An Introduction to Algorithms
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Lecture – 55
Network Flow (Contd.)

So we are talking about network flow flow network. So, just to recap in the last class we have started this topic.

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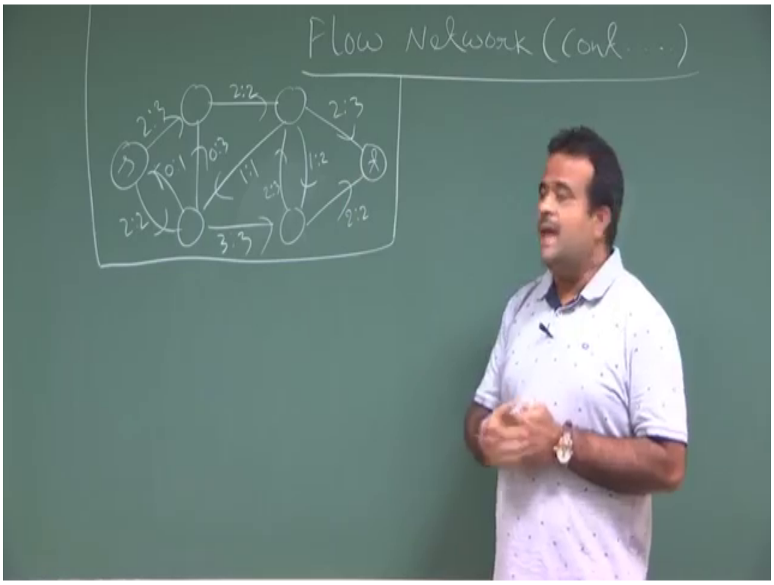

So, a network graph a flow graph is a graph you say directed graph. So, where we have a source node s and we have a sink node t. And each node each edges are associated with some capacity this is the capacity. So, this is these are the capacity basically of each edge. So, if there is no edge between any 2 vertices then the capacity will be 0. So, then this is the this is the flow now you want to this is the positive flow positive flow. So, in the last class we defined the positive flow it is a we want to flow some current from source to sink so that in the intermediate vertex there will be no consumption of the current. So that means, it is a function form it is a positive flow.

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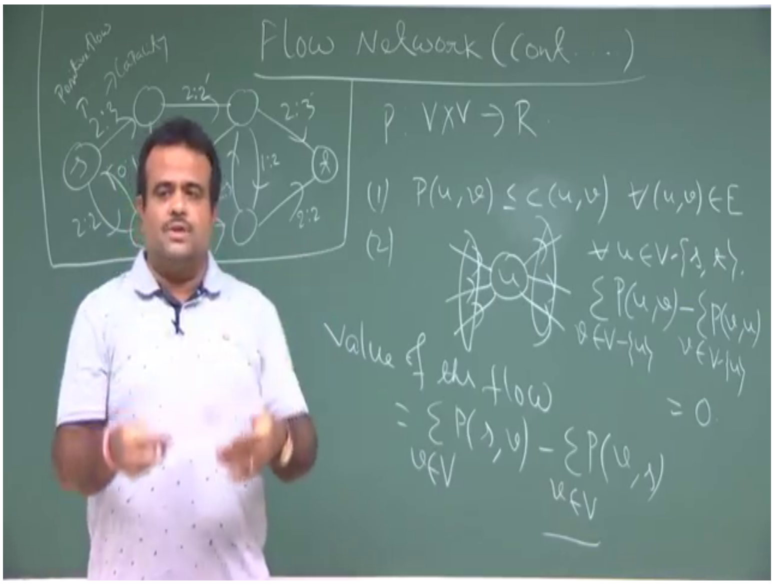

It is a function from v cross v to r such that is satisfy few condition like first condition is the flow positive flow should not be greater than the capacity.

So, p of u v must be less than equal to c of u v for all u v belongs to E. And we have seen the flow conservation; that means, if we take a vertex u. So, the flow coming into this vertex and flow going in from this vertex this sum this sum should be same. So that means, summation of p u v. So, this is flow going out v belongs to v minus u minus summation of p v u v belongs to v this should be 0. So that means, no flow consumption should be there. So, this is called this is one of the property this is for flow consumption. Then the value of the flow we defined.

So, the value of the flow is basically, the flow which is going out from the source. And this is true for this flow consumption properties true for all u which is not source and sink, but for source and sink the we have passing the current from the source and sink is. So, this is being the current sort of. So, so this is the value of the flow value of the flow is basically that the effective flow effective current we are passing from s. So, that is basically summation of p s comma v this is for all v belongs to v minus summation of p v comma s. This is the value of the flow. So, this same amount of flow we are though is going to the sink also.

So now we have seen another concept which is called flow cancellation. So, that is that is like if you have a vertex say from this 2 vertex u to v. So, there is a flow to amount to this is positive flow 2 is going there one is coming here. So, instead of that if you just make it 0. So, effective flow is this is called net flow you different on net flow. So, effective flow is we can make it 0 we can make it 1. So, this is the effective flow form u to v. So, this is basically flow cancellation. So, this is basically without loss of generative we can assume that flow is going to only one direction. So, either from u to v. So, from v to u. So, if there is 2 direction we will just cancelling the flow and we will take the net flow. So, that is the concept.

So, let us defined the net flow.

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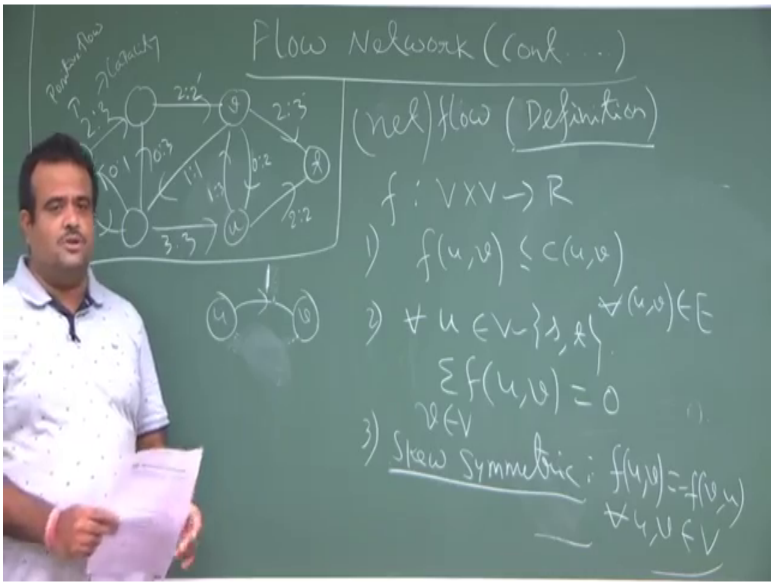

So, net flow so, this is the definition of net flow. So, basically net flow is a function f form v cross v to r such that it satisfy 3 condition again the same capacity constant say for u v. So, it should be less than c of u v for all u v belongs to E. So, net flow should be less than the capacity and flow consumption is 0 at every point. Because every point has effective flow is 0. So, that we have to write. So, for all u which is basically coming from this said we have summation of f of u v is basically 0. This v is belongs to this.

So, here we have one sum in the positive flow we are having 2 sum, but here we have one sum because we are just considering the net flow we are considering we are doing the flow cancelation if there is both side flow. So, this is the flow consumption flow conservation proper property and another property is called skew symmetry city, this is the new one skew symmetric property. This is telling so, suppose there are 2 vertex u to

v. So, if there is a flow from a f from u to v then there is a flow, then we then there is a negative flow form v to u, this is the symmetric skew symmetric property.

So that means, f of u v is equal to minus of f of v u for all u v belongs to v. So that means, if there is a. So, if there is a flow then from reverse direction it is minus of that. So, effective flow. So, if you say for example, for this vertex for this is u this is v. So, there is a effective flow from u to v is 1. So, what is the flow form if this is 1? One unit flow is going form this is the effective flow net flow. Then what do you say what you can say from this to this we have minus 1 flow is going something like that. So, this is the skew symmetric property of this. Now this is the definition of the net flow and you already have a definition of the positive flow.

Now, we will see this 2 definition are equivalent. So, how to get that? So, we want to see this 2 definition are equivalent positive flow and the negative flow. If you have given the positive flow we can have we basically have a net flow if you have a net flow we can have a positive flow.

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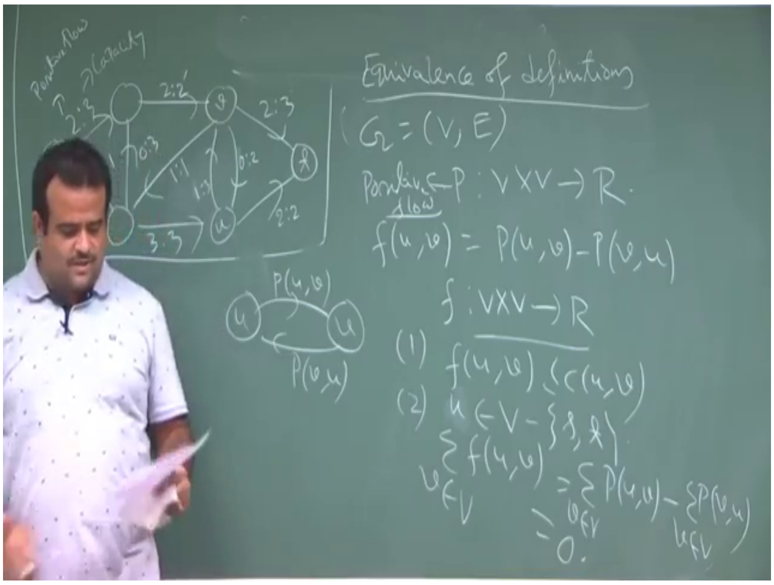

So, so this is called Equivalence of 2 definitions one is net flow another one is positive flow. So, so first one suppose we have a graph, we have graphs net we have a flow we have a flow graph. Now suppose we have a positive flow p. Let suppose you have a positive flow on this graph. Now you have to have a net flow. So, how to define the net flow? So, we just defined f of u v is basically p of u v minus p of v u. So that means, if there is 2 vertex u to v. So, this is basically p of u v this is basically p of v u. Now the net flow is we defined this net flow by this way.

Now, we can easily see. So, this is the function form v plus v to I. Now we can easily see we can easily verify this is a net flow net flow function, because we can see this is satisfying 3 conditions like capacity constant. So, this is basically f of u v is less than c of u v, because this is anyway less than the capacity and we have subtracting some quantity if which is this is greater than equal to 0. So, this is true and another property is the flow conservation property So that means, if you take a vertex v from this which is not source are saying then we want to calculate summation of u comma v, v is basically summation of this is say u comma v.

This is basically nothing these we want to. So, it to be 0 this is nothing but summation of this summation of p u comma v minus summation of p v comma u. So, this is all u and this is all v. Now we know this is 0, from the definition of then positive flow. This is the positive flow we have given a positive flow function, then we different a function like this and this function is a net flow function. Now we want see the keys skew symmetric. So, if we have a so, skew symmetric property means the value slow value from u to v is basically negative of.

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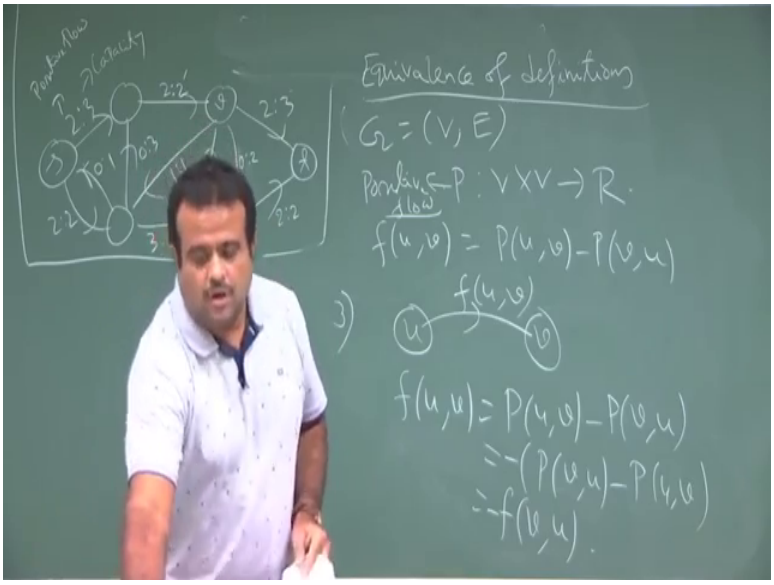

So, we have u and we have v. So, so if this is so, this is the third property.

So, what is the flow value from u to v? So, this is basically so, f of u v if basically how we define p of u v minus p of v u. So, this we can take minus of p of v u minus p of u v. So, this is nothing but by this definition this is nothing but minus of f of v comma u. So, this is the skew symmetric property. So, these 3 properties are satisfying. So, this is a so, this is nothing but a net flow. So now, you want to so if we have given a net flow how we can get a positive flow. So, that is the other way around.

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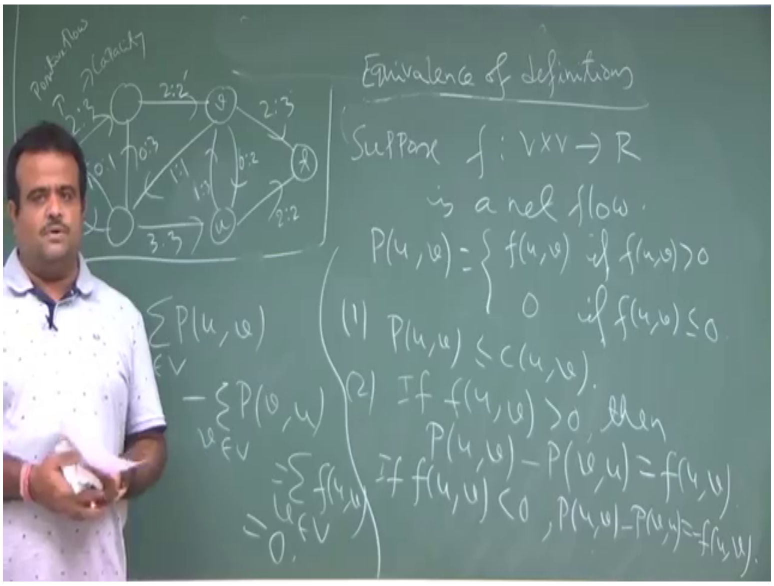

Suppose we have given the net flow say, then suppose if this is a net flow function. So, this is a net flow net flow now if this is a net flow. Now the question is how we can have a positive flow function. So, we define like this p of u v is because positive flow is always positive. So, we defined this f of u v if it is 0 like sorry, if it is positive otherwise it is 0 if f of u v is less than 0. So that means so, if we have 2 vertex u v the if there is a flow f, now if f is positive then that is our p of u v. But if f is negative then we do not have then p of v is 0 basically.

So, that is the way we defined this. Now the question is how to prove this is a positive flow? So, for positive flow we have taking so, the condition like capacity constant. So, this is trivial because p of u v is either 0 or this, and both are basically less than c of u v. Because capacity if there is no ways then the capacity is 0 no edge between a 2 vertex u v. Otherwise it is the positive. And the second one is basically we need to show flow conservation. So, for flow conservation what we need to do? So, we did so, if this is greater than 0, then we know p of u v minus p of v u this is basically f of u v. Because in that case this is 0 this is this is 0 because this is positive means, this is p of u v is this 1 and minus of that is negative so, that is 0. So, this is ok.

So now if this is less than if p of u v is less than 0, then we have that is minus, we have p of u v minus p of v u is basically minus of this. So, this is basically by the skew symmetric property. Now, therefore if you take the sum like summation of p of u v this v is wearing from v minus summation of p of v u. So, this is nothing but summation of f of u v. And this we know this is nothing but f of will be these we know 0, because net flow consumption at any point is 0. This we have already this coming from the definition of net flow.

So, these 2 these 2 definitions are basically equivalent. So, given a net flow given a positive flow we can have a net flow given a net flow you can always think of a positive flow. So now we will deal with the net flow because it is the simplification simplify notation in set of both the direction we have the net flow effective flow. So, that is the basically simplifier simplification in the notation. So, we will deal with the net flow good. So now, suppose we have a net flow, then how we will define the value of the flow? So, this is another definition.

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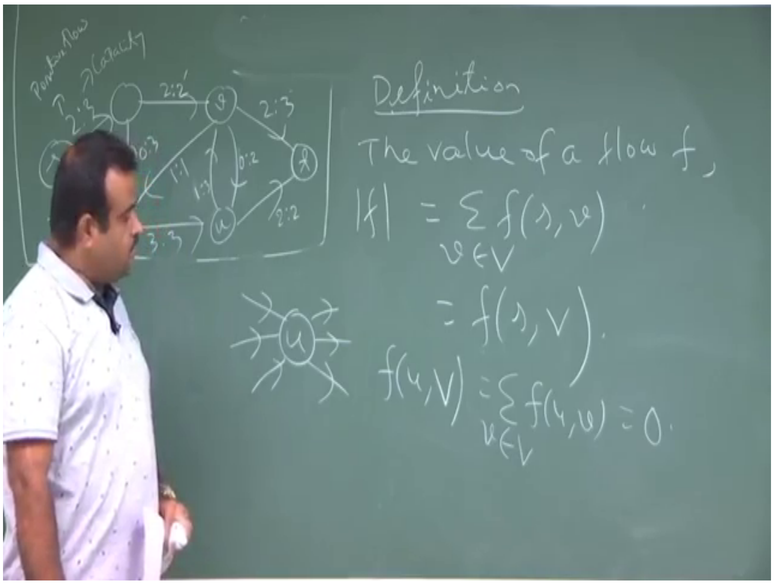

The value of the flow the value of a flow f is denoted by and it is basically given by summation of f of s comma v. And this v is coming from v.

So, this is basically in said it is we denote by this. So, this is basically the net flow. So, we have a source vertex s. So, you have other vertices over here. So, net flow we are just talking about net flow. We are not talking about positive flow, in that case there will be a flow from this side. So, we are just considering the one side flow. That is the effective flow or the net flow. So, that way. So, we just sum. So, these are the net flow. So, we sum all of this all of this net flow. And that will give us basically summation of f of s comma v this. So, this is the this is the flow we pass from the source to the sink. And then so yeah. So, that is the think. So now, this is the set theory (Refer Time: 18:57) set theory notation f of s comma v.

So, similarly we can I mean the flow consumption. So, if you take a vertex v. So, the flow consumption is basically on this vertex is 0. So, how we different that? So, that is basically in terms of net flow f of u comma v is 0. So, this is basically in said notation this is basically proper. So, this is nothing but summation of summation of s of u comma v. So, this is in said notation. So, this is 0. So, this is basically the flow consumption at any node is 0.

Now, we will have some properties of this properties on this net flow some properties.

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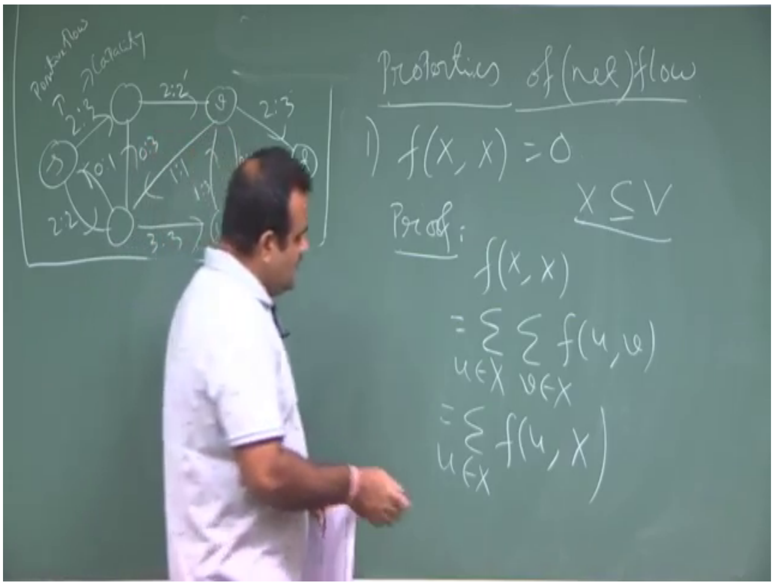

Properties of flow specially in net flow we have taking about now our flow is net flow. So, you gave given f we want to talk about some properties of f. The first lemma first property of f of x comma x is 0, where x is a said. So, we are we have dealing with the said. So, how to show this? How to prove this? So, how to prove this? S of s comma s is 0. So, to prove this suppose what is s of s comma s? So, this is basically double sum. So, this is s is a subset of v. So, s is a subset of v it contain some of the vertices, s is subset of v.

So, basically this is nothing but summation of double summation of f of u v. Now u is belongs to x and v is belongs to x, both are belongs to x. So now, this we can write as summation of f of u comma x. So, basically so this is basically the net flow which are going form s s to s. So, this is nothing but 0 because you know the net flow of any vertices is 0. So, sorry this is basically coming from double summation. Now we take the now we take the sum of this.

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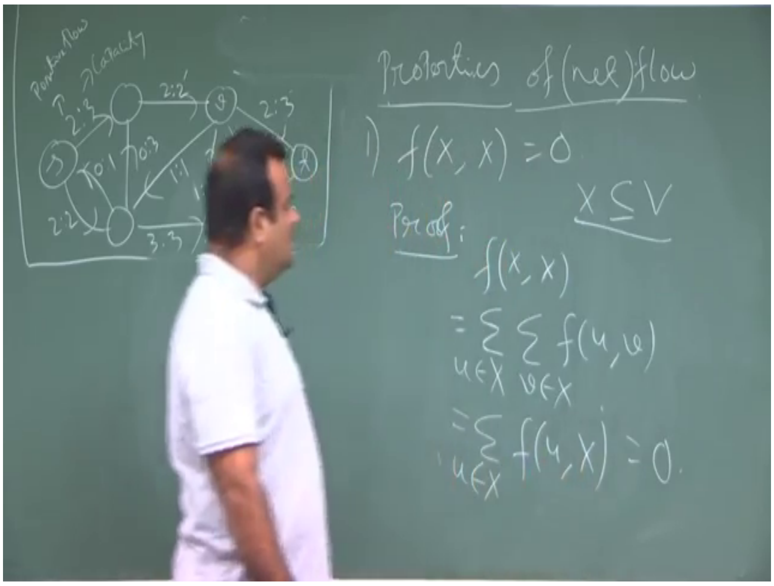

So, this is basically summation of u belongs to x f of u comma x.

So now this is all the all the flow from x to that vertex. So, this is basically will be same as. So, this is net flow will be same as the all the flow which are out going from this vertex. So, this is basically is 0. So, net flow is basically 0. So, this property is now we have. So, how we are doing yea. So, you have to be first.

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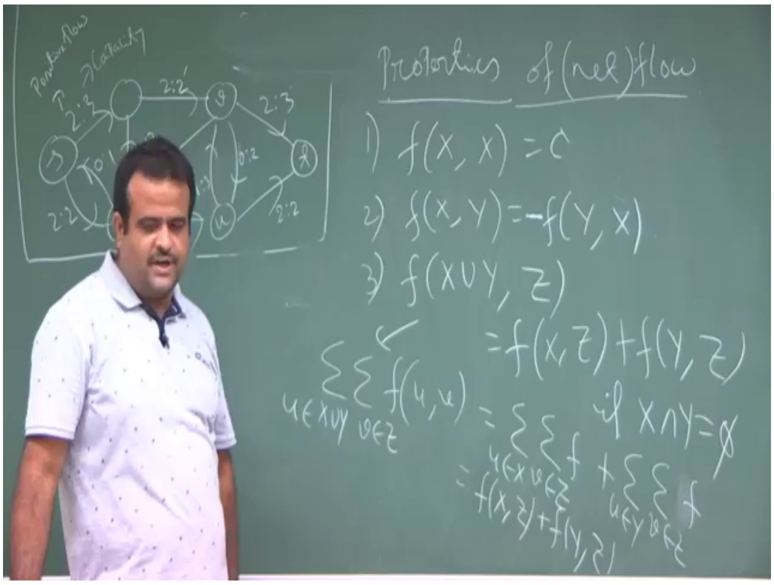

So, this is basically another property f of x comma y basically minus of y comma x. So, this is this is basically coming from skew symmetric skew symmetric property of the net flow.

Then the another property is basically this is minus. F of x comma y comma z is nothing but so, this 2 are disjoint sets x comma z plus f of y comma z where x y are disjoint set. So, this is the property we have. So, here x y at disjoint set. So, how to prove this third property? Because see this is x is a x x or y all subset of v basically. So, how to prove this? To prove is so, if we have a disjoint subset. So, basically it is coming from. So, we just yeah we just take the sum. So, this is what this is nothing but summation of double summation of f of u comma v where u belongs to and v belongs to z ok.

Now, u belongs to this means it is basically disjoint set. So, it is basically we can write this sum as summation of u belongs to x v belongs to z, if plus summation of u belongs to y v belongs to z v belongs to sorry v belongs to z f. So, this is nothing but f of f of x comma z plus f of y comma z. And this is true because so, this is true because they are they are disjoint set. So now, we will we will use this result or this theorem or this lemma is to prove another theorem which is basically called which is basically telling us the flow I mean the net flow the flow of a network. So, that will discuss they will start from here in the next class.

Thank you.