Computational Arithmetic - Geometry for Algebraic Curves

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Week - 04

Lecture – 08

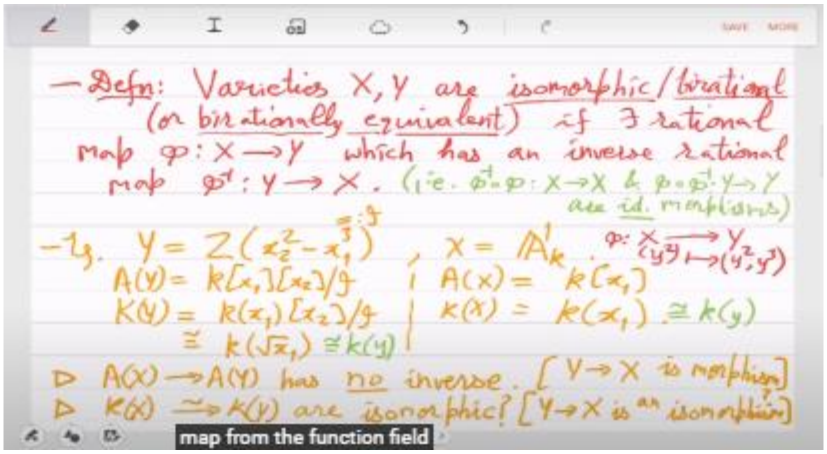
Tangent Space and Singularities

Any questions you have? Yeah, yeah, with this I have as we discussed rational map will be given by some open sets, it can even be just one. No if they are if they are if the description is given by a many open sets then they should be you have to use the equivalence relation. So, intuitively I think this is because open set essentially is everything. So, the closure of that will be the whole of x that the definition is very relaxed, but if you make it dominant then you see that on the LHS U is almost everything and on the RHS image of U is also everything. So, probably the correct thing is dominant rational map which we usually think of. No why that is not.

No, no the definition was just saying have an inverse. So, x and y are called bitrational equivalent or maybe we can also say just bitrational. If there is a φ and there is a $φ^{-1}$ I mean which can be further described. So, which what it means is that So, what should you do? So, you apply φ on x you get to y and then you apply $φ^{-1}$.

So, this should be identity and the opposite yeah yeah or identity maps or maybe I should call morphism right. I will need the space so that should yeah so for this you only need definition on open patches not the whole of X. Yeah, so $φ^{-1}$ is not really what we usually think it is it is some other map χ, but φ composed with χ or χ composed with φ behaves as a as the morphism identity on certain patches okay I think we should then start. so in this it was correct that x and y are I mean the where I the curve y is bitrational to the affine line that is correct because of the field being isomorphic the function fields being isomorphic. But do you know what is the morphism at the level of the point which you have So what is this map φ? So point on x will just be a single coordinate or on y it will be 2 coordinates.

How do you expand this 1 to 2? So you can actually get it from the field isomorphism. So this kx function field is isomorphic to this let us say parameter y, ky and this KY function field is isomorphic to the same function field small ky, but at the level of x1 there is a square root happening right. So, actually I will just give you the point level map it will be point y^2 here is mapped to y^2 y^3. So, note that y^2 , y^3 is always a point on the curve, they pick y because $x1^3 = x2^2$. right and in the affine line saying that we are going over the points y^2 is basically going over all the points because every point is a square, small k is an algebraically closed field right.



So, this bitrational map is essentially giving you parameterization of the curve y ok. So, you can actually deduce point level map from the function field isomorphism this is it. Then we had affine open we are calling this x sub f affine open which is which we showed we actually showed a bitrational isomorphism or equivalence between the affine open and this 0 set the hyper surface. So, xf is actually it is this distinguished open set, but it is also bitrational to a, in fact it is also an isomorphism. So, you have morphism both ways.

So, it is isomorphic to a hyper surface in a bigger ambient space a fine n + 1 space. Sir, can you also explain the motivation where motivation for rational map is. So, basically we want something that we can compute and for working with the function field is easier to compute. So, we I mean as we saw here the coordinate ring testing for this isomorphism is a bit harder, but when you are in the function field then you can actually identify you can do this the thing we did in the end which is you basically break up your function field into the purely transcendental part and the algebraic part which will be finite and then studying your variety and comparing your variety with other varieties reduces to just this finite part. So, you actually have and which helped here by showing which shows actually that you have bitrational equivalence to any hypersurface.

So, these are things you could not do with morphism, but you can do it with rational map. Yes, yes. yeah exactly we so working with I mean you want to reduce the theory of varieties to the theory of fields. So, think of it as a reduction yeah and I think it was a good suggestion that I should not use isomorphism, but just call it birational equivalence.

So, it this I will just say is birational to a hyper surface.

Since you came late, so I gave here the point by point map between the affine line and the curve. So, it is basically this parameterization. So, y^2 is being mapped to φ^2, φ^3. this is the bi-rational equivalence between line and the curve, it comes from the function field isomorphism written carefully, no we do not, well I am mapping points of x to points of y in that way, so any point in x can be thought of as y^2, yeah so sure but everything will work. it is still a bi-rational map.

Yeah, so you can think about both you can think about the map happening in the affine n space or you can think about the map happening in the function field that with that translation take some effort, so you have to do it carefully. yeah any questions till now, then we will start studying smooth points, so non singular varieties. So, what is singularity or and the opposite of it is smoothness, when will you call a point P on a variety X to be smooth or non-singular. So, yes the picture that you should remember in this topic is the following. So, you have the XY axis and you have some curve.

So, you can identify I mean you can pick a point P and what you are interested in is the tangent at this and the way to define tangent at least in real analysis is you try out several points that gives you a chord and keep moving closer to P and in the limit you will have. the chord which is path which intersects with the variety or in this case the curve only at this point P right that which is called the tangent. that is true yeah it can it can yeah the curve can come back and the tangent can again intersect it sure. Yeah so but I mean we have a notion of distance in analysis so in the neighborhood of P you only see a single intersection which is P. I do not want to go into multiplicity because I mean it is very difficult to define any of that.

So, local is the first thing you will try because you have a some idea of metric. So, in the neighborhood of P the tangent should intersect the curve at exactly one place which is this point that you are interested in. Yeah so this is the variety x and point P and that is called the tangent. So tangent is a line here and in general it will be a hyper plane right in 3D it will be a plane and so on one dimension less. So that is the thing which you want to generalize without drawing any pictures.

and without having this metric of distance. So, how do you do that? So, let X be a fine let I x be the ideal defining it. So, it is the ideal in A which is the coordinate ring of the affine variety and pick a point in x. So, Yeah so with this intuition of line what you want is actually a vector space right you should define your instead of defining tangent now what you should define is a tangent space. So it will be a vector space and for that what you should do is you should look at only the linear parts of these generators Fi's.

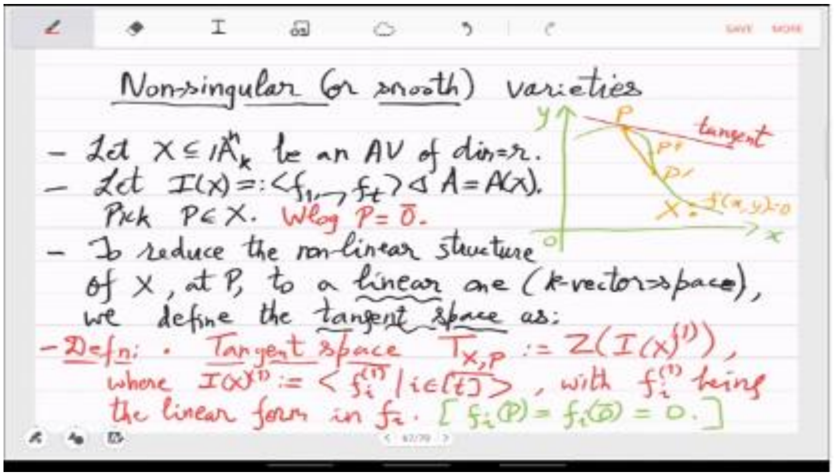
so the idea is that you want to reduce the non-linear structure of X at P to a linear vector space, so it becomes a K vector space. so we define the tangent space as the linear space over the field base field small k for simplicity I will assume here which is without loss of generality that p is the 0 point. p is the origin in that case the definition is the following of the tangent space. So tangent space we will write it as T sub x, p. and it will simply be the 0s of ideal linear part.

So, this Ix1 is the linear part of the ideal which I defined further as f i is linear part and yeah I do not need. So, actually the ideal generated by these linear forms. Now you should remember here that Fi 0 is 0, so Fi P is Fi 0 = 0, why is that? Well because we are assuming that the point P is 0. I mean coordinates are all 0 and it has to satisfy each of these Fi's which means that Fi at 0 is 0 which in other words Fi is constant free right. So Fi starts with a linear form then there is a quadratic form then there is a cubic form and so on to arbitrary degree up to degree of this polynomial Fi.

So we are only looking at the linear form part that is being called Fi1. and the common 0s of these Fi1s is or the ideal generated by Fi1s is called Ix1 and the common 0s is the transient space Txp. Is it clear? Why is it different like this? Why, well because it will match the picture. I am multiplying by define partial derivative.

So, we will come to that. So, that is the. So, this is the tangent space let us check the properties later and. So, tangent space is what should. So, how does it compare with x. see x was the zeros of f , zeros of f1 to ft the tangent space is , zeros of the linear parts of them.

So, what has happened is locally you can potentially see a bigger space you will get more points you will get clearly you will get points which are outside of x the variety x right. So, in the picture also that it is happening that the point p you are getting on the tangent line, but all the other points are outside the curve right. So, a similar thing is happening here also. So, instead of drawing this Txp we just define it as the like as a different variety, but it happens to be a vector space. That you can see that 2 points in Txp can be added it is again a root coordinate oh then it is 0, then Ix1 is will be 0, sure.



No, no, no it is no, no we are not expecting anything this is the definition it is fixed now. The only thing we have done is we have picked P to be the 0 point. So, your only valid question can be what happens for other points. So, for other points we will actually shift the whole system F1 to Ft by that point.

so that 0 is a root. So for any other point you just shift the system so the whole variety will basically be shifted by the point P so that you come to the origin. So any other variety any other point on a given variety you can work like that yeah but the definition for 0 is this there is no other option it is a unique definition as we will see it will satisfy everything we want. we will see an example, but let me first talk about the coordinate ring of this it is called the dual space. So, the linear functions on the tangent space gives us the dual space which I will denote as Txp wedge and this is as stated A1 mod Ix1. So, A1 is basically the linear forms in the polynomial ring modulo the ones which you are considering 0 which is Ix1.

So these are the linear functions which are defined on the tangent space. So this is the dual of the tangent space. It is slightly different from the coordinate ring of tangent space but it is something more familiar as it is the dual of a vector space. So, an example has to be given because there are many questions. So, let us see the example of affine line.

So, in this case and the point we are taking is the 0 point. So, there is only one coordinate it is 0. So, what do you think is T x p? what is the tangent space of the affine line at the origin. So, this will be 0 of Ix1 and as Madhavan said Ix for I mean in this case Ix was itself actually 0.

So, Ix1 is again 0. So, this is 0s of 0 which is everything. the affine line itself, right. So yeah, so it is not I mean geometrically it makes sense for the affine line tangent at the origin is the whole line, right. It is consistent with our geometric picture, this is good.

What is the dual of that? so dual is mod i x1 which is 0 so it is a1 which happens to be just k times x1 right or the vector space span by x1 over the base field k right.

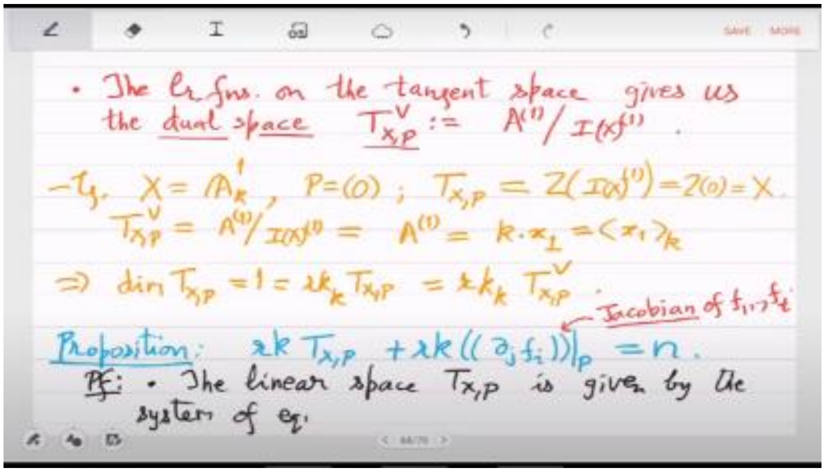
in this case you can also look at the dimensions. So, what is the dimension of the tangent space? So, that is for the affine line 1, what is the rank of the tangent space as a vector space over the base field k that is also 1 and what is the dimension of the dual that is also 1 and maybe I should see rank. So, the tangent space dimension and that of the dual both of them is 1, you can talk about dimension or the vector space rank. yeah so we didn't see partial derivatives here we will do that next, but any questions till now I hope this is fine the fine line example set the stage for everything else you can do the same thing in other examples. In particular this example you were asking y^2 equal to x^3 also.

We will do that I think later. Let us first go to the relationship with partial derivatives. Yeah, I want the dimension of the variety. Yeah, because I define Txp to be the 0 set. yeah you prove all those things. See you have this is the best case possible right you have linear system and you are looking at the 0 set I mean it has to be an affine variety.

So, we can prove this we should prove this proposition now to see the effect of derivatives So the rank of this tangent space and the rank of the derivative matrix, so dou j f i right, you have f 1 to f t and you have variables x 1 to x n, so you look at this this matrix of first order derivatives, partial derivatives because F i is invariant. So, this is also called the Jacobian matrix of the polynomial system. What is the meaning of rank of this matrix? So, you are looking at the rank over the function field, it is a functional matrix. So, you look at the rank over the polynomial ring or the function field, yeah but I do not want that, so I want local information around P, so actually I should be evaluating this at P, so the matrix at P, look at the rank of that, so this rank plus the tangent space rank should be what? what do you think for example, in this affine line case what happened. So, rank of the tangent space came out to be 1 and rank of this matrix is 0 derivatives were all 0 right.

So, the sum was 1 which happens to be the dimension of x well it also happens to be n. So, in general it will be n that is what we want to show. Yeah I know I spoke too fast, so we will actually look at the matrix at the point P then it will become small k. Yes you do not want to look at the functional matrix because you are only interested in the variety up to the point P around P, so you should actually evaluate it, but the matrix name is correct. So, if you just look at the functional matrix it is called the Jacobian of the system which was what F1 to Ft.

So, look at the Jacobian of the generators. Of course I mean the weird thing here is that the generators are not inherent to the affine variety. So you can pick some other generating set also. Jacobian will change but still this theorem will remain true. It does not depend on generators. It is something, it is an inherent property of the geometry not of the representation.



How do you prove this? so now you do the school calculation the way you calculate tangents right. So, we have to do this in higher dimension now. So, the linear space Txp is given by the system of equations. So, differentiate the I mean what was there in the definition it was these f i 1's. So, what is f i 1? FI1 happens to be this let us check this is this correct.

So the difference is that in the LHS I have only linear part of Fi but in the RHS I am using derivative jth derivative on I mean it is basically you are differentiating by xj. So do by do xj you are applying that on Fi the full Fi it has also quadratic term and so on. But these quadratic terms when you differentiate you still get a variable there and at p which is 0 that contribution is 0 right. So, you can see that this is a it is an identity and that as you go over all the i's that is your system defining Txp. So, Txp is essentially the solutions xj of this system which gives you the claim the proposition.

So its solution space x ‘ ϵ k^n is of linear rank equal to n -, so these are n variables minus the rank of this matrix. the defining matrix of the linear system. Is that clear? Solution space is exactly the definition of tangent space. So, you have that rank of Txp plus rank of this matrix has to be equal to the ambient space dimension.

Fine, next proposition will be about the dual space. So, how is the dual space related to the germs. So the tangent space somehow is approximating the variety and the functions on the tangent space which is the dual then should somehow be approximating the functions which are defined around p these are the jumps right. So what is the relationship? Can you guess? So the dual which is a1 | ix1|. how is it related to the germs. Now, what if you evaluate this function or any function in this dual space at P the answer will be 0 right, because you are only looking at linear forms.

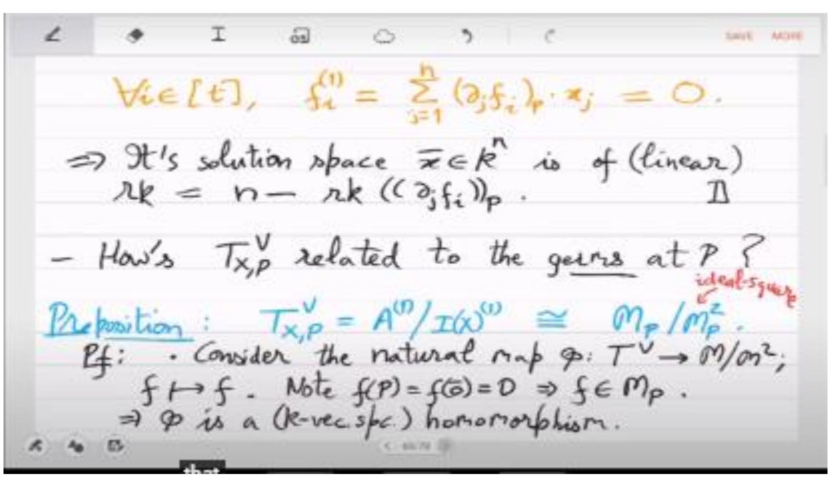
So, at P you will get the answer 0. So, this should somehow be related to MP. because MP are the germs which vanish at P right. So, is this correct the isomorphism yeah it cannot be because on the LHS you have linear forms in the RHS you have arbitrary potentially arbitrary polynomials yeah. So, you just do MP ^{2}. So, that is the geometric understanding of the dual space So why is this true? By the way MP ^{2} is the ideal square.

It basically you take any two elements in MP multiply them and look at the ideal generated by these products. It is called MP ^{2}. is ideal square. So, when you do ideal square what you get is the quadratics are considered 0. You take a I mean the a linear part of f in MP and a linear part of g in MP when you multiply them they become quadratic.

So, this is somehow a linear object now. How do you show this? So, consider the natural map φ which sends let me just call it T dual → m | m ^{2} | everything is obviously with respect to P here in this context. So, the natural map will be you take a linear form in LHS and just view it as an element in MP right, which is well we have to check whether it is actually defined. So, it just sends F to F, is it a well defined map? So, | MP ^{2} | is it well defined? So, f is in MP right. So, since f is already in MP you can also view it as an element in MP | MP ^{2} |.

So, it is clearly a well defined map. What I claim is that this map is actually an isomorphism. No, no the thing is that we have picked p to be the 0 point. So, every linear form vanishes at 0.

We are using that. So, it is a trivial thing. Once you have the correct point. So, first thing we have to show is that it is injective. It is a ring homomorphism. Why is it injective? In fact, it is even a k vector space homomorphism.



We view it like that. Let us check injectivity. So, say φ (s f) = 0. which means what? Which means this F which was in MP is actually in MP ^{2}, which means what? As F is a linear form we get that f should actually be 0 in the polynomial ring. Is this clear? MP square does not have any linear forms. So, if your f is in there then it must be 0 which means that it is 0 in your domain also that is the tangent space dual that is injectivity is that clear.

So, next thing is subjectivity. So, let δ = f / g. So, what do you want to show? You want to show that any element in MP over MP ^{2} let us call it δ that is a pre F in φ^{-1} (δ) exists. So, we will actually construct that. So, let us pick an element in MP and without loss of generality we can assume that G at P is 1, right because rational function δ around p, defined around p, g at p cannot vanish, it will be some constant, so we can assume it to be 1. And I already want to give you the pre-image δ ‘, can you guess that, what is the element in the dual of the tangent space which φ will take to this δ.

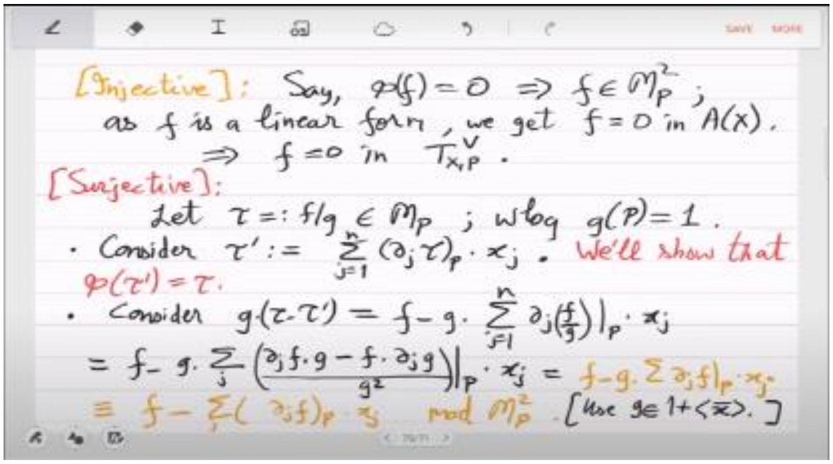
You have to just define a linear form right. Yeah. So, its coefficient should just be the derivatives of δ. So, we just use the derivatives call that δ prime that is our best guess. So, dou j δ at p times xj. So, we will show that φ of δ ‘ is δ that is our goal if you already see this then we do not have to do the proof do you see this. So, basically δ | mp ^{2} | a δ prime yeah δ prime | mp ^{2} | if you view that it is the same as f by g.

Let us do this. So, let us consider I mean we want to show essentially that δ prime and δ are the same right modulo the | m ^{2} |. So, let us take the difference so consider g times δ - δ prime because there is a denominator in the g. So, let us normalize by that and we are interested in δ - δ prime. So, this is what this is equal to f - g times sigma of let us write that at p times x j and we can continue this differentiation. So, you get f - g times j dou j f times g - f times δ j g / g ^{2} at p times x j.

That is the Leibnitz rule and what to do next, well this g ^{2} at p is just 1, so we can ignore that and yeah, so the other thing I can ignore, so remember that g ^{2} at p is 1, so you ignore g and g ^{2}. what can you do with F times dou jg evaluated at P, well it is 0, yeah so we simplify all this and we get this. do j f at p times x j is that clear. And I can continue this now | m ^{2} |. So, what I can do is I can take the g inside why is that? Yeah or you can simply think of this, so g has in the constant part 1 and the remaining things are linear or higher degree.

So the constant 1 multiplied with this while the linear and above part multiplied with this, so that will multiply with xj and that will give a quadratic part. So actually mod mp square what you get is yeah actually I can do it also directly you get this is that clear. So, here we are using the fact that g = 1 + m.

we are using this property. So, g is 1 + in fact even the ideal x' . So, g has the constant term 1 and everything else is this monomials in x. So, they will multiply with xg and then they will vanish mod mp ^{2}. So, that is where we are What about F? F is still there. We have only eliminated G.



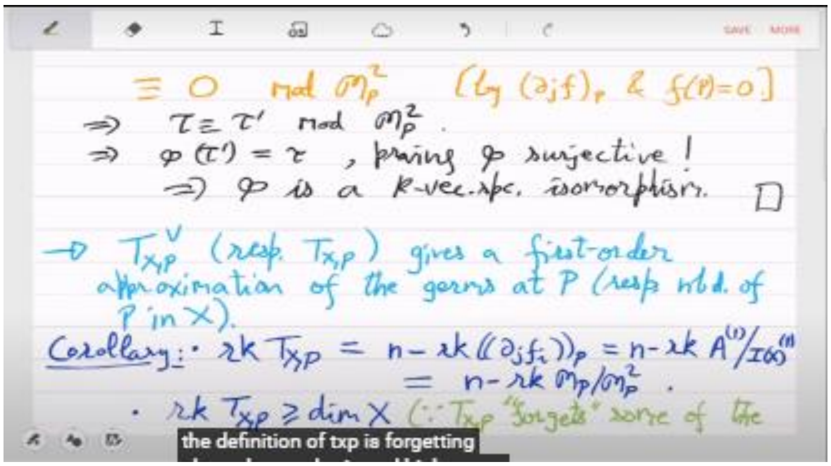
Essentially in this calculation G becomes 1 and the reason is this. We are going | MP ^{2} |. What next? yeah how do I deduce that no no oh yeah sorry yes sure yeah good so this is. yeah right. So, f was constant free yeah that also we are using. So f was constant free and the other things for these xj coefficients but xj coefficients are equal to drjf at p.

So we have got into 0 which basically means that δ and δ prime are the same. Is that clear? | mp ^{2} |. So this is the magic of squaring the the germs which are vanishing at I mean the mp ^{2} mp ideal if you square it you get this nice property. You have an isomorphism now you have this subjectivity and hence isomorphism. So, φ of δ' is indeed δ proving subjectivity so this means that φ is an isomorphism right.

So, yes what we have learnt in these two propositions we can now summarize. k vector space here small k. So, what we have learnt is that the dual of the tangent space respectively the tangent space itself gives a first order approximation. of the germs at P respectively the neighborhood of X. So, geometrically the tangent space is approximating the affine variety X around P. and algebraically it is approximating the for the germs which are defined around P and in fact giving you a first order approximation because we have reduced this question to linear algebra setting.

That is how you can read these two propositions. Any questions? I can also write a corollary So the rank of the tangent space = n - the rank of the matrix which is equal to n - the rank of a 1 | i x 1|. which we have shown also equal to MP / MP ^{2} is all these things are the same. So, that is one property the other property is that rank of the tangent space is at least the dimension of x is this clear why is that. because around the neighborhood of P we have potentially added more points than x had, right.

So, dimension can only increase it cannot decrease. Tangent can become a higher dimensional object. Since T x P forgets higher order constraints some of the higher order constraints that defined x. Yes, so I hope you can see the formal proof from this. So, basically if the definition of Txp is forgetting about the quadratic and higher of the constraints.



So, constraints become simpler potentially you may have more points around p. So, tangent space could have dimension bigger, but never smaller. between x subset I mean our basic case for a curve the tangent space is either a line or the I mean affine for fun space the line or the plane affine to space so sure I mean it is a subset. No but tangent space is I mean the only thing which is common between TXP and X is this point P. So, what you are claiming is too strong can't do that. No here it is only a statement it is only a quantitative statement it is not we are not saying anything about the points.

We are just saying that the dimension of X if it was R then the tangent space will have dimension at least R it can also be R + 1. No, you can also compare with the dimension it is the same thing. Dimension of Txp and rank is equal I think I mentioned that before. This here in orange I mean as a variety the dimension is 1, but also as a vector space the rank is 1 it is the same thing.

So dimension of tangent space is at least dimension of the original variety you started with. So what we want is we do not want the tangent space to go further away from x. So we want these two things to be equal. So if the dimension of tangent space is greater than dimension of x, then the tangent space has grown bigger, much bigger than the neighborhood of x at p which implies that in this case in that case tangent space has kind of lost all information. You do not want that to happen, you do not want the tangent space to lose all local information.

So, hence we will be only interested in cases when there is an equality. So, for that we define smoothness. So x is called non-singular at P if rank of the tangent space is equal to dimension of x. we can also call it that x is smooth at the point, p is a smooth point or p is a simple point. These terms are used interchangeably. It is a non-singular point, it is a smooth point or it is a simple point on the affine and x itself is called non-singular if there are no singularities.

So, if at every point studying the tangent space gives you good enough information locally then we call the affine variety to be non-singular or smooth. Yeah, but Jacobian may have will have 0 rank. No, no, so what is your example here? x is the fine n space dimension is n. Dimension is n, yes. No, no, no give me the counter example for x and p such that this inequality is false for the affine n space you are right.

So, if when dimension of x is n then what you are saying is the tangent space is also n, but it because it cannot be n + 1. So, you can see that that happens I mean in other words if you have dimension n in affine n space then that has to be a non singular variety which is correct there is no conflict. Now, let us take some examples that you were asking before. So, let us take so non affine space example let us take this curve.

in the affine 2 space and let us take the point we will take it to be 0. What can you say about the point in this affine variety? So, the best way is to use that derivative matrix to compute the rank. So, let us look at that. So, rank of dou j f here you only have 1 what is this? So, this is just a vector. So, it is equal to rank of well - 3 x 1^2 and 1 at the point P which is rank of 0 1 which is 1. which implies that the rank of tangent space = 2 - 1 which is 1 which is dimension of x which means that p is smooth.

So P is a non singular point on this curve because you can check that the tangent space is not bigger it is just exactly correct matches the dimension. If you replace by h2^2 now then the rank will be higher right.

Good. So let us do that. So let us do the same thing here. x2^2 - x1^3. So, notice that we had shown before that this is actually bi-rational to the affine line which may hint that this that every point should be smooth on this, but what we will show is that it is not the case. We will actually show that this point this is a singular point it is yeah it is not smooth it is not simple. Let us do that calculation again. So, you get - 3 x1^2 and now 2 x2, but that will give you 0, which means that since this rank has reduced tangent space will increase.

So, rank of the tangent space is now 2 - 0 which is 2 which is greater than the dimension of x right. So, tangent space has grown too big. So, which is p is a singularity. Yes. So, you show that. this is the unique similarity, this you have to show I have not shown this, but you can show that this point P the 0 point origin is the only similarity here, I can draw a picture for that then you will be convinced.

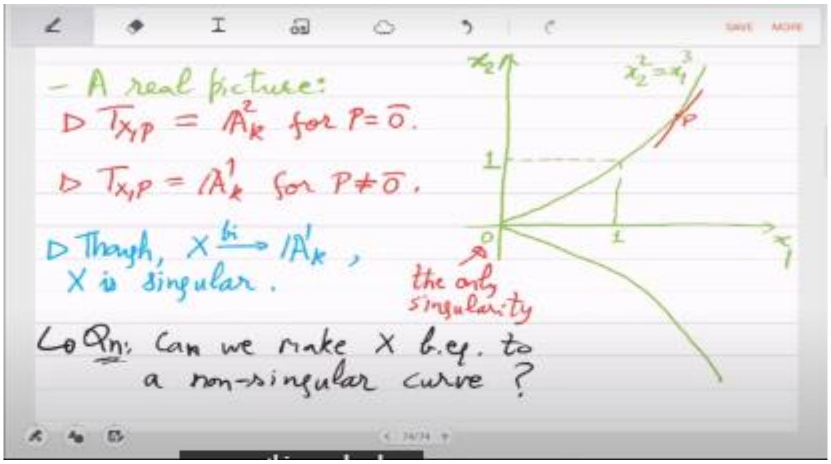
Let us do that. Let us see a real life picture of what we just did. So, you have x, y and origin and what is happening here. So, it is something like this. or maybe I should say x1, x2 actually and we are looking at $x2^2$ equal to $x1^3$ right. So, as x1 increases x2 also increases with this curvature and it has to be symmetrical along the across the x1 axis right.

So, same thing below. and at 1 it is 1. So, if you draw tangents anywhere things will be good like tangent here is this. You can see that anywhere you draw tangent it is a line except at the origin. So this is the only singularity. Well because I mean here you cannot even draw a tangent and when you do the calculation you see that actually the whole space is the tangent.

transient space. So, this and otherwise it is the affine line. Is that clear? So, this is the unique singularity you see this bit from the picture and prove it formally also. Any questions? Does it capture it? I think morphism should capture it. I think what happens is, yeah it is hard to say actually, why did the bi-rational map violate singularity, it is just do the calculation and see what happens, but it is there is no good explanation for that, but we can record that. so though this x is birational equivalent to the affine line x is singular.

Yes, I mean this is, there is something that is genuinely about x here. Just having a bitrational map to the line is not enough. I mean, so you can see in the picture this x looks very different from a line and that is inherently two dimensional information. And somehow that two dimensional information is completely contained at the origin. so if I flip this bottom part brought it above then it will become non similar or smooth curve, but here not. So, the thing is that we want to do that we want to remove the similarities at least for curves we in this course we want to remove the similarity and get a birational non singular curve.

So, that is the question we will solve next. Can we make x bitrational equivalent to a non singular curve? so this can be done and we intend to show ok. So, topic will be resolving similarity and we will restrict to curves only because this is hard or impossible on non curves.



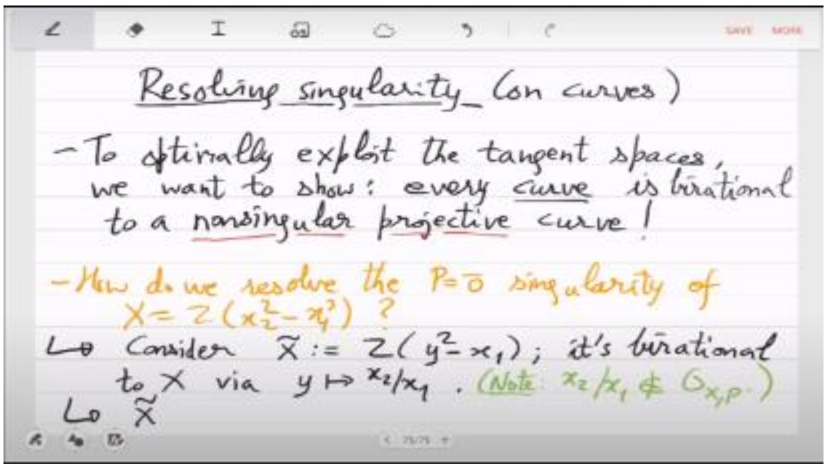
This will be a very dimension 1 proof that we will see. So, to optimally exploit the tangent spaces we want to show that every curve is birational to a non singular projective curve. So, this is this is another twist that we will go from affine we will be forced to go from affine to projective when we are resolving singularities.

When the idea is somehow yeah you read you can we will show how to resolve a single a unique similarity if there are two points which are singular then we will resolve it in different ways. and then we will have to glue the two ways together and that gluing naturally happens in the projective space. Projective space is by definition a glued object. There were many affine patches and we have glued them together so that we get the projective variety.

that is what we will have in the end, this will take some time. So, let us see the resolution in this example, the p equal to 0 singularity of x equal to z this curve, how do we resolve this unique singularity, any ideas? So, we will consider very different, but related curve which is y^2 - x 1 and I claim that these two curves are bi-rational, which actually is the same proof as we gave for bi-rational equivalence with the affine line. it is the same map, it is bi-rational to x via association y mapping to x2 by x1. To what? Yeah that is a good question why are we not doing that. So the thing is that then what you are doing is you are just drawing tangent at a point at the singularity and you are saying that now I will forget every other aspect of the curve and only look at this line. So when there are very two points of singularity in very different places then which line will it you want to cover all aspects of the curve ultimately.

So, you I mean it will be a very easy route to just take tangent space, but then the problem is that you will not be able to glue them. So, you want to stay as close as possible to the curve the tangent space takes you actually very far away in that in that respect. No no, so no no you want this theorem to be proved. Every curve is bitrational to a non-singular projective curve. If you just took tangent spaces at different points of singularity then you will not get the projective curve, then you will be moving to something else.

For example, then with just two points of singularity you will get the affine and two which is not a curve actually you have increased the dimension. So, you do not want to do that you do not want to go to non curve you want to stay in curve which is why this is a highly non trivial result it actually on the face of it looks unbelievable that from a curve you can modify it to a different curve such that the fields will be isomorphic for function fields and every point will be smooth. So that is what this trick will achieve, what this is doing is of course around the origin in that neighborhood it is kind of abstracting out this undefined quantity 0 by 0.



So, x2 / x1 is not a germ, right this is not a germ at 0. this is not defined in OXP. So, it is taking this undefined function rational function on x and it is calling it y and then you look at y^{2} - x1 still you have a bitrational equivalence and you can see that x tilde is smooth everywhere. you can check that no, no this calculation we already did that was the x2 - $x1^{3}$ you do that for x2 - $x1^{2}$. No, but that is a mistake if you go by the definition 0 is a smooth point the calculation we did I mean which is why drawing pictures may be wrong because. Really. so we are seeing y is square root of x1 so in the real space x1 look at this What do you think now? Every point has a tangent space dimension 1.

So, curvature has changed. So, yeah let us write it there now. So, x tilde is non singular and birational is a non singular curve and birational to x. That is the example, we will now do this in general, we will do it in a way which is abstract enough to be able to glue without drawing pictures, because drawing pictures will be impossible. when you do it for two different points you have to you can imagine the problems right the curvature has to be changed in a way. So, that in both the points good things happen and overall it is a curve and also bad things should not happen in a third point.

So, you cannot control this by drawing pictures. So, we have to do this systematically and that is what we will start next time. that is where your algebra will become more complicated. So, we will achieve this next time by introducing discrete valuation rings. So, we will develop DVRs to achieve this for multiple singularities. So, discrete valuation rings we will study next time.