An Introduction to Algorithms
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Lecture - 47
Application Of Bellman Ford

We talk about some Application of Bellman Ford algorithm. So, we start with; so Bellman Ford algorithm can be applied to the area like how in the linear programming solving the linear programming problem, but not the general linear programming problem it has some which is called linear constant; how to solve the linear constant. So, we will talk about that and also we may discuss some of the other applications like how to detect a negative stufological ordering of a graph. So, those we may discuss. So, let us start with the linear programming problem. So, what is a linear programming problem in short it is called LPP.

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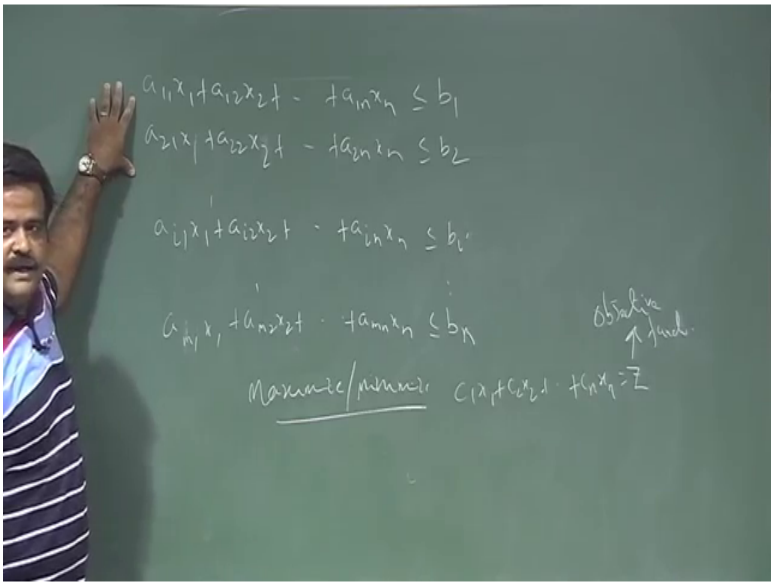

So, what is a LPP problem? So, LPP problem is basically we have a some linear constant like say a 11 x 1 plus a 12 x 2 a 1 in x n less than equal to b 1 a 21 x 2 plus a 22 to x 3 sorry a 21 a 2 so on so x. So, these are the variable. So, they are a 2 in x n less than equal to B 2. So, the i-th term is a i2 x 2 plus a i n x n less than equal to b i dot dot dot a i n x 1 or may be m if there are sorry f m one if there are m such linear constant. So, a m1 x 1 plus a m2 x 2 plus dot dot dot a m n x n less than equal to b m ok.

So, this is a given system of linear constant. Now our x 1 x 2 x n must satisfy this and it must maximize or minimize there is some objective function; it should maximize or minimize some function like c 1 x 1 plus c 2 x 2 c n; how many are there? Up to n c n x n. So, this function is called jet objective function; this is called objective function. So, we have given system of constant like this which is the basically the system of inequalities and then we have to. So, which where to find the x i x 1 x 2 x n we satisfy this system of constant and then which will optimize this; I mean either the objective form objective function could be in maximize form or minimize form, so which should optimize this ok.

So, this also can be written as the matrix form.

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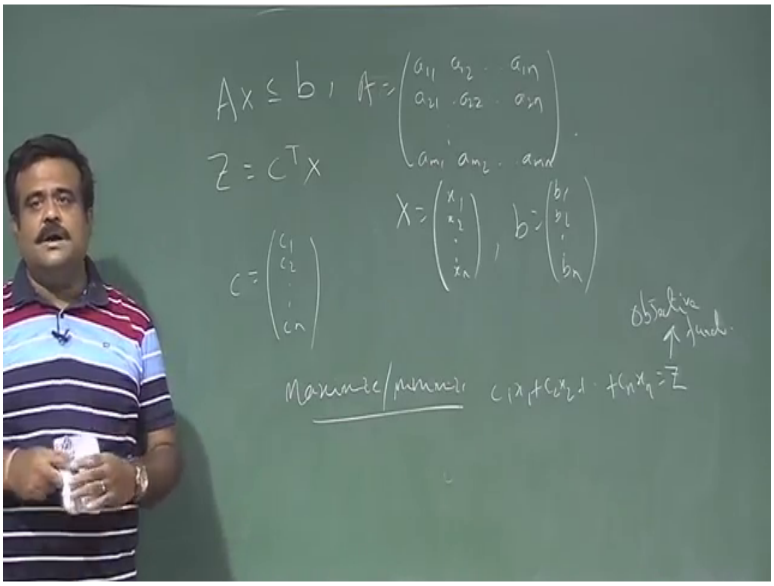

Like A x less than equal to b and this Z also can be written as c t x where a is the matrix a 11 a 12 a 1n, a 21 a 22 a 2n, a m1 a m2 a mm, and x is basically the vector x 1 x 2 x n and b is also the column vector b 1 2 2 b n and c is basically also the column vector c 1 c 2 c n. So, this is the LPP problem in the matrix power. So, you have to either this is the objective function which need to be either maximize or minimize. So, this linear programming problem this is the general form of linear programming problem.

So, there are some algorithms to solve this linear programming problem. So, one is simplex method, another one is some graphical method to solve the linear programming problem. Now, here we are restricting our problem LPP problem to a very particular case here. This each of this row is content two element 1 minus 1. That means, this A matrix each row of the A matrix is.

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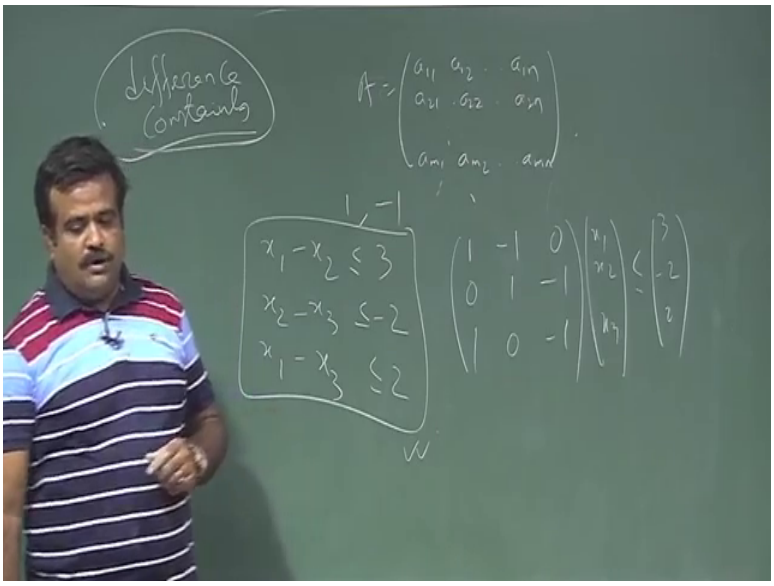

So each row of the A matrix is contained only 1 and minus 1- 11 and 1 minus 1; That means, it will be like this. So, x 1 minus x 2 is less than equal to say 3 then x 2 minus x 3 is less than equal to minus 2 then x 1 minus x 3 less than equal to 2. Suppose this is our, so what is the corresponding matrix? Matrix is basically: so 1 minus 1 0, 0 1 minus 1, then 1 0 minus 1. So, x 1 x 2 x 3, and what is b this is less than equal to what is b b is basically this c minus 2. That means, each row is containing only 1 1 and another minus 1; so very very special very particular case of LPP. So, these are called linear constant or difference constant. So, these are called different constants.

Now, we will just concentrate on this particular case or different constant how to solve the different constant. So, this we will just look at how to solve the difference constant. And then we have after getting the solution we have some optimization function maximizing or minimizing some objective function there. So, that will discuss.

Now, can you give me the solution for this function; for this constant difference constant. So, any value of x 1 x 2 x 3 which can solve this will we apply the simplest method or just by looking at this or graphical method also you can apply, but any way if you just try it.

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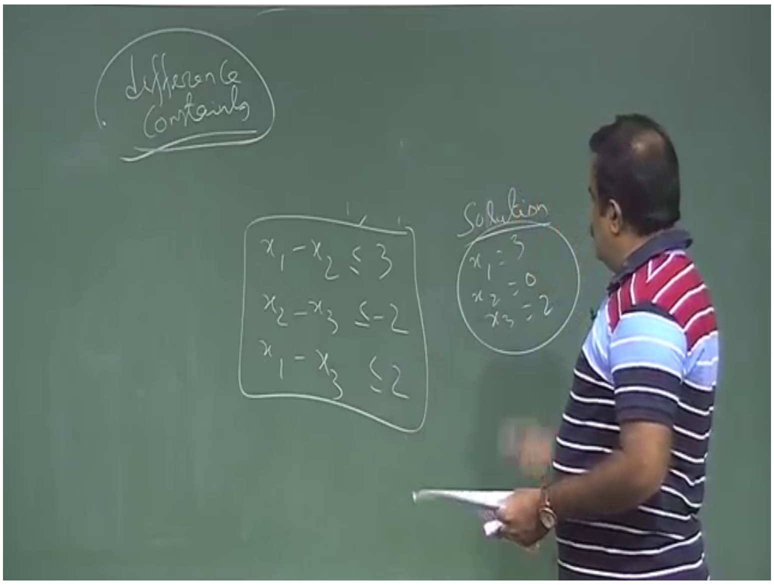

I think the solution will be x 1 equal to 3 x 2 equal to 0 and x 3 is equal to 2. So, one can verify if you just put it you satisfying all this. So, it is satisfying all this; this is a solution for this difference constant system, ok.

Now, we want to see how the Bellman Ford algorithm can help us to get the solution of such different constants, and how we can make use of Bellman Ford to have the solution of this thing. Now for Bellman Ford we need a graph, now we needed a directed graph now the question is how to construct a graph out of it. So, how to construct a graph from the given different constant, I mean system of different constant?

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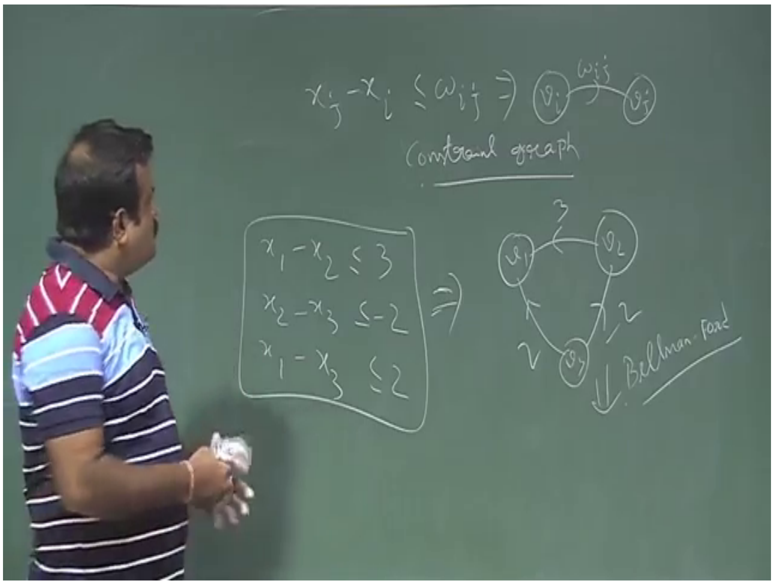

So, that to construct a graph suppose we have a different; we have equation like this x x a x j minus x i is less than equal to w i j, ok. So, this will corresponds to a part of the graph like v i or you can just say s i also; this is just a notation I mean notation of denoting the vertex. So, this is basically w i j. So, so x i if this is a constant this is called constant graph constant. So, give a constant difference constant we have this corresponding graph. So what is the graph corresponding this? So, from this we have what? So this is basically x 1 or x 1 we can denote by v 1 v 2 then we have v 3.

So, we have x 1; x 1 minus x 2 is less than equal to 3. That means this is basically x 1, so w i j so this is basically x 2 minus x 1- so this is 3. And then x 3 to x 2, so x 3 to x 2 we have a minus 2 and then x 3 to x 1 you have plus 2. So, this is the graph, this is the constant graph corresponding to this system of different constant, ok.

So, we have the graph; now once we have a graph this a directed graph with the age weight and this age weight is could be negative also, because Bellman Ford can handle the negative cycle. So, there is no restriction on the negative weight edge. Now given this graph we have to do. Now, once we have the graph we can apply the Bellman Ford algorithm. So, we can apply the Bellman Ford on this. Once we apply the Bellman Ford it will give us the either it will report a negative cycle or it will give us the shortest path weight, ok.

Now, suppose it is giving us a negative cycle; that means, there is a negative cycle in the corresponding graph then we will we have to show that there will be no solution of the system of constant. That means, system of constant is infeasible.

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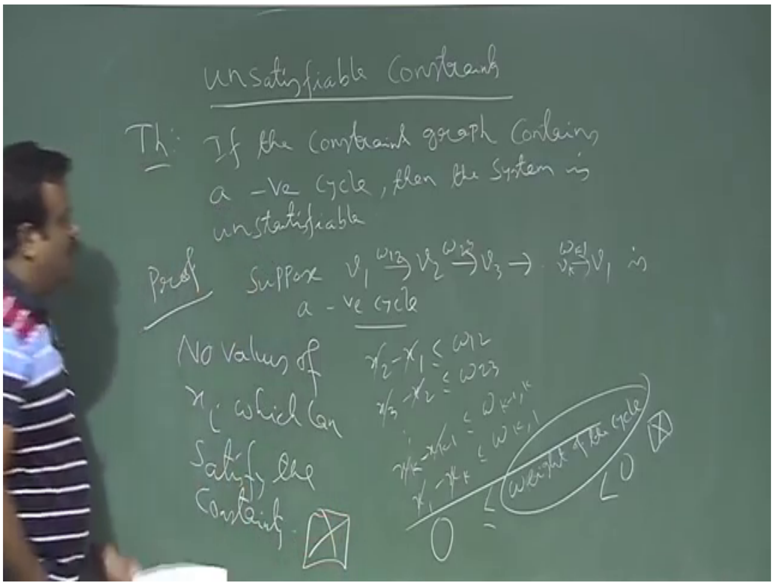

So, this is called infeasible or unsatisfieable constraint. So, what it is telling? It is telling that this is theorem if the constant graph contain a negative cycle and that can be determined by Bellman Ford- content say negative weight cycle then the system is infeasible; then the system is not having the solution- then the system is unsatisfiable. That means, there is no solution, ok.

So, how to detect that whether there is a negative cycle? So, we convert this system to a constant graph, then we run the Bellman Ford, so once we run the Bellman Ford we can easily see that whether there is a negative cycle or not, because Bellman Ford will report the negative cycle if there is. So, then if Bellman Ford is reporting negative cycle then there is no solution exist for this system. So, how to prove this?

So, to prove this suppose there is a negative cycle in the graph; suppose this is a negative cycle in the graph v 1 v 2 v 3 dot dot dot it will reach to again v 1 is a negative cycle, ok. Suppose this is a negative cycle, now in the graph in the constraint graph. So, from this what is the corresponding constraint we have? So, for from this we have the constraint like x 2 minus x 1 is less, so this is basically w 12; sorry w 23 and this is basically w if this is v k. So, v k to again v 1 so w this weight is w k 1 comma.

So, this will be corresponding to x 2 minus x 1 is less than equal to w 12 and this will corresponding to x 3 minus x 2 is less than equal to w 23 like this and dot dot dot x k minus x sorry x k minus x k minus 1 is less than equal to w k minus 1 comma k. And then the last one is x 1 minus x k is less than equal to w k comma 1. This is coming from this graph. Now if we add it up this thing will cancel out this, this, this, this, this, this, this, this. So, this part is 0, and this part is basically sum of this weights. And sum of this weights means weight of the cycle. So, this is basically weight of the cycle.

Now this is a negative cycle. So, this weight is negative this is less than 0, so this is the contradiction. That means, we are not having a solution for this system of constraint. If we have a solution then we should not reach to this contradiction. So, we are not having getting a solution. That means, there is no values of x i can satisfy this constraint, because this is 0 and this is negative which is not possible. So, there is no values of x i which can satisfy this constraint. So, this is the proof by contradiction. So, there is no values of x i which can satisfy the constraint. So, this is the proof.

That means, if there is a negative cycle in the graph then there will be no solution will exist. That means, then the system of constraint is not feasible, it is a unstaisfiable, ok So, now we have to poof the other way around. Suppose there is no negative cycle. That means, Bellman Ford will give us the solution, Bellman Ford will give us the single source shortest path then from there how we can get the solution for this constant, how we can get the solution for this system of creation. So, that you have to show in the next. So, that is the satisfying of the constant.

So, if there is a negative cycle in the graph then this system of constraints different constraints are not feasible; that means there is no there is no such x i satisfy this, ok. So, let us just write this.

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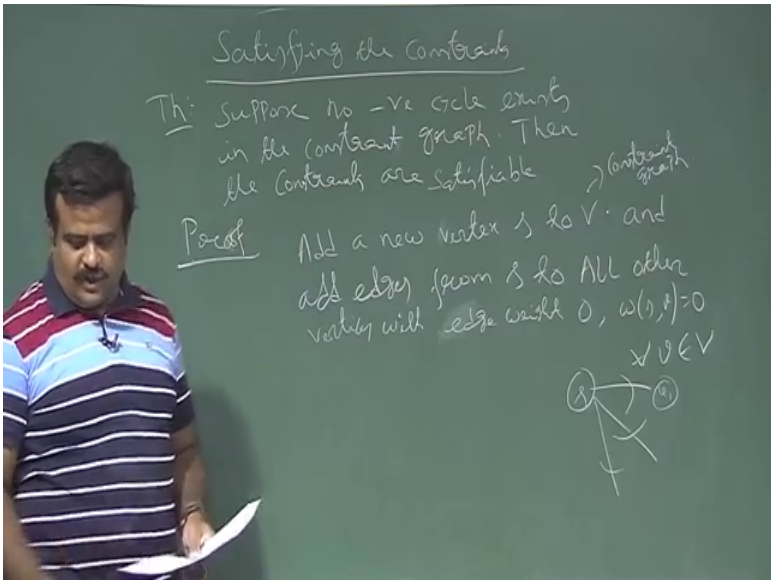

This is the feasibility case or satisfiable satisfying the constraint. So, here there is no negative cycle so that means, we have a solution. So, suppose if this theorem is telling suppose no negative cycle in the graph exists in the constraint graph then the system is feasible; then the constant at staisfiable, then there is solution for this constraints then constraint are satisfiable, ok

So, this is the case where we have the solution for this system. So, when there is no negative cycle we can expect the solution. So, how to prove this? So, to prove this we just act some vertex in the graph. So, what we do? We just add a new vertex. So, what we have? We have a system of constraint from there we constant graph and in that constant graph what we are doing we are adding a new vertex, so just adding. So, add a new vertex, so that will be a starting vertex source vertex because Bellman Ford needs a source; source vertex this is a single source shortest path algorithm.

Add a new vertex s to v; v is the constraint graph; v is constraint graph which is coming from this constraint different constraint vertex v. And a an add the edges form v to any other vertices and add edges from; sorry s to all other vertices with edge weight 0. So, that remains w of s comma v is 0 for all v. So, form s we add the; so v 1 v 2 v 3 we add the vertices to all other edges to all other vertices.

So, let us just draw that graph. So, basically we have the constraint graph.

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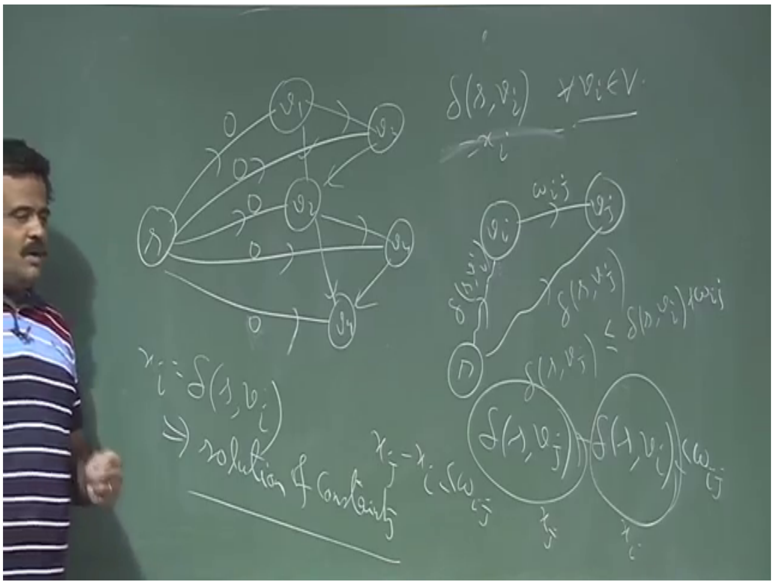

Say like we have some vertex v 1 v 2 v 3 v 4 v 5 and they are connected. So, depending on the constants they are connected somehow. Now what we are doing? So, this is constraint graph coming from that difference constraint. So, we are adding a vertex and we are from this vertex we are adding the a edges to all other vertices. And this we are adding weight to be 0 this newly added edges and this is for all other vertex. So, this is the weight, ok

Now, if the original graph is having no negative cycle; that means, this new graph just adding a vertex x will not having a negative cycle also because we have just added the 0 weight edge. So, by adding 0 weight edge it cannot connect the negative cycle, because the original graph is having no negative cycle. So, though no negative cycle in this graph. So, if there is no negative cycle we do the Bellman Ford algorithm and we run the Bellman Ford algorithm. So, no negative cycle means shortest path exists. That means, we will get delta of s comma v i for all v i belongs to v. So, this we are getting. So, we are getting the delta of s comma v i for all v i belongs to this, ok.

So, now this we are assigning to. So, this exists, because there is no negative cycle so Bellman Ford will give the solution so these exist. So, this we will defined as x i. Now what we have? We have this say v i and v j, and we have this s vertex over here. So, from v i to v j what is the weight? Weight is w i j and this is from v i to v j; v i to v j which is w i j. Now, we have a delta of s comma v i we need to put Bellman Ford this is delta of s comma v j ok.

So, now by triangular inequality what we can say; we can say this delta of s comma v j this is basically weight of the shortest path from s to v j and this is one the path. So, this must be less than equal to delta of s comma v i plus w i j; delta of s comma v i plus w i j now we can take this side. So, delta of s comma v j minus delta of s comma v i is less than equal to w i j. These are the solution so this is this is the system of constraint this is that i-th constraint and this is the solution. So, if we just assign this to be x i and this to be y x j; that means, x j minus x i is less than equal to w i j. So, these are basically this deltas are basically solution of this constraint. So, this deltas delta of s comma v i these are the basically gives the solution of the constraints. So, these deltas are giving the solution of the constraints.

So, what we are doing? We are having a constraint graph, we are just adding a vertex x and we are adding the edges with the 0 weight. Then we run the Bellman Ford and since there is a no negative cycle in the original graph. So, there will be no negative cycle in this new graph. After running the Bellman Ford we will get the deltas, because the shortest path will exist. And these deltas are nothing but the solution of this constraint that is coming from this triangular inequality this is the triangular inequality. So, this is coming from this result and this is the constraint and this is satisfying the constant this deltas are basically satisfying this constraint. And so these deltas are basically the shortest path solution of the difference constraints.

Thank you.