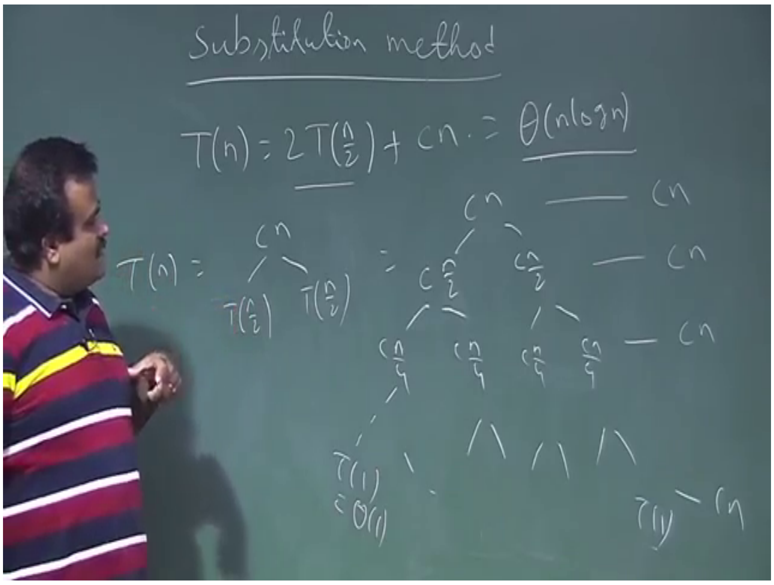
An Introduction to Algorithms
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 05
Substitution Method For Solving Relurrence

We talk about Substitution Method. This is the method to solve the recurrence. So, we have seen a recurrence which you have obtaining from the say merge sort also.

(Refer Slide Time: 00:32)



We have seen the time complexity of merge sort if we want to sort n array of size n then the recurrence we got T n by 2 plus theta of n. So, this is merge cost and this is cost for recursively solving two sub arrays.

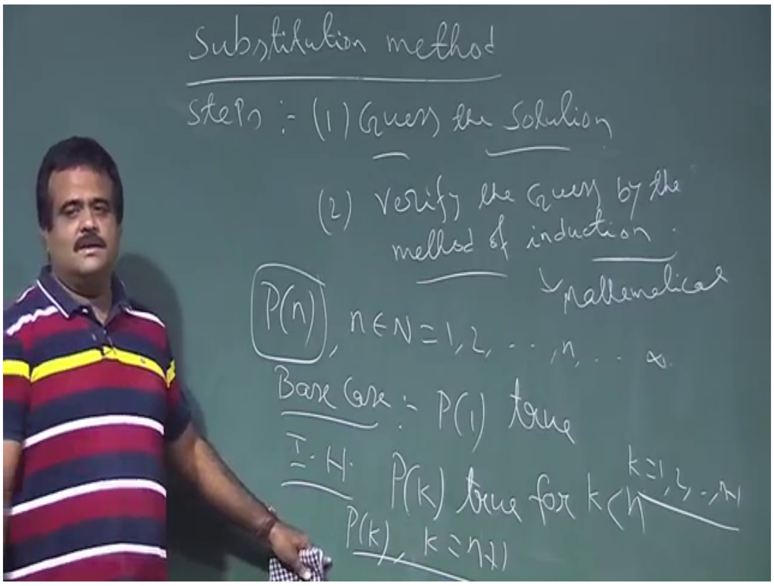
Now the question is how we can solve this type of recurrence. So, we have seen one method which is recursive tree method. So, in that recursive tree method we have seen that so if we take this as c n then we have seen T n is basically c n T n by 2 plus T n by 2, then this is again a problem of size n by 2 we further reduce this problem into sub problem. So, so this is basically. So, c n by 2 c n by 2 and it will be T n by 4 T n by 4 T n by 4 T n by 4 like this. So, this way it will continue up to n is to the constant T 1.

So, c n by 4 all are c n by 4 in this level. So, it is any at T 1 all the branches of this t and n a T 1 and T 1 is basically the problem of size one which is theta 1; cost of that two solve that problem is theta 1. Now, the time complexity runtime is basically sum of this all nodes, so to make the sum we have seen we can take c n sum of the levels and we know the height of this tree is log n. So, basically it is giving us the n log n algorithm.

So, this is the recursive tree method which we have seen in the last lecture this method as some drawback in the sense. So, this is not a (Refer Time: 02:53) method in the sense. So, we have seeing this is c n c n c n, so is the proof that in the i-th level also in the general in the i-th level also it will be c n. So, that proof we are not doing here. Unless we have to proof that by some way like method of induction or something or we can just say this is giving a rough idea of the solution.

So, this recursive tree method recursive tree method is not a full proof method in that sense. So, today now in this lecture we will talk a substitution method which is basically a method by induction; mathematical induction.

(Refer Slide Time: 03:50)



So, we will prove our solution by mathematical induction. So, basically it as following steps. So, this is a general method so it as following steps. So, we first guess the solution. Basically we have given a recurrence then we first guess the solution. That means, we assume this is the solution. So, then by the method of induction we verify our solution, verify the guess by the method of induction; so mathematical induction.

So, induction means anything we want to proof in terms of n; n is a natural number any property in terms of n. Actually there are two version a mathematical induction: first version is suppose we want to proof some property that property is P n in terms of n where n is a natural number natural number means 1 2 up to n up to infinite. So, we have to proof some property in terms of n for that mathematical induction is telling there is a base case, in the base case we prove that it is true for some lower value of n that is we can proof that the P 1 is true.

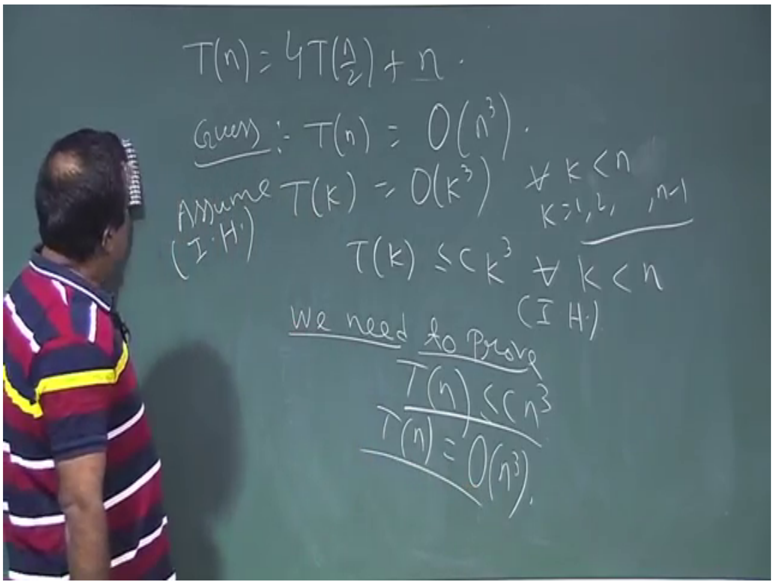
Then we have induction hypothesis step. So, we assume that P k is true there are two version for all k up to n. That means, P k is true for k is equal to 1 2 up to n minus 1. Then if we can proof that P n is also true; that means if we can proof that P k is true for k is equal to n plus 1 if we can show that then we are done then we have a base case it is true for 1. That means, it is true for this is one version another version is; if we assume that P k is true for n is equal to k is equal to n and then if we can proof it is true for n is equal to k n plus k is equal to n plus 1 then it is true for all n.

So, we will take this version of the induction method so we assume that our result is true for all k up to n minus 1; that means, before n. And we need to prove that it is true for sorry it is true for n. That means, we need to show P k is true. If we can proof that then we are done, then by the method of induction we can say this is true for all n. So, this is what we know the method of induction.

So, we will use these to verify our proof by the help of induction. So, we first given a given a recurrence; we have given a recurrence we first guess the solution and then we will try to justify our we will try to verify our guess whether this is this guess is correct or not and that will do by the method of induction. And also we have to to solve the constant based on the base case.

Now let us take an example then it will be more clear.

(Refer Slide Time: 08:07)



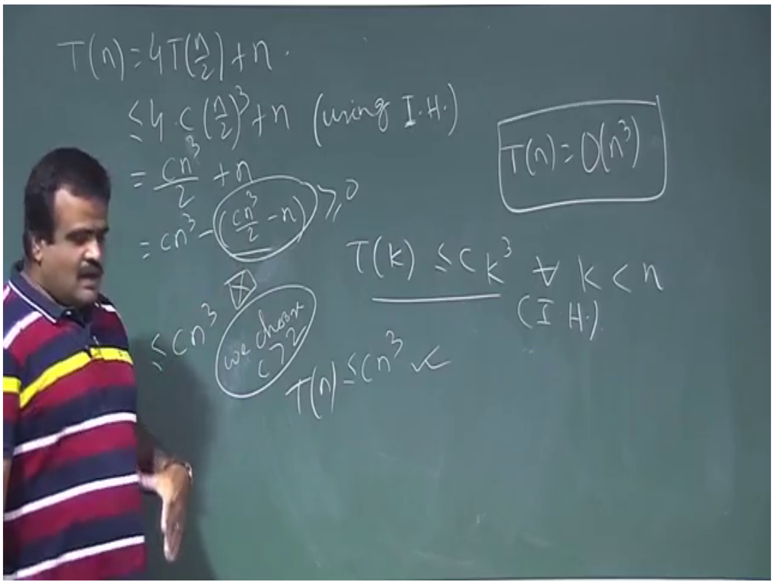
Suppose we have a recurrence like this T n is equal to 4 T n by 2 plus n. Suppose this is our given recurrence and we need to solve this recurrence by substitution method. So, step one: guess guessing step. Suppose we are guessing the solution is big O of n cube suppose this is our guess. Now we need to verify this guess whether this is correct or not by the method of induction; so for that what we do we assume this is the assumption. We assume this is the assumption or this is also called induction hypothesis. So, we assume the T k is basically order of k cube for all k up to n minus 1, so this is strictly less than; so we can say up to n minus 1 or if we strictly less than.

So, this is basically k is equal to 1 to up to n minus 1. This is our assumption we assume that our result is true for all k up to n minus 1. That means, what this means this is T k is less than c k cube this is for all k up to n minus 1. So, this is our induction assumption or induction hypothesis.

Now we need to show, we need to proof need to proof or need to show that T n is also less than c n cube. If we can show this then T n is basically big O of n cube. By assuming this if we can proof this then by the method of induction we can say this is true for all n; all natural number n. So, that is the way we pause it. So, let us try that so we need to show this. This is not at done we need to show this.

So, we had this. So, this is our, we had this induction hypothesis, induction assumption we keep this.

(Refer Slide Time: 10:45)



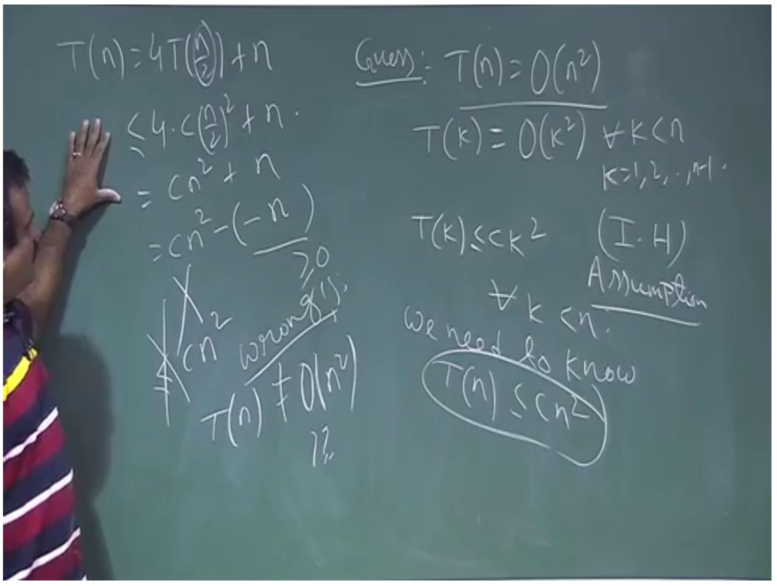
Now, we write this our recurrence, our recurrence is basically 4 T n by 2 plus n.

Now, we use this induction hypothesis. Now this is n by 2; n by 2 is less than n so n by 2 is k which is less than n. So, we can use this hypothesis. So, this is basically less than equal to four into c n by 2 is k k cube plus n, this is using the induction assumption or induction hypothesis. So, this is basically equal to; so c n cube by c n cube by 2 plus n. Now this we want to be less than equal to c n cube, if we can show this then we have done. Then this is the. For that this we must able to write c n cube minus something and this quantity should be greater than 0. If it is greater than 0 then we have proved, then we can say this is less than c n cube.

So, let us check what is this quantity? This is basically c sorry c n cube this is c n cube by 2 minus n. Now c is in our hand, c is a constant which we can play with positive constant. So, if we just choose c to be; if we choose just we choose c is greater than 2 then if we choose this then this guy is greater than 0 and then this is less than this so done. That means, by taking this assumption we have shown this is less than c n cube. By taking this assumption we have shown this result is true for k is equal to n also. And we can easily check the base case. Then, so the result I true for all n. So that means, T n is basically big O of n cube.

So, the solution of this recurrence is big O n cube. This is the method, but basically we are using the mathematical induction. Now we want to, because we have guess this is order of big O n cube, now we want to guess whether it is n square or not because this is our we can have any guess. So, then we have to justify our guess by the method of induction. So, this is our recurrence. So, big O n cube big O n square is also basically big O of n cube. So, we want to see whether we are getting this solution as big O of n cube or not.

(Refer Slide Time: 14:05)



So, this is our new guess; now we want to guess. So, the T n will be big O of n square or not. So, this we guess. Again we have the assumption step or the induction hypothesis step what is that. So, we assume this is true for all k up to n; that means, k is equal to 1 2 up to n minus 1. We assume this is our result is true for up to n. That means this means T k is this is our induction hypothesis, this is our assumption or induction hypothesis or this is our assumption.

That means T k is basically less than equal to c k square for all k less than n. So, this is our assumption, now what we need to show? We need to show that, so we need to proof. Need to show that T n is order of n cube; that means, this result is true for k is equal to n. That means, we need to show that t of n is less than equal to c of n square, if we can show this then we can say T n is order of n square.

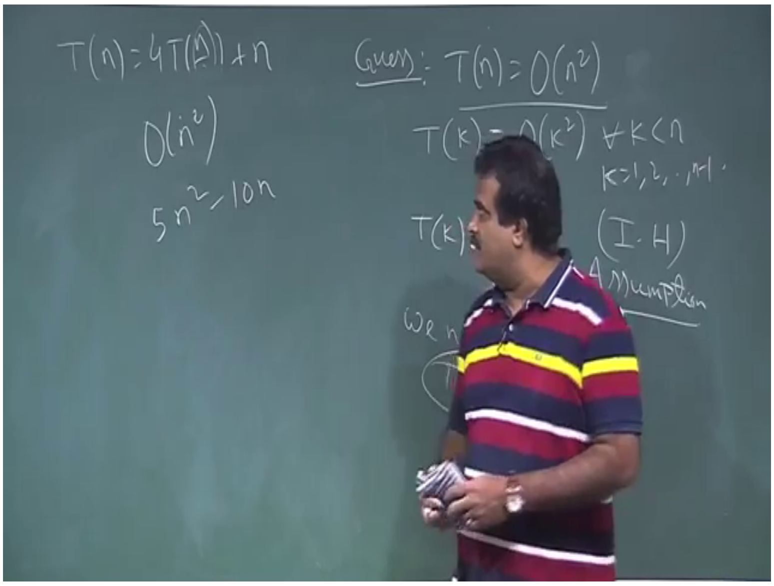
So, how to show this? This is our recurrence, we will use this recurrence, so this is our k n by 2 is less than so this is our k, so we can use this induction hypothesis. So, this will be basically; this k is n by 2 which is less than n. So, we can use this induction hypothesis this is 4 into c n by 2 square plus n.

So, this is basically 4 5 cancels c n square plus n. Now, we want this to be less than equal to c n square in order to get this result. In order to get this result we want this to be less than equal to c of n square. So, for that what we need? We need this should be written as c of n square minus of something, and that something must be greater than 0. But what is that? That is basically minus n which cannot be greater than 0 because n is natural number n start from 1, 2, 3 like this. So, the minus is cannot be natural number.

So, this is wrong, so this cannot be less than this. So, this is not true wrong. So, then what will be our conclusion? So, if this is wrong can we just say that T n is not equal to big O of n square or what, because we are not achieving this we took the assumption this is for up to n; now we try to prove it this will be true for n also, but we are not achieving this. So, can you conclude that this will be not solution of this is not big O n square or we have something to say over here; here in this bound.

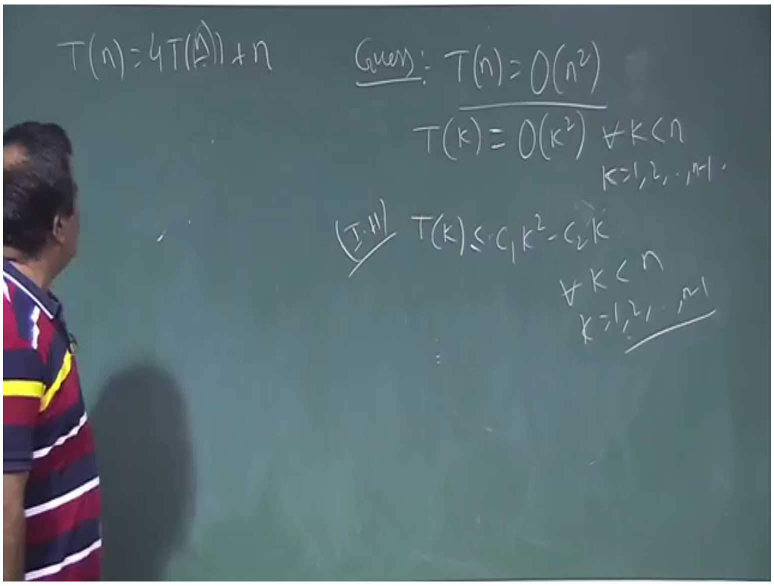
So, this way we cannot achieve this. Now the question is we can tighter this bound, we can have a tighter bound than this. See big O of k square; big O of k square means not only this. So, this way we are not achieving that that is clear. So, but we can tighter this bound like big O of n square.

(Refer Slide Time: 18:43)



So, big O of n square; so this is also big O of n square say 5 n square minus 10 n this is also big O of n square, because we ignore the lower order term then we ignore the leading coefficient. So that means, this bound we want to make it tight. So, we assume this is true big O of k square. Now here we just check instead of this bound we take little tighter bound.

(Refer Slide Time: 19:17)

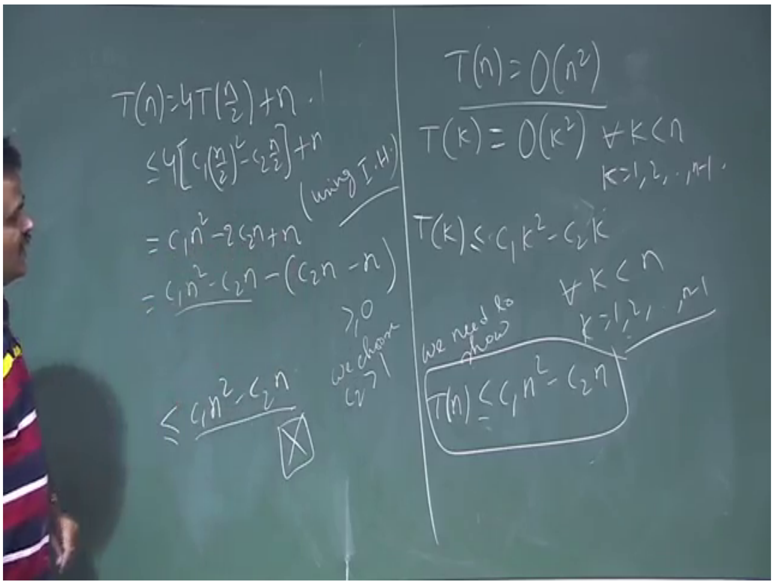


So, we take this to be c 1 k square minus c 2 k because that is also big O of k square, because this is the lower order term anyway you are going to ignore the lower order term. So, this is also big O of k square. So, we will take this tighter bound and then we will see whether this will help us to achieve that.

So, just we assume this is a big O of k square and big O of k square means this also earlier bound was this is the more tighter than the earlier bound. And with the help of this bound this tighter bound we want to see whether we can justify this, this is the solution is big O of n square is this clear. So, this is basically we just this is our induction hypothesis step.

So, this is basically true for all n less than n. So, this is basically k is equal to 1 2 n minus 1. Now we use this bound here to achieve that, so let us try that.

(Refer Slide Time: 20:35)



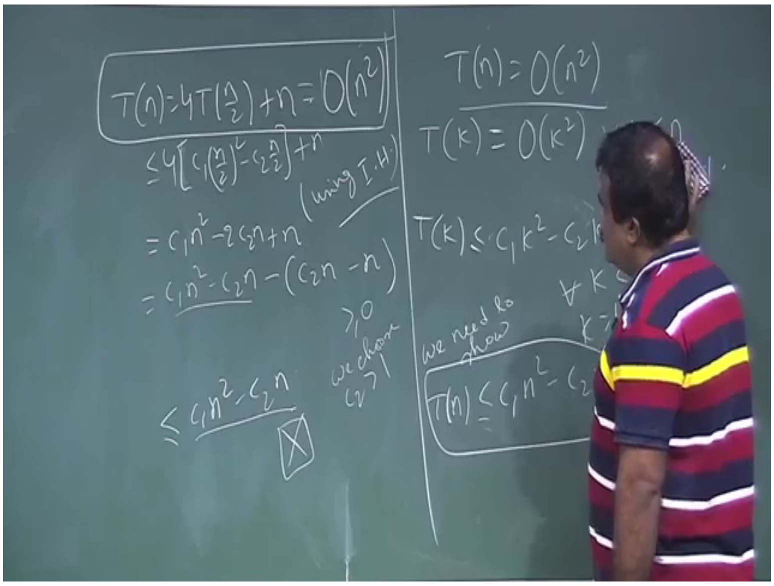
So, let us write our recurrence. So, this is our induction hypothesis, so let us write our recurrence T n is equal to 4 T n by 2 plus n. Now this is also k n which is n by 2, now we assume the induction hypothesis so this is basically less than equal to 4 c 1 k square. So, n by 2 square- so this we can take another bracket minus c 2 n by 2 plus n this is we are. So, this is basically we are using this, this is we are using the induction hypothesis the assumption; this assumption we are using.

So, this is basically what? This is basically is cancelling out, so c 1 n square minus 2 c 2 n plus n; just we are simplifying this. Now this is the assumption we made. So, what we need to show? We need to show that this is true for k is equal to n also. That means, we need to show; so we need to show so that T n is less than equal to c 1 n square minus c 2 n, if we can show this then we are done. So, that is we want to achieve. So, we want to show T n is less than this. So, in order to do that; so this is basically we want to show this is less than equal to c 1 n square minus c 2 n.

So, to show this we can write this as; this we must write as c 1 n square minus c 2 n minus something. So, this is the quantity we are looking for minus something and this term is greater than 0 then we are true. Then we can say this is less than this then we are done. So, what is this term? This term is basically c 2 n minus n. Now c 1 c 2 is the positive constant we can play with we can choose c 1 c 2.

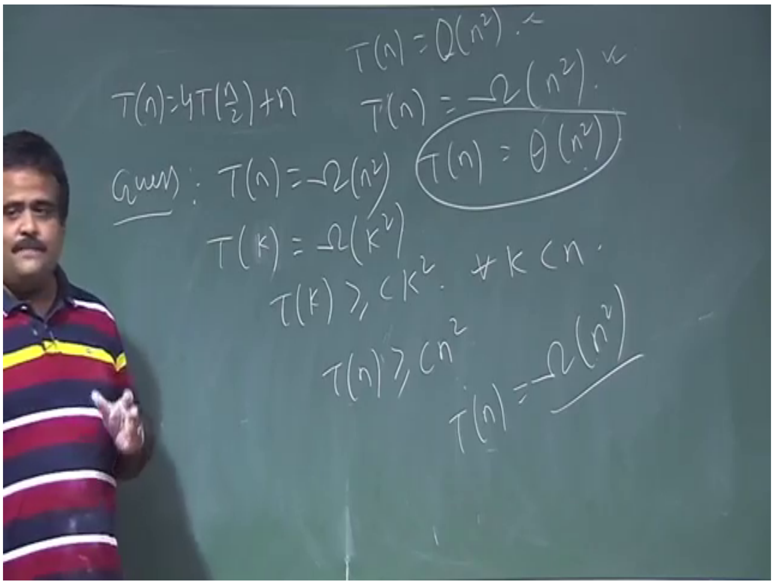
So, if we choose just c 2 is greater than 1 then this is positive; we choose c 2 to be greater than 1. Then this is positive then this is done then we achieve this is true for k is equal to n also. Once we achieve this is true for k is equal to n we assume this is true for k is equal to 1 2 n minus 1. Now we achieve that this is true for this, this is little tighter bound the earlier one this is true for k is equal n. So, the result is true for all natural number.

(Refer Slide Time: 23:58)



That means, the solution for this is basically big O of n square. So, this we prove by the help of mathematical induction. So, this is the big O of n square, now we want to see whether this is big theta. So, for big theta what we need to show? We need to show this is big omega. So, basically we want to achieve this solution, we have seen this is big O of n square.

(Refer Slide Time: 24:40)



Now, next we want to see whether this is big omega of n square and if both this satisfy if this is big omega and big theta then it is basically I big omega and big O then it is basically big theta of n square.

So, now the question is how we can show this is big omega of n square. To show this we will follow the similar path, similar way. So, we guess this is our guess; we guess that T n is equal to big omega of n square so that means, we assume that T k is basically T k is basically big omega of k square. And so we will do the same thing this is the mathematical induction, so we will have this is greater than equal to some c k square then we will try to; so this is true for all k less than n, then we will try to show that T n is greater than equal to c n square. Or else we have to tighter this bound. So, this way we will continue and we will get we can easily prove that T n is equal to big omega. This I am leaving you for exercise, just complete it. And we can show that this is a big omega of n square. Once we achieve this is big omega of n square I big omega of n square we have seen this big theta big O of n square, so both will give the solution as big theta of n square.

So, this is the substitution method. So, this is a full prove method in the sense here we are using by taking help of mathematical induction we are justifying, we are proving our recurrence is true for all n. So, basically it has two main steps: one is assumption- we have to guess the solution. So, where from we can guess the solution? This recursive tree can help us to guess the solution. Suppose this is our say recurrence, now initially we guess whether it is n cube one can guess whether n square n to the power 4. So, how to get a guess? So, for that guess one can take help of the recursive tree and the recursive once we get some solution of the recursive from the recursive tree then we can justify the solution by using the mathematical induction. So, that is the substitution method. So, for the guessing purpose we can take help of in the recursive recurrence tree or recursive tree

Thank you.