An Introduction to Algorithms

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Lecture - 56
Network Flow (Contd.)

So, in the last class we have seen this, properties of a net flow.

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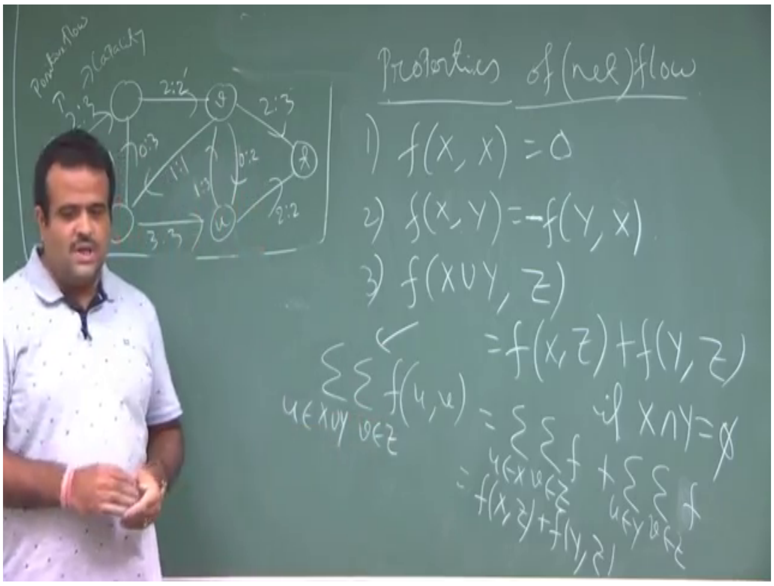

If we have a if a net flow then f of x comma x is 0, then f of x comma y is equal to minus of f of x y comma x and then if you have a 2 disjoint set, if of x union y then this is basically sum of this. So, this we have proved in the last class.

Now, we just want to use this to have the theorem which is related to the value of the flow. So, this is basically the value of the flow.

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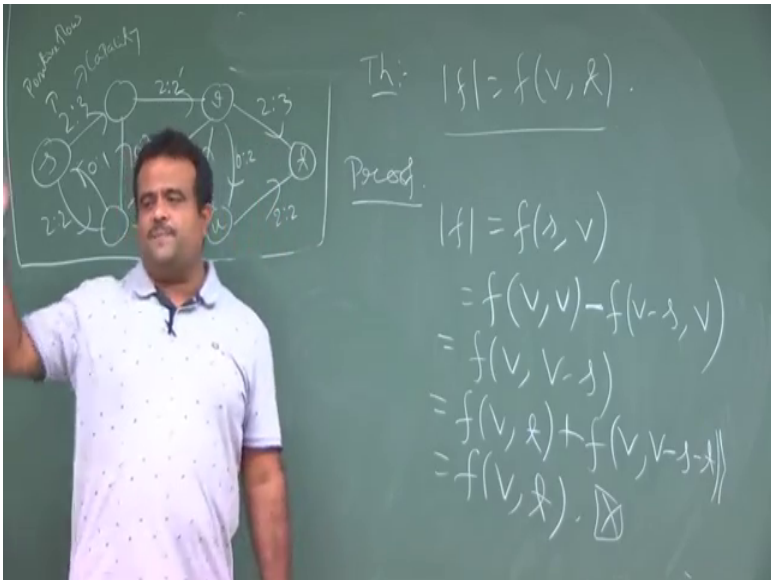

So, basically it is telling the theorem is telling we have given. So, value of the flow is basically f of v comma t. So, it is the total flow which is receiving at the sink point. So, this we have to this we have to prove. Now we know what is the value. So, how to prove this. So, we know what is the value of the flow we know the value of the flow is basically the flow which is passing out from the source. So, we have the source here. So, this is basically the flow which is passing out from the source and this is the net flow. So, this is basically summation of sorry s of s comma v. So, this is the definition of the value of a flow because here f is net flow not the positive flow.

If you have to take positive flow then minus of that, but here you are dealing with the simplifier notation which is net flow. So, then then how we do that. So, this is this we can write f of v comma v because f of v comma v is 0, just now we have seen then f of v minus s comma v because yeah. So, this is basically you can omit the brackets. So, this is coming from these properties we have discussed. So, this is nothing, but f of v comma v minus s. So, now, this we can write as f of v comma t plus f of v comma v minus s minus t. So, this is the way we define this i mean we are using the last 3 lemma we have discussed. So, this is nothing, but f from v comma t. So, this is the proof so; that means, whatever the flow we are passing at the same (Refer Time: 03:31) source that will be absorbed in the sink. So, that is the property you want, and that is quite obvious no because we want to say this is the network we want to pass the current.

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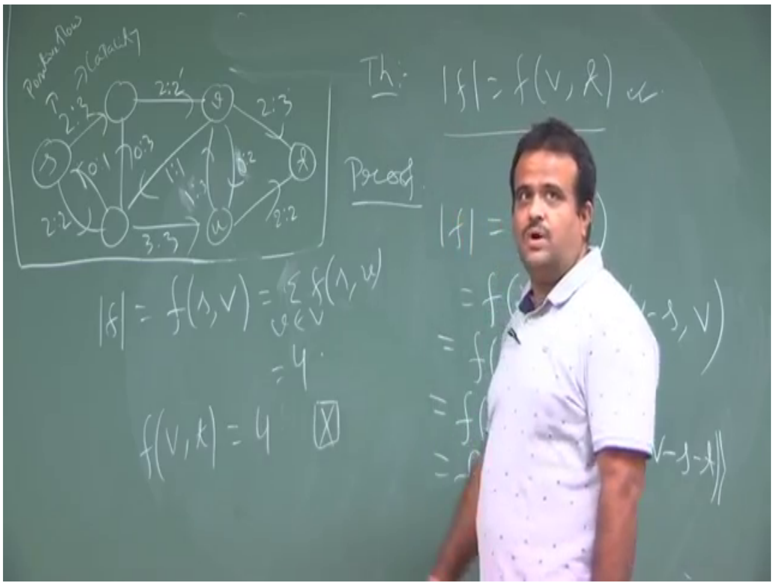

So, we want that whatever the current we want to pass from here net current that should be absorbed here and that is the value of the flow and this is the theorem is telling that will be absorbed there i mean that will be passing at that end c ok.

Now, if we take this example. So, what is the net flow what is the value of the flow. So, value of the flow for this network. So, this is a network we have these are the basically capacity and these are basically positive flow in the example we have discussed in the previous classes we discussed these are the positive flow now what is the net flow? Net flow is basically after we cancelling the flow cancellation if. So, we assume the without any loss of generality, the flow can the flow will be only one direction from u to v not from v to u. If there is a flow say earlier it was 2 it was earlier the positive flow was it was 2 it was 1. So, what we did once instead of flow in both direction we just want to flow in one direction and we are just talking about net flow. So, net flow is one because this was cancelling out. So, this is the flow cancellation we did. So, effectively we have this net flow.

So, then what is the value of the flow. So, value of the flow is the flow net flow you are passing from this. So, this is basically 2 plus 2 it is basically 4. So, this is basically summation of f of s comma v, v is from this. So, this is basically 4. So, this is the value of the flow and now we can check what is the flow at the sink flow at the sink is basically 2 plus 2. So, 4 this is the example of this theorem. So, whatever flow we are passing from the source that will be going to the sink. So, this is that way now we defined another concept which is called cut flow cut I mean. So, cut. So, what is the cut?

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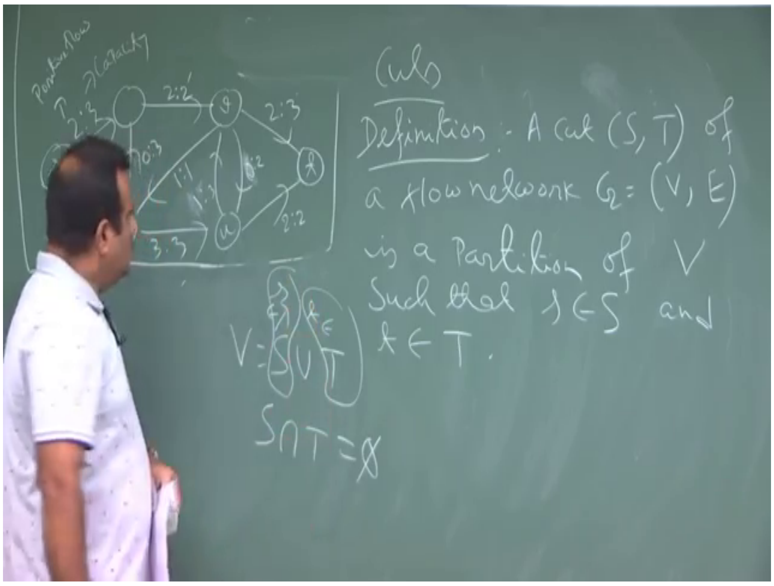

Let us just define the cuts. So, this is definition. So, a cut S T this is to set S and T capital S and T of a flow network G, which is basically V comma E is a partition of V such that the source belongs to sorry source belongs to S and the sink belongs to T.

So, any such partition basically we have we have a graph g and we are taking a partition; that means, we are dividing the vertices are there V. So, this is the V vertex. So, V is basically S union T and this is the disjoint set this is the disjoint partition and such that s will be here and t will belongs to here. So, this is the property. So, we take a class we take a plan capital S from using this I mean which containing the source S and we take another set capital T which contain the sink T.

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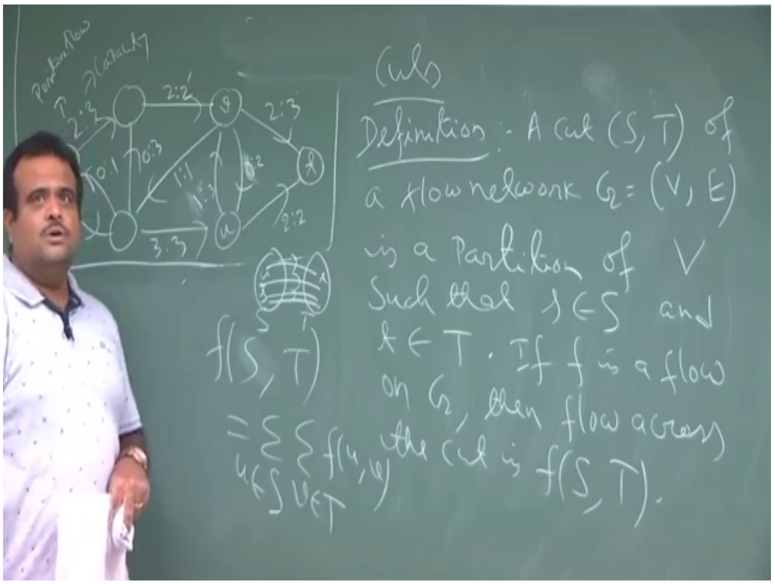

So, this will this is where any such partition is called a cut any such partition is called a cut. So, for example,. So, so this is a cut now how we defined flow across the cut. So, suppose we have a cut here, now if f is a flow. Flow means we now means the net flow always because we simplify the notation flow on G then flow across the cut. So, flow across the cut is basically f of S comma T, this is how we defined the flow across the cut. So, the total flow across the cut so; that means, basically. So, f of sorry f of S comma T is basically nothing, but summation of double sum this of u comma v where u is belongs to S set and v is belongs to T set.

So, basically we are partitioning v total vertex is v we will be partitioning into 2 parts S set T set. Now there are h form u to v I mean one part is in s, another part is in t we know s small s is here we know cap small t is here. So, now, this is basically sum of the flow from one set to another set where the these are called bridges like one vertex is in s capital S another vertex is in capital T. So, we take the sum of all such flow and that will give us the total flow that will give us the flow across the cut that is how we define this. So, let us take some example over here for this network say. So, these are the. So, for this network let us take some example.

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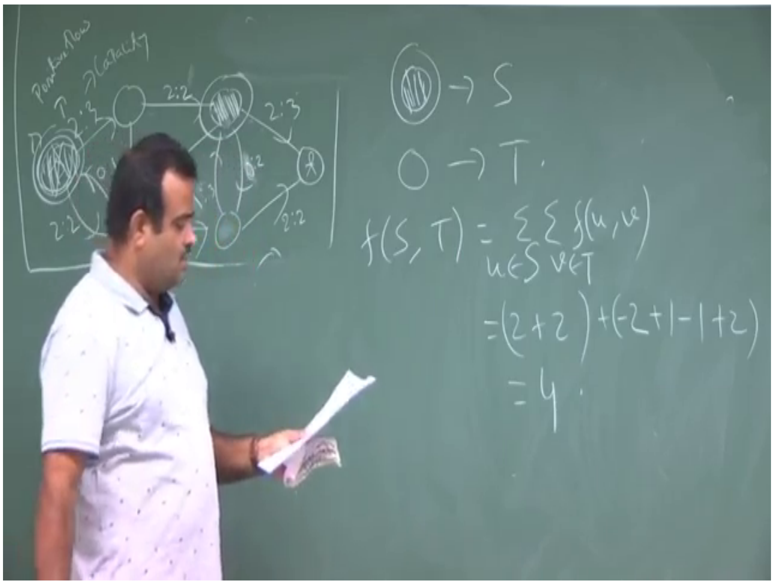

Suppose this is our source and this suppose our s is basically these are the study. So, this is s yeah. So, basically these 2 vertices are in s say, these are in S and remaining all are in T. We can take any cut this is one cut, only thing we need to consider that this is a partition and the source must be in capital S and the sink must be is capital T. So, that is the only thing we need to take care.

Now, what is the f of S comma T then? So, f of S comma T is basically summation of all the vertices all the flow between the edges such that one edges in S and another S must be in T. So, what are the edge basically who are in. So, this is one edge this is one edge and this is one way. So, this is basically minus 2 will come, then this is coming from here to here. So, what is the flow from here to here? So, this is basically flow is 1. So, the flow from here to here is minus 1, because this is the queue symmetric property of the flow and there we have a flow from here to here. So, any other vertices are in capital T. So, at here to here to here to here it is it is basically one. So, it should be yeah. So, now, we consider all the vertex all the vertex form this to this.

So, this is nothing, but 2 plus 2 for this 2 what I for this these and this and then from for we take this node from this node this is the 2 plus 2 and this is another edge, but this minus of that is 0. So, that is no contribution now we take another vertex from s capital

S. So, this vertex for these vertex. So, we have this is the flow from this to this is basically minus 2, then then we have flow from this to this is plus 1, flow from this to this is minus 1 and then we have flow from this to this is plus 2. So, this is basically coming out to be 4. So, that is that that we know the value of the flow. So, this is the result. So, if you take any cut and it will basically give us the value of the flow, if you take any cut. So, that is the theorem we are going to show. So, this is the observation. So, not only these 2, we can take any vertices and it will work.

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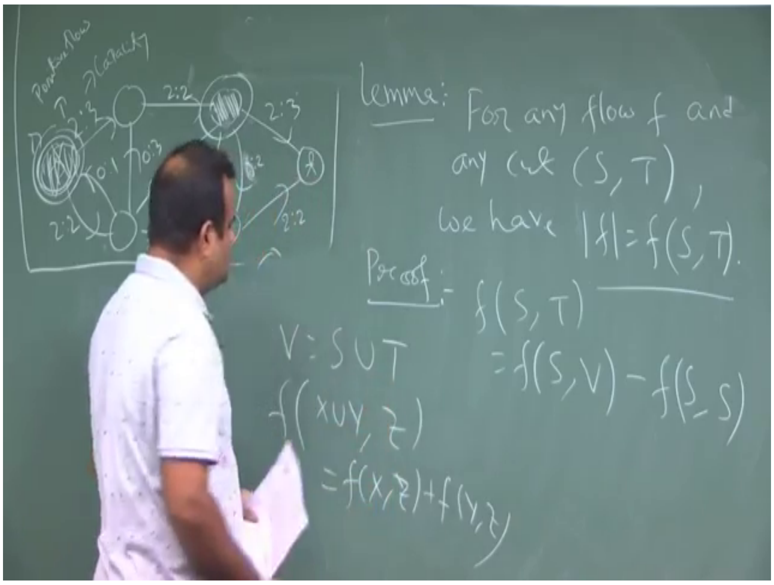

So, the observation is; if you take any cut. So, this is the lemma. So, for any flow if and any cut S comma T we have the flow value, value of the flow is basically value the value of the flow across the cut. So, this is basically s of f of S comma T.

So, how to prove that, you have seen through this example. So, if you take these 2 vertices in s, then the value and other in t capital T then the value of the flow across the cut this cut is 4. Not only this we can take any other partition and we can see the value of the cut is always 4 and that is the value of the flow. So, how to prove this? So, basically f of S comma T is nothing, but f of s comma we will use that properties f of S comma V minus f of S comma S because v consist of S union T and S union T are disjoint. So, we know this property f of we know this property na f of s comma this is basically f of x comma z plus f of y comma z. So, you are using this, where s x y are disjoint. So, by using this we can show this. So, this is 0 this we know f of x comma x is 0.

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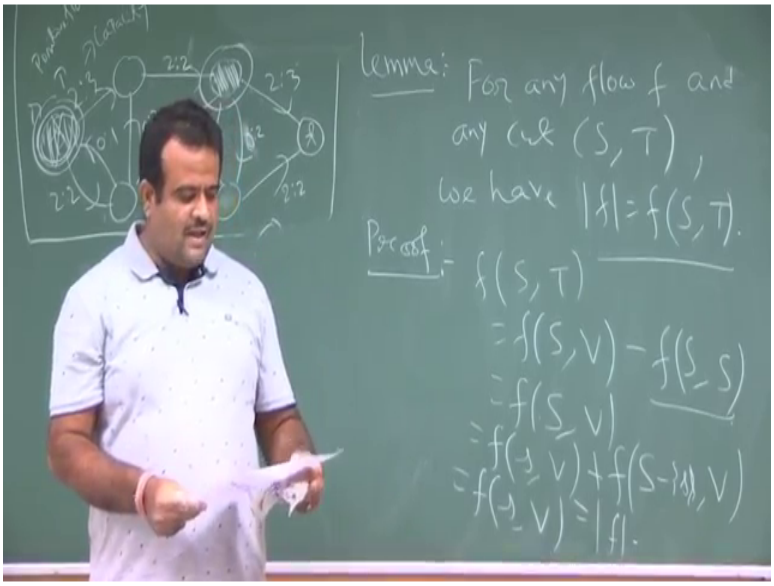

So, this is basically f of S comma V, now again we can write as f of s small s comma v plus f of capital S minus small s comma V.

Again that property of disjoint property. So, this is nothing, but f of s comma v. So, this is basically the value of the flow. So, this is the proof. So, if you take any cut then that will give us basically the value of the flow. So, now, we define the capacity of a flow. So, capacity across a flow. So, this is basically some of the capacity on the it just in the flow sorry the capacity of a cut.

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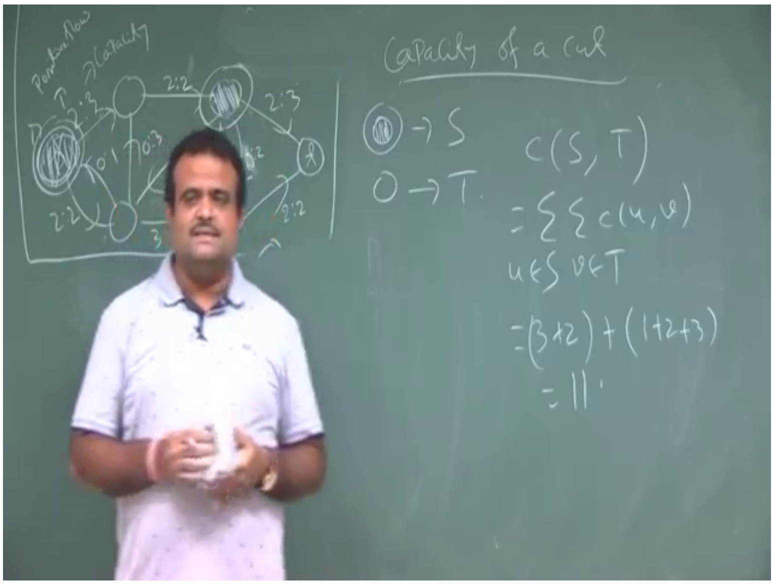

So, how we define that. So, for example, if we take this this flow network where these are the vertices in S and other one T. So, the capacity of this basically the capacity. So, we add up the capacity of each of the vertices. So, this is basically we denote this by c of S comma T this is nothing, but summation of double summation of c of u v, where u is belongs to s and v is belongs to T. So, this is here. So, if you do this to 3 plus 2 then plus what are the values going 1 plus 2 plus 3, 1 plus 2 plus 3. So, this is basically 11.

So, this is the capacity of this cut. So, now, we are having we want to use the cut for. So, basically what we are looking for we are looking for max flow we have a network we want to flow maximum current. So, max flows for that we are bringing this cut, and we will see that there is a relationship max flow with the capacity of this cut. So, that will give us a algorithm basically. So, mean cut max flow.

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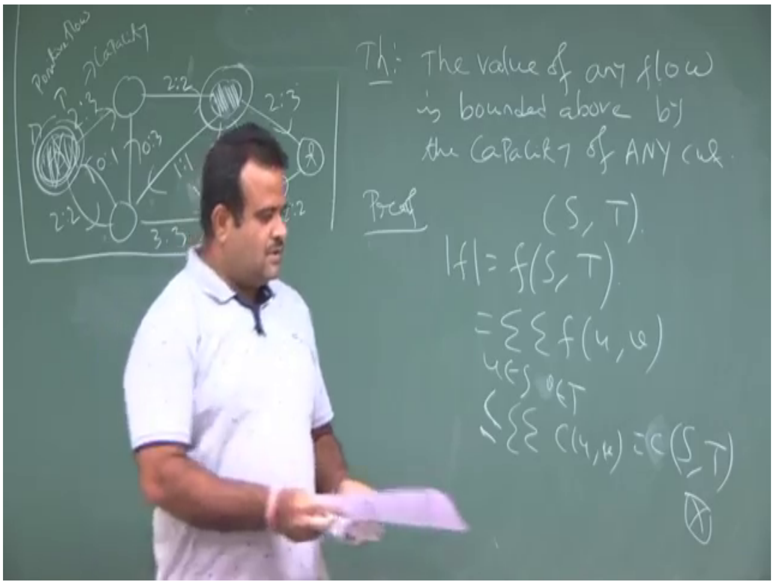

So, let us just write that theorem. So, this is sort of upper bound of the on the max flow. So, the value of the value of any flow is bounded above by the capacity of any cut, that is important any cut. So, the value of, so max value of the flow. So, this is the upper bound. So, how to prove this? So, let us take a cut over here say S comma T be any cut now we have seen the value of the flow is nothing, but value of the flow across the cut and this we know this is basically summation of f of u comma v, v belongs to s and a u belongs to s and this v belongs to T.

Now, this this is the flow. So, this must be less than all the capacity on that h. So, this is basically summation of double summation of C of u comma v and this is nothing, but our c of S comma T. So, this is the proof. So, this is the proof. So, this is the basically the value of this. So, now, we define another concept which is called residual network and that will give us the one algorithm to find the max flow.

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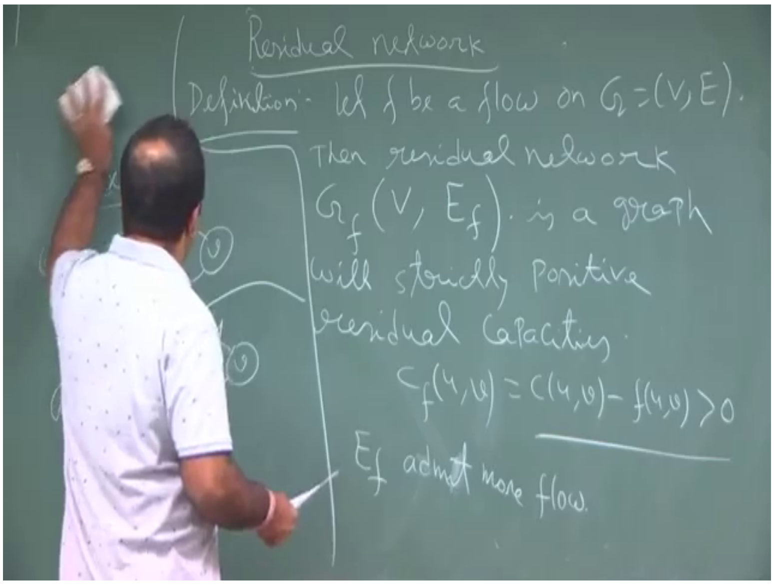

So, this is the last concept and then we will talk about augmenting path. So, this is the definition the residual network. So, what is the definition of residual network? So, suppose we have given a flow let f be a flow net flow, on a flow network G which is basically v comma E. Now we defined a residual network we define the residual network which is another graph which is denoted by G f and whose vertices are same, but only thing for the h we have E of f, because this is having different (Refer Time: 22:45) and we may have edges also different. So, this is the is a graph directed graph with strictly positive residual capacities so; that means, C of f of u v is equal to c of u v minus f of u v which is basically greater than 0.

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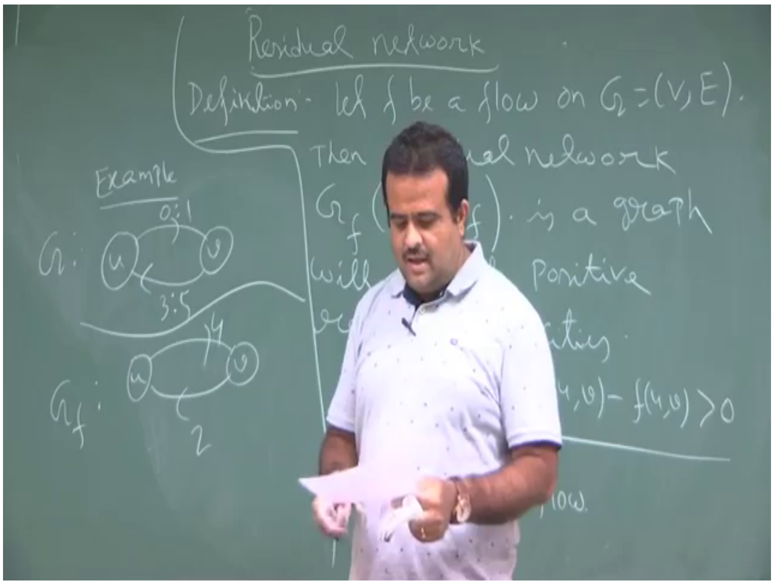

So, for example, for example, and so; that means,. So, if we have a vertex u. So, this is our G. So, suppose there is a u v. So, suppose this is the situation 0 is to 1. So, this we are dealing with the say positive flow 3 is to 5. So, suppose this is the original graph part of the original graph u v. Now in the residual graph we have same vertex u v, but only thing we are changing the edge how it is changing the edge? We want to see how much more current we can send, how much more flow we can flow on that. So, that is the idea. So, how much more current we can pass. So, that is the thing. So, that will become in in G f basically this will become like this u v. So, how much more current we can pass from this to this, this capacity is 5 we have 3. So, we can pass 2 and then from this to this we have my. So, from this to this, this is the net flow. So, this is minus 3 and the capacity is 1, 1 and then minus of minus 3. So, this is 4. So, this is the basically so; that means, 4 unit of current we can flow here and 2 unit of current we can flow.

So; that means, this E of f is basically admits more flow. So, how much more we can flow more flow. So, how much more current we can pass in there. So, this is the thing. So, now, we will define. So, the lemma is basically.

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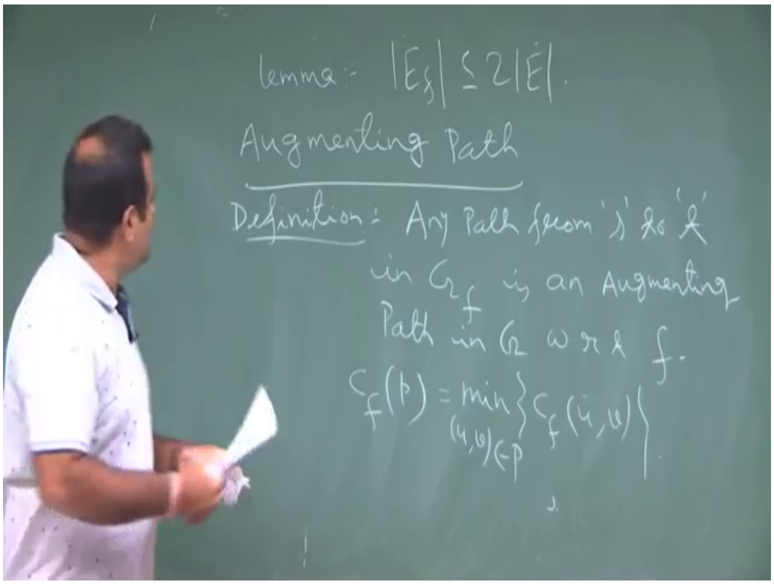

So, this is a lemma. So, this is basically number of edge must be twice off number of this. So, now, we will define augmenting path. So, this is basically path for the this residual graph. Now how we defined this definition. So, any path; path from s to t in G f is an augmenting path in G with respect to the flow with respect to the flow f. So, then then we can just write that c of f p the flow value can be increased among, because this much. So, we will take an example. So, c of f p is basically minimum among this c f of u v, minimum among the capacity in the in that path in that part and that minimum we can pass, this is the more value we can pass on that path.

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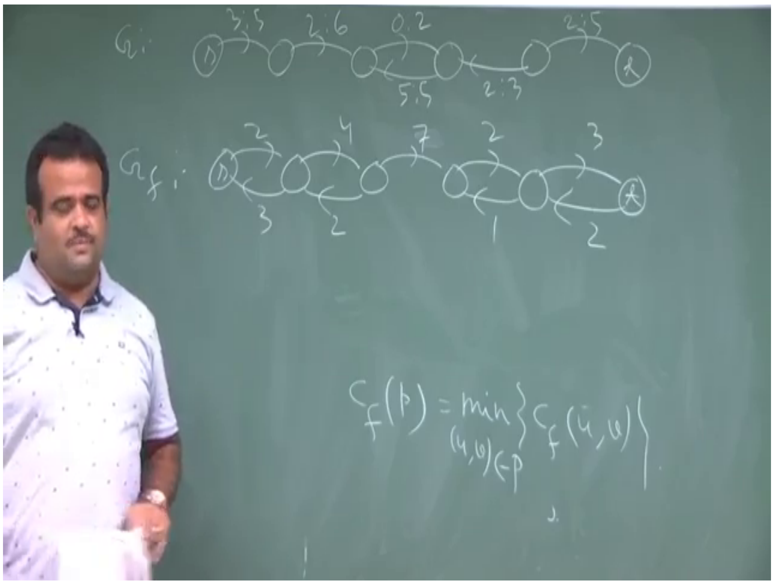

So, we will take an example let us take an example suppose we have a in the original graph we have. So, this is s and we have some vertices. So, this is these are the zeros, 2 these are the capacity and the flow, 55 i. So, this is t say 2 comma 5. So, suppose this is a path from s to t in the original graph, now what will be the then the path for from the in the residual graph that is that is called augmenting path, in G f how we draw G f. So, G f is basically this is s. So, now, how much more, we can put from here to here 2 now from here to here we can pass 3; now how much more we can pass here to here 4 and here to here 2 now here to here how much more we can pass 2 and this is basically overall all we have used. So, there is no way and then this is basically here to here no this is basically 7 because this is 5 current is coming. So, this way we can pass this 5. So, this is minus 2 minus or minus 5 this is 7 and here to here we can pass this 2 and here to here still here we can pass 1 and here to here we can pass 3 and here to here again pass 2.

So, this is the augmenting path from s to t. So, any path for me in the residual graph is augmenting path, but this is the augmenting path now what is the minimum capacity in this path? Minimum capacity is 2 here in this way so; that means, this 2 we can 2 value we can easily pass more on this network on this path. So, that is the idea. So, that will give us a theorem. So, this is called min cut max flow theorem.

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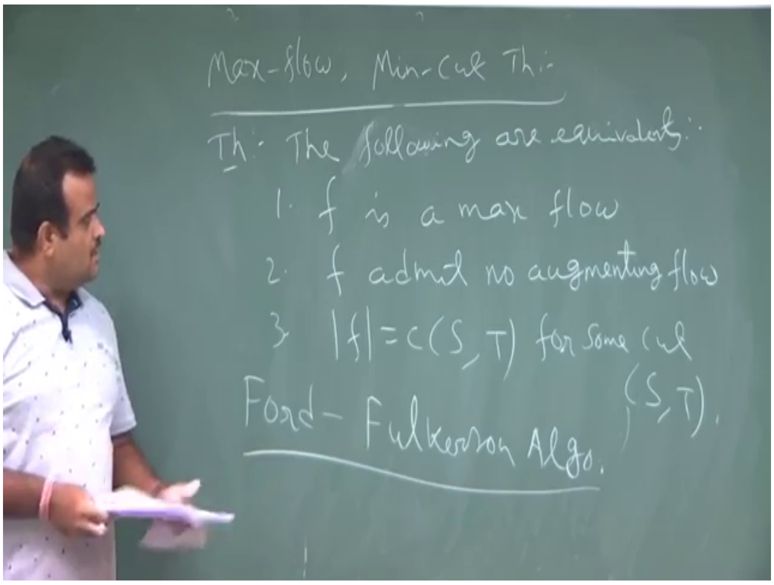

So, this theorem will have a give us the algorithm. So, max flow 2 minutes. So, we will just state the theorem we are not going to prove this in this course. So, the following are equivalent, the following statement are equivalent or following are equivalent what are they?

So, if f is a max flow is a max flow then f admits no augmenting path; that means, because if there is a augmenting path, we can pass some more current there. So, that is; that means, it is not a max flow then this is basically c of S comma T for some cut. So, there exist some cut. So, this is the theorem and there is a algorithm called ford Fulkerson algorithm based on this theorem, which will be which is the algorithm to finding the max flow. So, this is this algorithm is based on this theorem. So, we are not discussing this now, but this will be I mean this theorem I mean this algorithm is based on the theorem and we are not proving this theorem in this class.

Thank you.