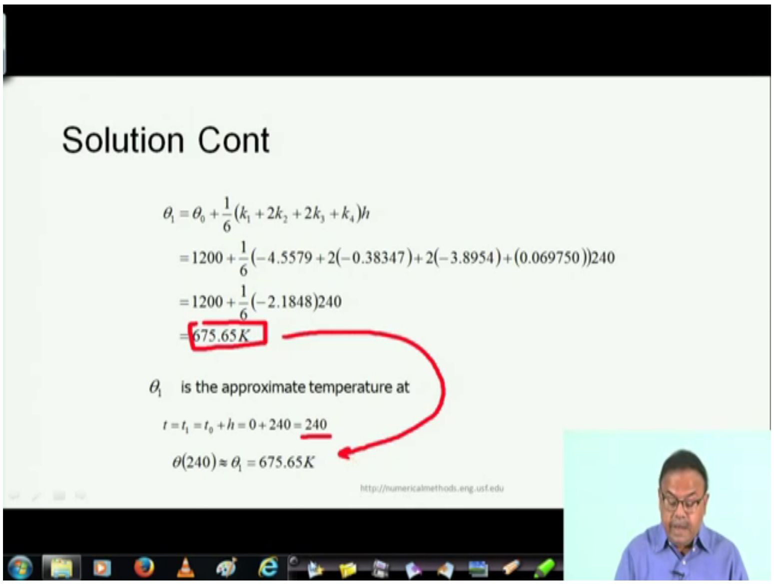

So, the solution therefore, the approx then theta 0 was 1200 initial condition and here I put in the values of this times h h plus 240. So, I have taken it at the midpoint. So, I find that that the temperature sorry the temperature that would be would be 675.65; that is the approximate temperature at 240, all right. So, this will be the value at 240. Next I have to find out at 480. So, what would I do?

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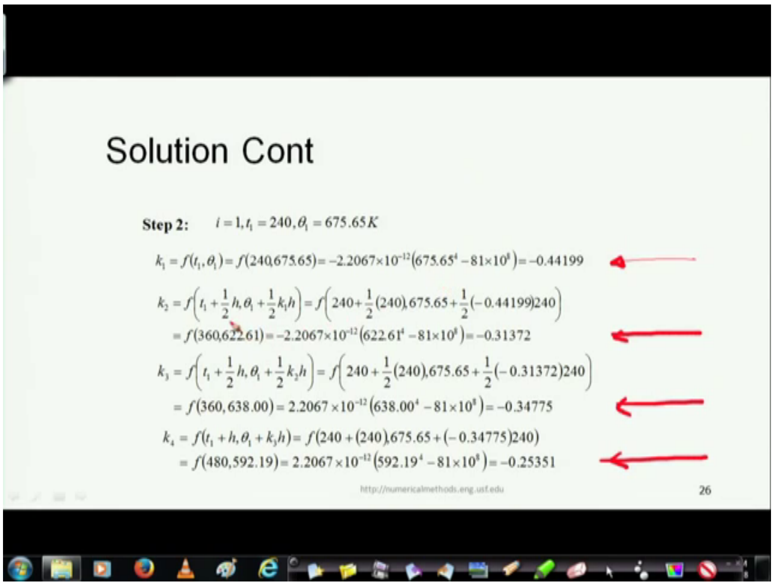

Step 2; I have taken I have taken had half point, I found out now my initial value theta 0 is changing.

Now, again do the same thing find out k 1 find out k 2 find out k 3 find out k 4. Now with this initial condition using the same function all right h is again, what is what will be h; h will be again 240 because I have to find it out at 480 right at 480 degree temperature.

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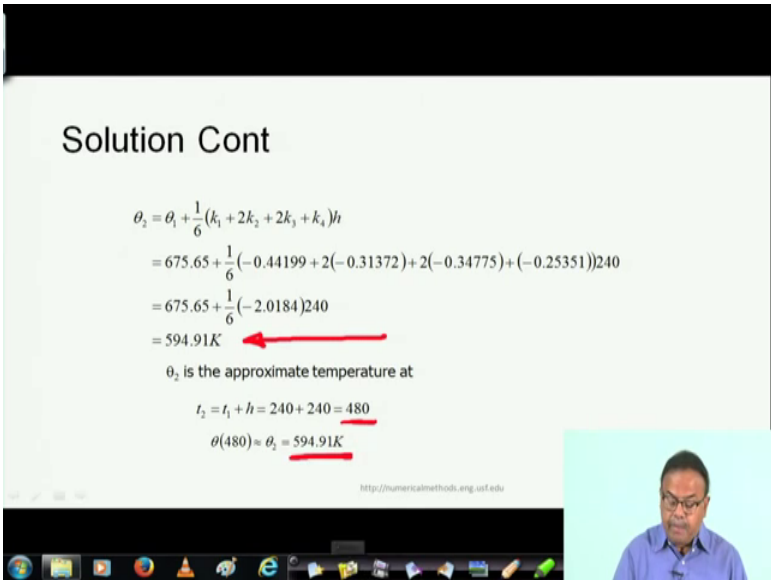

So, now, I find out that using theta one is now 675.65 that is approximate value that I got earlier and I compute this I find out using the same Runge Kutta method that this is the approximate temperature at 480 degree at 480 seconds would be 594.91 degree Kelvin ok. This is how we apply Runge Kutta method and very useful.

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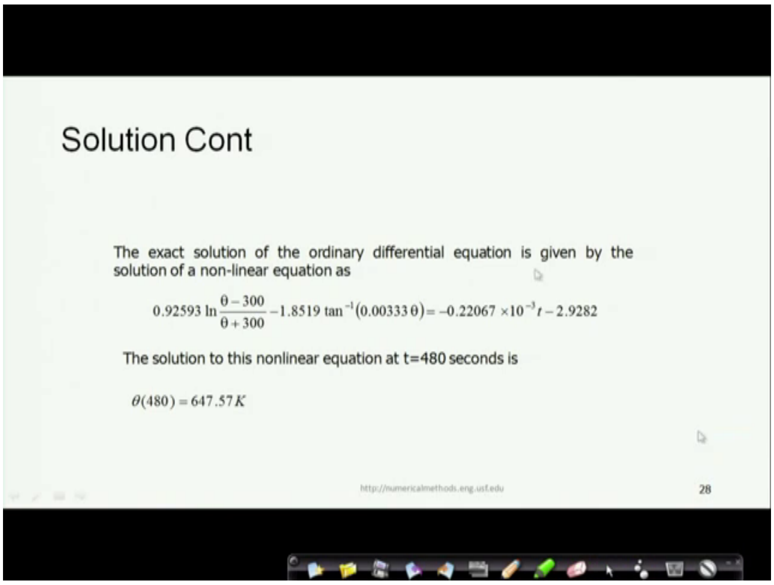

So, the exact solution of the differential equation is 647.57, if I solve it now, it was and we got it how much 594.91 and 594.91.

So, it was not very far, not very far around 50 degree Kelvin that is certainly an approximation.

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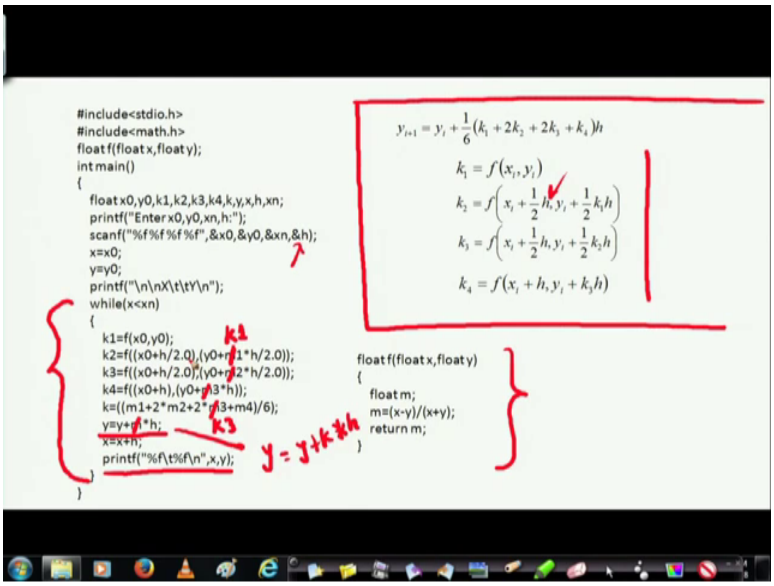

So, now we have understood; what is Runge Kutta method.

Now, I want to write a c program for it again you might find the approach to be mathematically very novel maybe in cricket, you may find some initial difficulty in understanding, but I am sure you will understand it very fast, but you will see that the encoding, it is a c program is or as a program is really simple. Now you are learning see you can in future will be using MATLAB and other things, you will be able to solve it very easily, I am showing you the c solution here on this site, I have kept what we learnt till now, right y i plus 1; that means, in our case the temperature at 480 degree is temperature at 240 degree plus this and k 1 is f x y k 2 is f x i this one f x i plus half h plus y i plus half k 1 h etcetera, etcetera, etcetera.

So, now let us look at the program we have got the math function everything ready. Now this function some of this function has to be written off this function is here its being shown as a very simple function it can be any function the differential equation function that earlier function that we have shown that log function that you have to write ok. So, here is an example of a simple function x minus y y x plus y just what elastity function whatever is your differential I mean sorry, the differ the devidates given that will come in this function. So, now, you see here I did how many times at the age value x 0 x 1 value all these things I read now the key key things comes here this is the implementation of the Runge Kutta method. So, very simple, you see, I am computing k 1 k one is f x 0 y 0.

So, I am coming to this function computing x 0 y 0, then I am going back computing k 2 for x 0 plus h by 2, h has been read here scan f h has been read ok. This h has been read, then y 0 plus m 1, I am sorry, here it should be k 1 k 1, this would be k 2, all these m s you read it as k, ok. This is k 3, this is k 3, this is k 2 times h. So, actually we are computing this thing straight way, I am computing all these and then y is a sign y plus k 8, sorry, this whole thing is k 8 here this will also be k.

So, it will be this statement will be y will b y plus k h y plus k h right and xx plus h I am incrementing x and going on all right I am doing it for 2 intervals you like I come over here. So, I ultimately come to this print f and I print the value of y for a particular x that is the Runge Kutta. So, this this is straight way amenable to some c program and for each of these fs, you are calling you are calling the function every time, all right.

So, this is the Runge Kutta method for solving a differential equation. So, I had encourage you to look at other methods of a integration like Simpsons. One third rule is a very popular method and differentially equation Runge Kutta. This is known as Runge Kutta fourth order method because we are taking 4 terms, here, there are even, but this works very well for most of the engineering problems. So, I will encourage you to write programs on this and later on hence fourth, we will move to another interesting aspect that is known as recursion a new style of programming which will take up in the next lecture.

Thank you.