An Introduction to Algorithms
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 42
Shortest Path Problem

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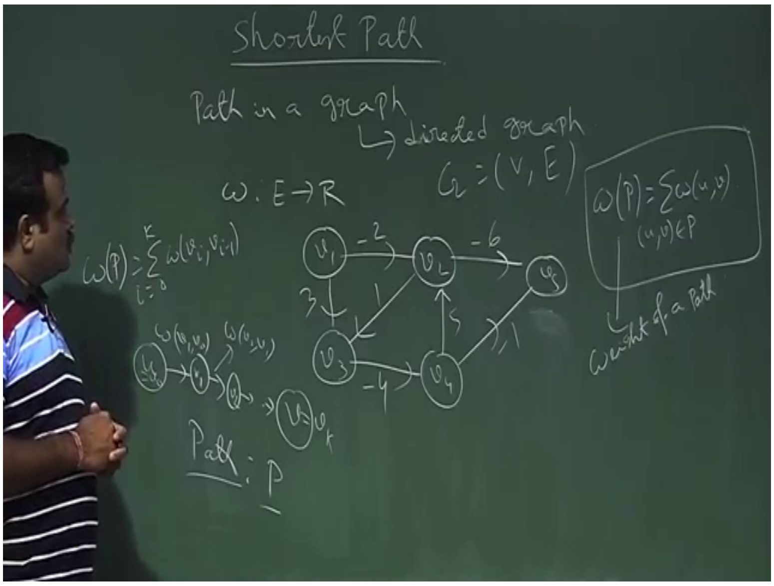

So, we talked about shortest path problem in a graph. So, basically what is a path in a graph, this graph is a directed graph undirected graph sorry directed graph or digraph. So, we have a directed graph V comma E where this E is a edge having the direction, so this a directed graph. So, let us take an example of a graph. So, say v 1, v 2, v 3, v 4, v 5 say, so there are say 5 vertices, and say these are the edges, suppose these are the edges. So, this is our graph and we have a weight function on this. So, weight function we have weight on each of the edges, so that is the weight function.

So, suppose we have a say minus 2, 3 minus 4 5 then we have say minus 6 and we have say minus 1. So, suppose we have yeah, so this is the graph suppose this is the graph. Now, what do in the path in a graph. So, path in a graph means. So, we take a vertex v u and we take a vertex v. So, path from u to v is basically if we follow the direction of this edges, there must be some vertices this is v 0 then v 1 then we have v 2 dot dot dot then we have v k. So, this is a path physical path we denote this by P.

So, we start with the vertex u and then we take a edge, then we go to another vertex v 1, v 2 like this. So, ultimately we have to reach to v. So, this if there is a physical path then that is denoted and weight of the path is basically. So, we have the weight of this w v 1., v 0; this is basically w v 2 v 1 so like this. So, weight of this path is defined as summation of the weight of the edges basically. So, weight of the edges means sum of the weight of this edges. So, summation of w v i, v i minus 1 and this i is varying from 0 to k. So, this is the weight of the path. So, basically w P is basically summation of w u, v where u, v is belongs in that path. So, this is how defined weight of a path. So, you basically take the sum of the all weights in that path, so that is give as the weight of the path.

Now, the question is now this is the problem of finding the shortest path. So, when we can have a shortest path from a vertex u to v. So, first of all there has to be a physical path. For example, if there is no path from u to v then there is no question of shortest path. So, the physical path has to be there from u to v. So, like if we say if we ask for a path from say u is say for this graph, u is say v 5, and if we take any other vertex say v 2. So, if we ask the shortest path, because there is no path from this because there is no outgoing edge from v 5. So, v 5 to any other vertices, there is no path. So, there is no question of shortest path.

So, to have the shortest path first of all we need to have the physical path from u to v, if we have to have the shortest path from. So, what is the shortest path suppose so there has to be a path from u to v. So, there may be many paths from u to v. So, among this path whichever will be the minimum weight that will be the shortest path from u to v. So, among these say there could be several path from u to v among this path if a path is giving us the so that is basically

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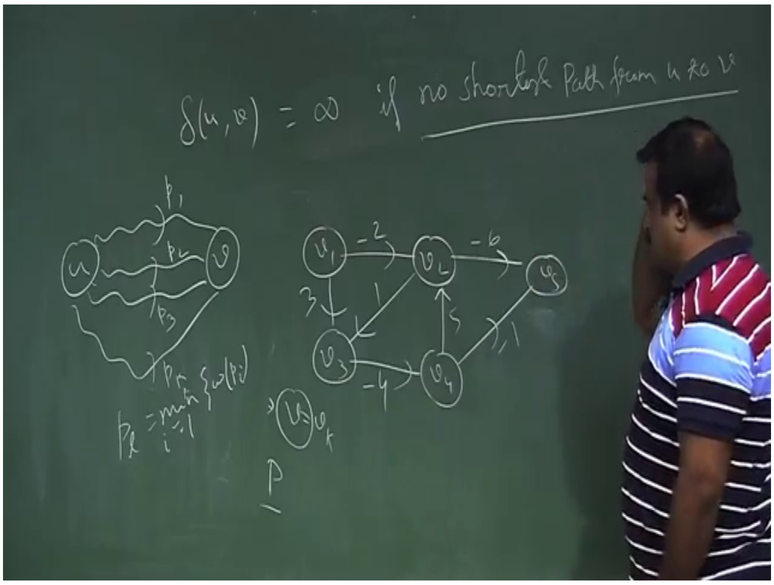

So this is basically w p is basically minimum among all w P such that. So, this is basically denoted by delta u v. So, weight of the shortest path from, so you have a vertex u then vertex v. So, this is basically minimum among all path from so summation of so this w u w is basically w P is basically summation of w u comma v where u v in this path, so minimum among. So, if we have many path from u to v, so P 1, P 2, P 3 like this. So, if you have P k if there are k many paths from u to v. So, then you consider weight of the path. So, if it exist the minimum this is not infimum, this is minimum. If the minimum exist, then we take that minimum suppose this P l is the minimum among all these P l is the minimum of all w P i, i is from 1 to k say then P i is our shortest path from u to v.

Let us come to these example suppose for these example and that weight of the shortest path is denoted by delta u v. So, if there is no shortest path from u to v then we denote this by infinity. If no shortest path from u to v, so no shortest path means one option you have seen that there is no physical path from u to v, and there will be another option which is called negative cycle we will come to that. Even there is a physical path from u to v then also you may not have a shortest path, so that we will discuss.

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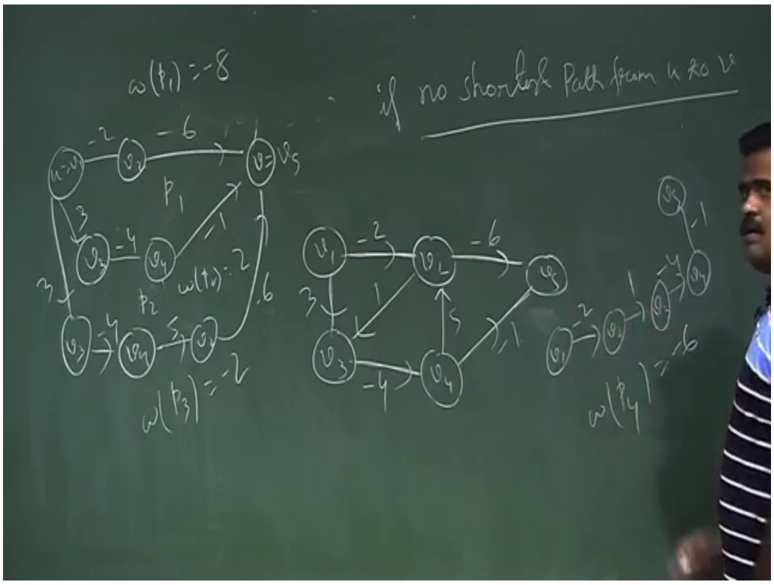

So, let us take this example. So, let us take this example suppose we want to find this is a graph suppose we want to find the shortest path from say v 1 to v 5. Suppose v 1 is u, u is v 1 and v is say v 5. So, we want to find the shortest path from u to v. So, how many paths are there from u to v. So, there is one path we can go from u 1 to v 2 then v 2 to v 5. So, this is basically one path u 1 to v 2 and then v 2 to v 5. So, this is you say P 1. So, what is the weight of P 1. So, this is minus 2 minus 6, so weight of P 1 is basically minus 8. So, any other path from u to v, v 1 to v 5, so yeah, so we can go from here to here then we can take this then we can take this. So, from here we can go to say v 3, then from v 3 we can go to v 4; and then from v 4, we can go to v 1, v 2, v 1 to v 3 v 3 through v 4 v 3.

So, what is the weight of this. So, this is 3. So, this is a P 2. So, weight of P 2 is basically what. So, this is minus 4, this is minus 1, so this is basically minus 5. So, this is 2. So, weight of this P 2 is 2. Now, is there any other path? Yes, we can go from here to here, then here to here, then here to here, then here to there. So, this is also one path. So, we go to v 1 to v 3 then v t to v 4, and then v 4 to v 5, yeah we can write this path separately. So, we can go to v 3 then v 4 then v 4 to v 5 sorry v 4 to v 2 and then v 2 to v 5 directly. So, what is so this is say P 3. So, weight of this path is how much. So, this is again 3, this is minus 4, this is 5 and this is minus 6. So, this is coming out to be weight of this P 3 is basically so minus 10 plus 8, so minus 2.

So, similarly we can have another path like this. We can go from here to here then here to here, then here to here, again we can have here to here, here to here, here to here, here to here, but that will not reduce the weight because this is positive cycle. So, it will just add the weight more. So, why to take a loop if there is a cycle here and this cycle is positive cycle. So, there is no question of loop it into that cycle because that anyway we are going not going to reduce the weight again. So, is there any other path, yeah, we can go from v 1 to v 2 v 2 through v 3 v 3 through v 4 and then v 4 to v 5.

So, let us just try that. So, we can go from v 1 to v 2, v 2 to v 3, and v 3 to v 4, and then v 4 to v 5 because if we go again v 2 then again v 6 that is minus 1, but still we have a minus 1 over here. So, this is P 4. So, what is the weight, weight is minus 2, this is 1, this is minus 4 and this is minus 1. So, this is basically minus 6. So, weight of this is minus 6 anyway we can think of you can find out some more paths by taking the loop also, but that loop is if it is that cycle is positive cycle, then there is no point in looping there. So, among this, there could be 3-4 paths which is taking care of this negative cycle, all these things. So, among this if we see this is the minimum weight path. So, this is minus 8, so that is the shortest path from v 1 to v 6.

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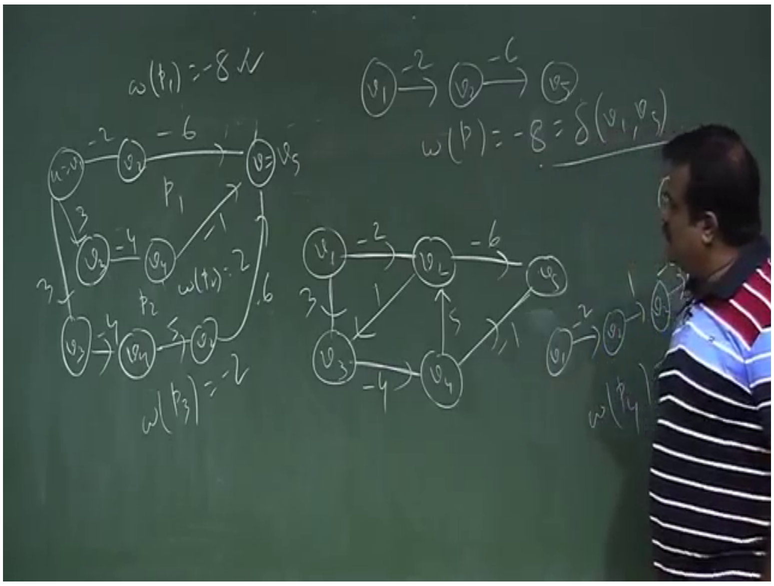

So, the shortest path from v 1 to v 6 is basically we go from v 1 to v 2 and then v 2 to v sorry v 1 to v 5 is basically like this. And weight of this path is basically minus 6 and this is minus 8 and this is basically delta of v 1 to v 5. So, this is the shortest path from v 1 to v 5. So, this is our problem. This is our problem given any two vertex we need to find the shortest path if it is exists. So, when the shortest path will not exists if there is no physical path from u to v, then there is no question of shortest path. If there is no path then there is no question of shortest path. But even though if there is a path from u to v, then also you may not have a shortest path. When if there is a negative cycle in the path then there is no question of shortest path.

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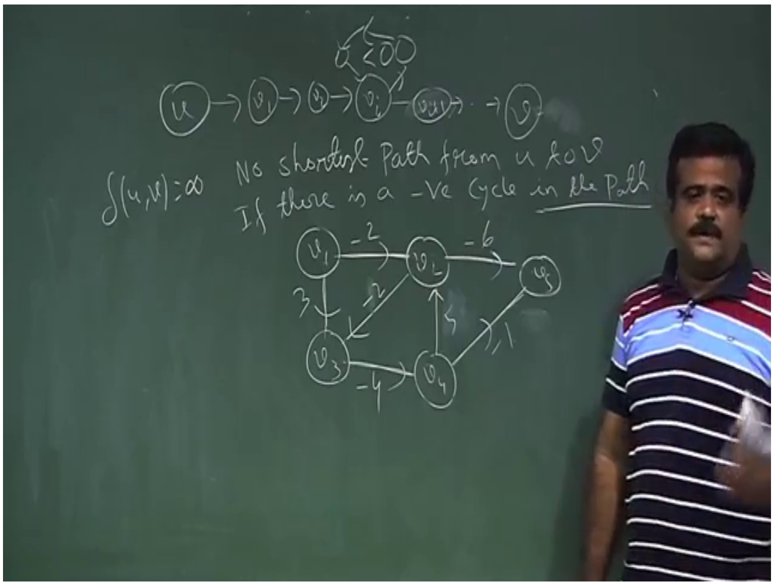

For example, for this case if we make this to be say minus 5 means instead of plus 5 or yeah or this is let us keep this plus 5 you just you make it to be minus 2. Then we have a negative cycle over here then what we do then this 6, 8 may not be a shortest path because why because if we demand this path is a shortest path then we have a another path, we can go from here to here, then here to here, here to here, here to here. So, it is reduce minus 2. So, as many times as we want we can have a hop there lopping.

So, if we have from v 1 to v from u to v suppose so this is say v 1, v 2 and suppose there is a vertex v i, and there is a negative weight cycle this is negative. So, this is say v i plus 1 dot dot dot v k. So, if there is a negative cycle in the path from u to v. So, there is a physical path from u to v, but there is a negative cycle then no shortest path will exists because this will not converge, because if you demand this minus 8 is my shortest path. So, what I will do I will have a loop over here, so that will reduce further. So, if you demand that will be my shortest path then I will make another loop over there that will because that loop is negative.

So, if we have a hop in that loop every time we are going to reduce the weight so that way this shortest there delta will not converge, so that way delta v will be infinity for this case. So, no shortest path, no shortest path exists from shortest path from u to v, if there is a negative cycle in the path. If there is a negative cycle in a path from u to v then there is no shortest path for u to v, so that also we have to take care that there should not I mean if there is negative weight then we should able to detect that.

So, now the question is so now, this is our problem now, this problem is called shortest path problem, finding the shortest path problem. So, how we can find the shortest path or length of weight of the shortest path? So, basically you want to find the delta. So, we want to see whether we can apply dynamic programming technique. So, for that we want to see whether there is a optimal substructure design in the problem.

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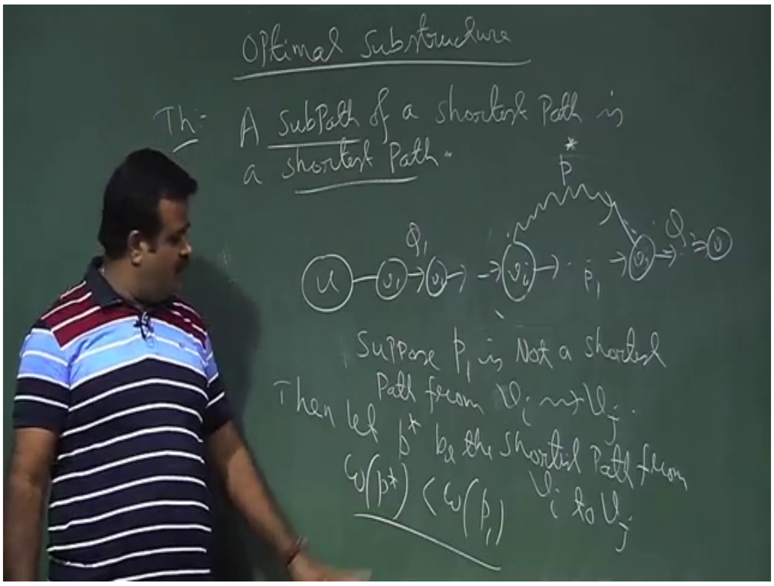

Optimal sub structure. So, this is the first hall mark of dynamic programming problem, dynamic programming technique. Any dynamic programming algorithm should have this properties. The first hall mark is optimal sub structure so; that means, what it is telling it is telling if we have a solution of a problem then it contains solution of the sub problems every solution of a whole problem will contain the solution of the sub problem. So, this is the first hall mark and the second hall mark is repetition I mean overlapping sub problems. So, there are many overlapping sub problems are there is let us check whether at least we have the first hall mark or not optimal sub structure.

So, how do you establish this. So, this is you can use in theorem for this it is telling any sub path, a sub path of a shortest path is the shortest path. So, how to prove this? So, what it is telling, it is telling if you take a path from u to v, suppose this is we take a shortest path this is the shortest path from u to v. So, say v i dot dot dot v j sorry. So, v 1, v 2 dot dot dot v i dot dot dot v j v. So, suppose this is the shortest path from u to v, this is basically delta of u v. So, this is the weight of the shortest path from u to v. So, this is a shortest path from u to v.

Now if you take a sub path, if you take this just this path from this is a sub path, this path only, so v i to v j. Now we have to prove that this will be the shortest path from v 1 to v j, so that is the optimal sub structure. If you take this path sub path from v 1 to v j in the shortest path of u to v then this will happened to be the shortest path from sorry shortest path from v I to v j. So, this is a sub path from sub path in this shortest path. So, how to prove that? So, to prove that you have to use what is called as cut and paste technique. So, this we are going to so this is the shortest path from suppose it is not a shortest path from u to v suppose. So, suppose this part is P 1.

So, this is the prove. This is suppose P 1 is not a shortest path from this is v 1 to v j, v i to v j, suppose it is not a shortest path from v i to v j. So, there has to be a path which is shortest path we are assuming there is no negative cycle and there is a physical path. So, we are so there is a path from v 1 to v j which is the shortest path. So, this you are denoting by P star. So, then let P star be the shortest path from v i to v j. We are assuming this P 1 is not a shortest path, so they. So, we are assuming that P star is a shortest path the another path. So, what we do. So, we take this as a this Q 1 this path from this to this, and we take that this path as a Q 2 from this to this. Now, we take this so; that means, what; that means, w of P star is less than or equal to w of P 1 because P star is shortest path from v i to v j and P 1 is other path from v i to v j. So, weight of the P star must be less than this.

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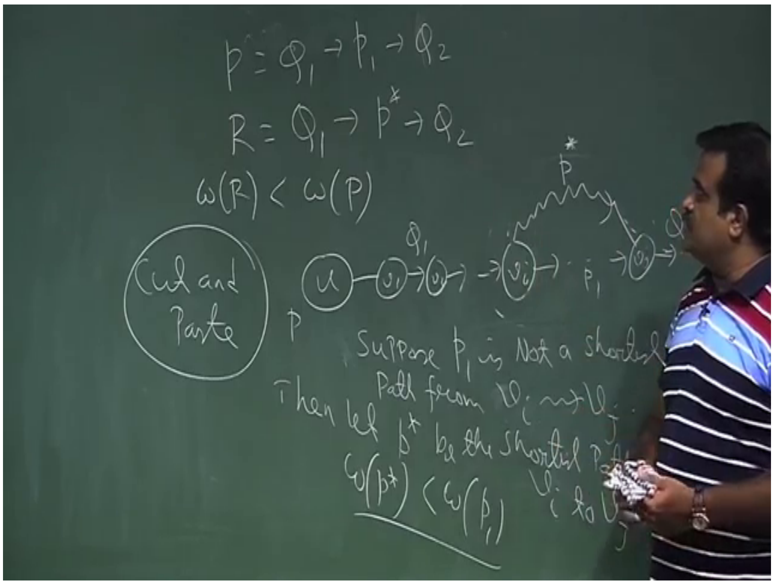

So, now we take the path from u to v like this. So, we take a path. So, this is our original path this is the p. So, what is P path P path is basically Q 1 and then followed by P 1 and then followed by P 2 sorry Q 2. So, this is our P path. Now, we take another path which is R say which we are taking instead of this P 1, we are talking the P star. So, Q 1 then P star then Q 2. So, now what is the weight of R, weight R must be less than weight of P because this P star is the shortest path of this, but this is not possible because we know the P is the shortest path from u to v, so that is the contradiction. So, this technique is called cut and paste technique, cut and paste cut on paste technique. So, we assume this is not a shortest path. So, we cut that one and then we paste a shortest path from that and then these to a contradiction, so that means this theorem is true. So, every sub path of a shortest path is a shortest path.

So, this is the optimal sub structure, so that is the good news, so that is the hall mark for dynamic programming techniques. So, even second hall mark is also we can easily verify. So, now we will think of dynamic programming technique, but before that we want to see whether we have the powerful on the greedy approach can be possible or not. So, for that, we will see the greedy algorithm can be applied or not.

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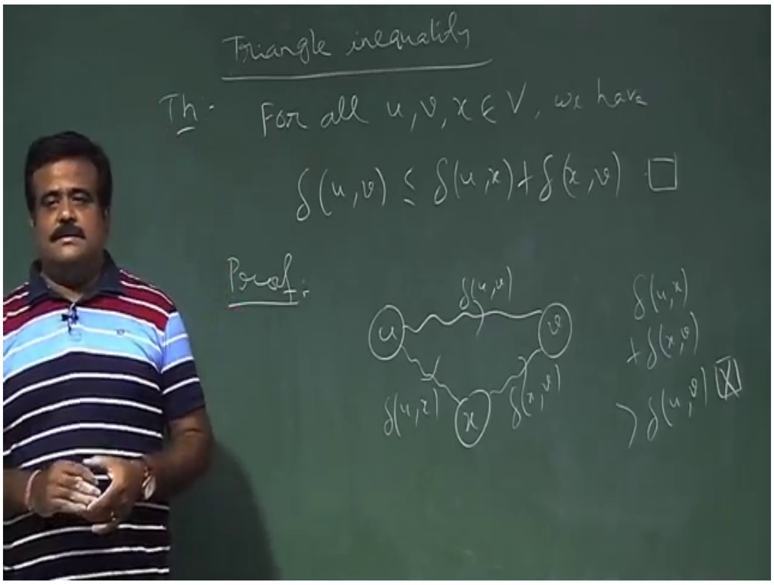

So, for that we take a theorem in equality another theorem which is called triangular in equality or triangle in equality. So, basically what basically this is telling us for all vertices u v and x belongs to v we have this in equality. So, shortest path from u to v must be less than or equal to shortest path from u to x plus shortest path from x to v. So, this is called triangular inequality. So, how to prove this? So, this will give us the algorithm which is the greedy approach or dijkstra algorithm we have to use this in equality there.

So, let us try to prove this. So, to prove this, let us take this is the vertex u, this is the vertex v and this is the vertex x. So, we are talking about shortest path from u to v. Now, shortest path from u to x and x to v. So, this is basically delta of u to x and this is basically delta of x to v and this is basically delta of u v. So, this theorem is straight forward from this picture because so if we take this so this is a shortest path from u to v and delta is the weight of the shortest path. So, weight of the shortest path must be less than of any other path.

Now, if you take this path from u to v if you go from u to x with the cost this we take the shortest path there then we go for x to v we take the shortest path there. So, this is the path from u to v, so and the cost of that path is delta of u x plus delta of x v. So, this is a weight of a path. So, this path has to be greater than the shortest path from u to v. So, that is the theorem. So, this is called triangular in equality. So, we will use this in equality to have the algorithm in our algorithm to finding the single source shortest path. We have a given a source s, and from there we will find the all shortest path from any other vertices, so that is the dijktras algorithm we will discuss in the next class.

Thank you.