Computational Arithmetic - Geometry for Algebraic Curves

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Week - 04

Lecture - 07

Rational Maps and Birationality

So, what we defined was mainly this algebraic construction inspired by geometry localization right. So, you can localize by a multiplicative set which means invert introduce fractions in the set big T and even more special you can take prime ideal p. and then the big set T will be complement of this which is a minus p that is a multiplicative set you invert all those elements that is called localization of a ring at a prime ideal localization of a at p that is how you get these restricted fraction integers. So, for example, you can decide that you will divide only by odd b. So, you will not have half, but you have everything else, you will have by 3 or by 5 and so on. You would not have by 6 for example, but by 7 you will have.

Then there are projective versions of this and we define this O (X) functor. So this is specially good for morphisms. So when you want to compare variety X with variety Y, then instead you should compare O(Y) with O(X) and the arrow will be reversed. So this is a contravariant functor.

We will be using this all the time. We define distinguished open set X sub F for a polynomial. is basically the zeros of F compliment in the variety X and this can be used to cover any open set that we had seen and we also saw algebraic version of it. So, O(X) functor on this distinguished open set basically localizes your polynomial ring. it localizes it by multiplicative set powers of f.

So, 1 / f gets introduced essentially in the ring. And finally, the germs are germs of over the variety x at the point p is basically the polynomial ring localized at the maximal ideal, which means that anything outside the maximal ideal can be now inverted, which makes sense because these are nice functions defined around at the point P, the fractions are defined. So, now what we will do is we will define a more I mean we will basically relax some of the axioms in a morphism. So, this will be a new way to compare two varieties. So this arrow we will call the rational map arrow.

The definition will be kind of the opposite that we gave for morphism. So we said that morphism is a map which for any open set v of y gives in the pre-image an open set and so on. Instead now what we will do is we will take an open set u of x. and start the definition from that. So, for varieties x y a rational map φ from x → y is given as take any set u, open set u of x and a map defined on that.

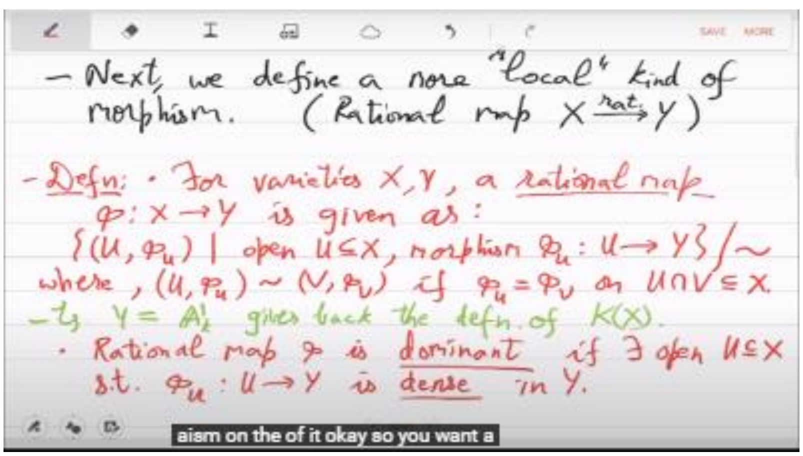
So u is open in x and φ u is a So what we want is we want now to start with the open neighborhood inside x and we just want a morphism from that neighborhood to y. So it is a more local property than what we had in morphisms before because morphism was kind of working with the whole of y instead of this now we have whatever the image of u is under φ u only that part is being used in this and yeah this set may be too big because there are I mean open set u may have a subset which is also open like u prime and so on. So, we want a notion to compare there. So, we mod it out by the equivalence relation with the understanding that if φ u and φ u prime are the same on u and u prime u ∩ u prime then we call them the same. So, if in a small enough open set you have equal morphisms then you call them call these pairs as equal in the set something that we have been doing with the rational functions all the time.

So, the same thing where u φ u is considered equal to or related to formally v φ v if φ u = φ v on the intersection which is again an open set of x this is happening in x. So with that we have these, in fact if we took y to be x what will happen? What are these objects u, φ u going from x to x? So in that case I think you will recover the definition of rational functions if you are mapping points of x to itself. but if you map from x → y where y is some arbitrary variety then you then we call it a. . That is true yes ok, yeah maybe I should write that.

Field meaning that it is a line. A 1. gives back the definition of kx, but now this allows us to work to actually compare two very different varieties, y may not be a line may not be the fine line and we will call this rational map to be dominant. if it kind of covers everything in the image. So rational map φ is called dominant if there exists an open U such that the corresponding map from u to y is dense in y.

So, I have to define this term in case you do not know what dense set means in a variety. I mean we basically want to say that it is almost everything. So, image of φ u inside y is almost everything in y. In particular if the image if φ u u is y then we will say that the map φ is a dominant map. No there is no φ, what is the relationship between φ and φ u? Yeah you can think of it sure it is a restriction yes.

But there is no common yeah so not for all no it is for every u. It is on every u, but the thing is what name will you give φ? It is not a morphism, it may not be a morphism, it is just a map. No but the definition is this, I am looking at all the pairs. Then you just have a morphism on x, you want something that is not a morphism on the whole of x. So, you want a morphism only on an open some open subset u in x.



So, you are saying that it should not be this set. It should be all in opens, some collection of opens. But then just 1 may be enough. Yeah just 1 is enough, but you still have the equivalence relation. So, what should I say sum open u yeah this I have to clarify it next time I see the problem I want this rational map to be much more relaxed than morphisms.

So, I cannot take u to be x, otherwise it will be the same thing as morphism. Yes, so even if this map is not defined on the whole of x. it will be enough it is defined on let us say even a single neighborhood if it is defined then we are fine. And yeah so why does that make sense I will motivate that pretty soon. Essentially there is this idea of dense sets and even if you do not cover the whole variety we will be happy if we are able to cover most of it.

So what is the meaning of most? So let us define that. so W a subset of X is called dense, if the smallest closed set containing W. is everything. So, in other words geometrically, so in the real plane for example, you have this W as an open ball and what is the smallest closed set which contains this ball? So you have to grow it slightly bigger basically you introduce the limit points or the boundary. So this will become slightly this will become slightly bigger will be including this limit points right.

So this will be the closed set which clearly is still small in the Euclidean plane this is still a very small part of the Euclidean plane. Now you will be surprised to know that that does not happen in our case. So in the Zariski topology, if you take an open set and you close it, you get everything. So our open sets are truly large.

Let us prove that. Dominant is just that image is dense. Yeah, φ (u). So it is being applied on y. So maybe I can make it consistent.

Let us call this y, not x. So inside y we call the image dense if the closure of that is everything. So this clearly seems a good thing to understand to compare x and y because image may not be everything but the closure of it is everything so you have a good understanding, you have a good comparison. That was the point of this rational map definition. will be studying dominant rational maps. Let us prove this property first.

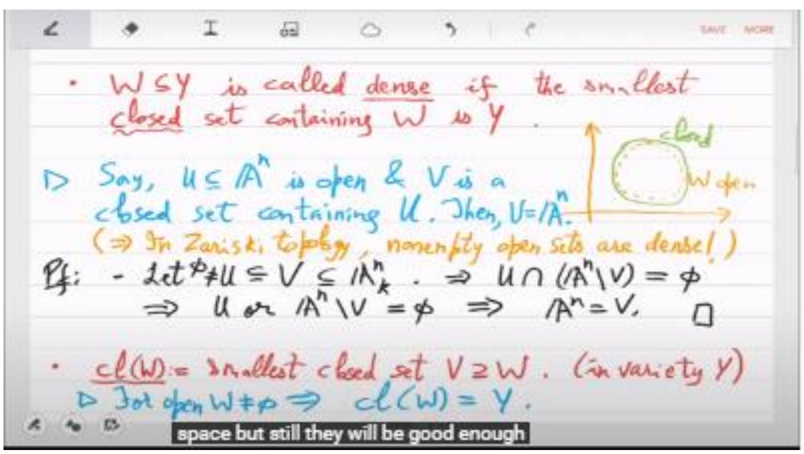
Say U is an open set and V is a closed set. containing U. Then what we will show is that V is everything. How do you prove this? How do you prove that closure of an open set is everything the whole space? In other words, let me interpret this also. So, in Zariski topology, open sets are dense So every open set if you include the limit points you will get everything, that is what we want to show.

So let, so by hypothesis you have U contained in V, U is open, V is closed. and k also we will assume I mean k will always assume to be algebraically closed k is equal to k ‘.

So, from this what you deduce immediately is that u ∩ a to the n minus v is empty right. So, you have an open set u and you have another open set complement of v they do not share a point. can that happen? That cannot happen right we have kind of sketched a proof that intersection of two open sets is always it always has a common intersection.

So, which means that one of these has to be empty. The only possibility is that since u we have picked to be non empty it means that e ^{n} = v is that clear. So, because of this Zariski definition actually gives you a topological space where every non-empty open set the limit points is all that is missing you get the whole space. Yeah, that is an amazing thing of this definition. We could have also defined closure.

Let us do that. So, closure of W is the smallest closed set V that contains W. but we have shown above that if w is not empty then the closure of w is the whole of y. So, in this in our course always closure of an open set will be fill be everything, but still the definition of closure makes sense because I mean you can use it when you did not start with an open set. You may start with an arbitrary subset and you can make it closed. So, in that case you would not get everything, but if you made the mistake of starting with an open set then you will get everything that is what we have shown. So, for arbitrary trouble we might still use this notation to get the closure. Yes. You take what? The interior.



No, interior is the set itself. Yes. Oh, you are defining interior to be that set minus the limit points, sure. Yeah, but then you should not even define it because of this, because this proposition gives you empty. Yeah, so in other words what Rishabh is saying is that in a closed set if you look at the limit points it is everything. So the boundary is the whole set. Yeah, so it is a very weird space to be in which is why all these pictures we draw they are all false because I mean this picture you do not see that right, you do not see the closure to be the whole Euclidean space.

but still they will be good enough to give you actual proofs. Now rational map has a nice computational meaning with this we can define isomorphism of varieties. So varieties are called isomorphic or birationally equivalent that will be the correct term. So, we will call varieties x and y be if there is a rational map both ways. So if there exists a rational map φ x → y which has an inverse rational map $φ^{-1}$ y → x.

no no it is implied right, dominance will be implied. If you have a rational map which has an inverse, so for now I only say this we will say varieties isomorphic or bi rationally equivalent if there is a rational map from x → y and inverse of that y → x. Remember that this the way we had defined rational map it may not hold everywhere, right. You only needed this to hold somewhere on some open patch. So you have to read this with that understanding.

I mean so for all you know φ is not even defined on the domain x, it's only defined on a patch. So let us see an example which shows you the difference why this was needed, why did we not stop at morphism. So let us take the example we have been taking many times before the zero set of $x2^2$ - $x1^3$ and let us take x to be the affine line. So, what is the coordinate ring of this? This also we have seen its x 1/ k x 1 you would go to a finite ring extension, finite degree ring extension which is if I call this ideal i, so mod i. and what is k y big k y that is make everything invertible except 0 right, localize at 0.

So, you will get k is this bracket x 1 function field and mod i which is the same as k xl and also you have introduced now square root of xl. Actually you introduced $\sqrt{xl}^3$which is basically introducing √xl. So, I can just write it like this. This is the function field transcendence degree 1, in fact it is a pure transcendental extension of k. what happens in this on this side, so here e of x is k xl and k of x is just all the functions.

So now what you can see is that, yeah I should have said one more thing, should have given you the algebraic meaning of this rational map definition. So, what you can immediately prove is that if ϕ is a rational map from variety $x \to y$, then you can define now homomorphism or morphism from the field of y to field of x. Maybe I should write it here. So, this is a rational map if and only if k y arrow k x is a homomorphism. Why is that true? So, this is quite straight forward you just look at the definition of rational map and compare it with the definition of rational functions.

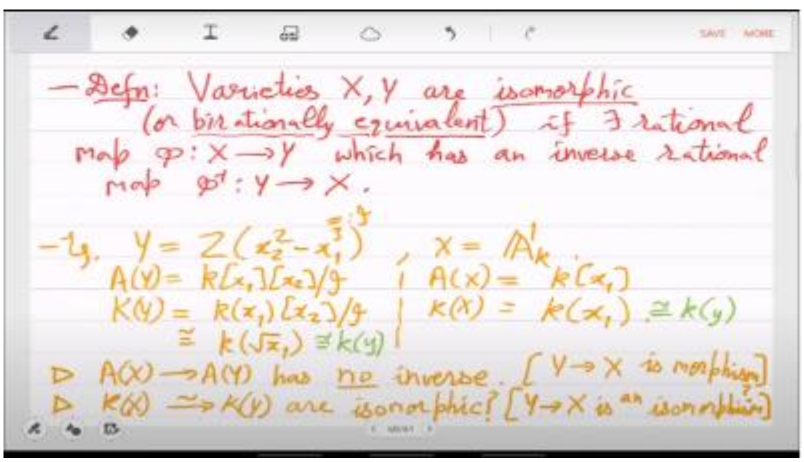
Basically you take any function f which will be like g / h on some open patch living in k y and you can see that through y you can pull it back on x. because the say this f equal to g by h was on v. So, you look at ϕ inverse v maybe not here, here also it should be possible. Yeah you can show this basically you pull back the rational function which was on y pull it back on x and you will get some open patch on which in x on which it will be defined the composition will be defined. So, just like the morphisms from $x \to y$ related to the coordinate ring the rational maps from $x \to y$ they get related to the function fields.

I will come back to this property again maybe give a proof sketch, but let us continue with the example. So, algebraically we can actually study these maps from $x \to y$. So, for that you have to compare Ax with Ay. So, clearly or you can see that Ax and Ay are not isomorphic. So there is a morphism from Ay arrow Ax exists, but it has no converse.

Have I messed up the arrows? yeah I think I am missing the arrows a y to a x yeah so a x arrow a y is the trivial trivial embedding of this polynomial ring in fun variable into two variables, but it has no inverse. So, which means that this y arrow x is a morphism that much you deduce but there is no morphism from $x \to y$ right because if there was a morphism from $x \to y$ then you have to have a homomorphism from a y to a x but that actually there is no reverse map. Right yeah sure. Yeah, for now I am only reversing the arrow, I am not thinking too hard on the geometric picture of this, but ky and kx they are actually isomorphic. So, kx arrow ky is actually there is these are isomorphic, there is an isomorphism.

Yeah they are both isomorphic to the function field kt. No of what? No so what I am saying is just this no this is isomorphic to k let us say y and so is this one is algebraic over the other but aren't they both isomorphic to the same thing. So what this means is that the rational map exists both ways. So y → x is an isomorphism is that wrong. No, but the geometric statement that I am making here is this variety y bitrationally isomorphic to the line.

It is not. Yeah, but it is non-smooth. No, because function fields if they are isomorphic then you should have this bitrational equivalence. Yeah, maybe I should work this out next time if there is some confusion, but these function fields do look isomorphic to me. No, no obviously if t is a formal variable they are isomorphic. No, no so. you just replace, you just associate that function to make √p replaced by t, means take it square.



So you have a function over here. And obviously the reverse error also exists because... Just say you have 2 + √p to be a function, then you can map it to 2 + t. So the map that you are saying is goes to t that is the only possibility right.

It is just the yeah it is simply the association you just associate square root t with t. Yeah, so in general when you will have g √t /

h √t. This will be mapped to gt / ht. No I think what he is saying is that what where does − √t go to? that seems to be creating a problem. Yeah, that is true actually − √t has nowhere to go in the sense that the only place it can be sent to it should be minus t, but is that a problem? It is not a problem yeah.

yeah and in general as I said before you just write in this basis variable square root t and map it to a different basis which is t. So, this seems to be the isomorphism. So, which is why I feel now it is a bit strange, the variety y is actually bitrationally isomorphic to the affine line, which you could not have deduced by looking at coordinate ring, but by looking at function field it is possible. Yeah, so this is a more relaxed way to compare two varieties. It gives you something which is actually exactly algebraic but geometrically it's a bit strange.

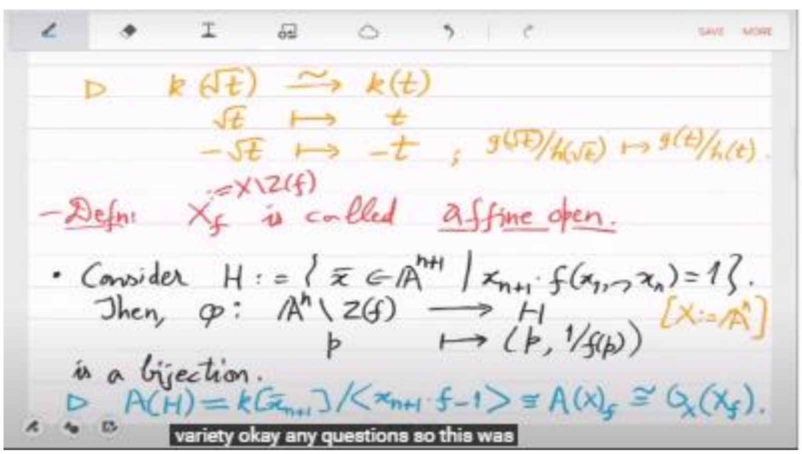
Yeah, maybe I have to come back to this next time let me move forward unless you see a problem by the end of the class. Okay, what I want to define next is what is called an affine open. So, we have seen this object xf the distinguished open set we will also call it now affine open. So, we know that this is an open set because it was defined to be x = 0(f).

f is a polynomial single polynomial right. So, it is an open set we attach this word affine to signify that it can be seen as a 0 set itself. So let me try to do that, yeah I want to give an isomorphism from this to a to a 0 set. So, let us consider the following. Consider each to be the set of points in a bigger space with n + 1 affine n + 1 space.

like this e raised affine n space − Z F, I want to map this to H. as follows. So, any point I will map it to. So, x1 to xn coordinates will take the point p. I need one more coordinate which is xn plus 1th which I will just set to be 1 / fp. Okay so a point which is not a 0 (f), we can actually then invert 1 / fp.

So we are mapping n coordinates to n + 1 coordinates like this and you can show that this map is actually bijective, right. Any point in any set of maps to H. it is injective and it is also surjective. So, this is a bijection.

So, in this sense Xf can actually be seen as roots of some system. So, it is a special, so these distinctive open space are actually special they are open, but if you expand your ambient space then you can see them as being defined by exact equations by equations exactly. So, we will also call xf because of this affine open because it is like a variety. although the confusion here is that it is not a variety in the affine n space, it was not close there, there it was only quasi affine, but in a bigger space it is affine. What is its coordinate ring? So, that you can see by the definition itself that the coordinate ring of H is simply the n + 1 variate polynomial ring modulo the defining system. So, that is the coordinate ring which is nothing but actually localizing the original polynomial ring by f.



You have just introduced 1 / f. So, this is actually isomorphic to the original polynomial ring of wherever you came from if it was x then x localized by f which is also isomorphic to Ox (xf). So, these three things are the same. in the original variety X let us say embedded in the affine n space Ox (Xf) is the was the same as Ax localized we have seen this before and now the third thing we are seeing is this bigger affine space and H is the affine variety there. and I guess I have set in this example x to be a to the n probably let us write that not this a to the n. It is an example for the whole affine space, but you can do the same thing for any variety So, this was just the definition of the same thing x sub f just viewed differently.

We have seen before that any variety x has a base consisting of these affine opens. that we have seen before and let us go through that property I sketched. So, varieties x, y are isomorphic if and only function fields are isomorphic let us prove that. where isomorphism remember is defined by a bi-rational equivalence. So, this is the definition of bi-rational b b e q it is it is not by morphism, but by rational maps.

If we take the definition to be just morphism I mean x → y you have morphism and you have inverse morphism from y → x then this k x is isomorphic to k y then it follows Yeah yeah sure sure, but many more things will follow many more things will follow. Morphism is far more powerful. I mean in particular what will follow what will it will be equivalent to Ax being isomorphic to Ay. It is a property at the level of coordinate rings. So, obviously if coordinate rings are isomorphic then their field of fractions will also be isomorphic, but you do not want to work with the coordinate ring somehow you want it is easier algorithmically also to work with the function field.

Working with rings is a bit harder or working with fields is a bit easier. So, let φ be the isomorphism be the birational map its this and its inverse both are rational And yeah the idea is the same that I mentioned before that you pick a dominant rational map u φ u and let and pick a rational function on the RHS which is ky. Now, what you want is the reverse arrow, right. So, going from x → y, you want to go from ky to kx.

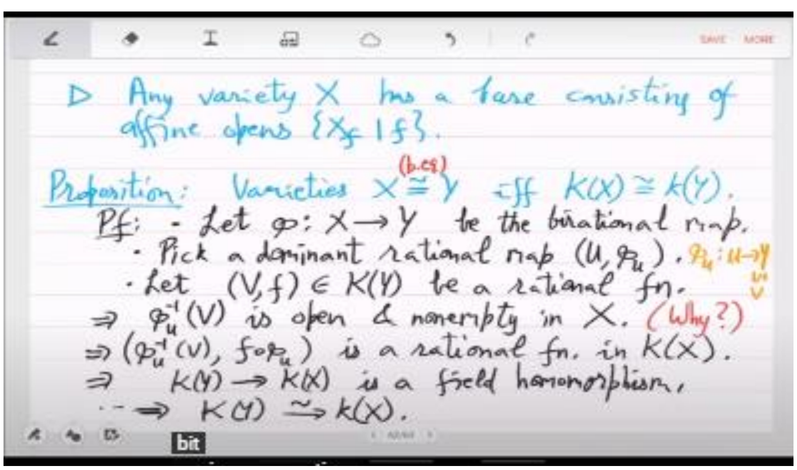
So, you want to pull this back. You want to pull back Vf on x now. So, yeah what is φ u $^{-1}$v? Well, so first thing I need is why should this v be in the image of φ u? Is that clear? No actually you do not need the whole thing to be in the image but just the pre-image I want to claim that this is open. Is it clear why this is true? Is open and non-empty in X. Why is that? This is the thing which needed some proof actually. In the case of morphism this was already an axiom, but in the case of rational map it is not.

So, you have to prove that this preimage on any open set is non-empty and open. How do you prove this? From u to y that is all you are given. and it is also dominant.

So, we have this, this is what we started with and now v is open in y, v intersection what? No, you cannot take intersection. φ u u ∩ Ic, did we know that ϕ u u is open.

Okay fine, yes so finish that part this is needed. Yeah once you have this you are almost done because now from the RHS you have moved to LHS on a over x. So you have this open set in the pre-image it is non-empty and on that you can pull back f. So the pull back of f will just be composition like we sketched in the definition of morphism so φ u $^{-1}$ v , f (φ u), this is a rational function on x, so it is in kx. So, what you are essentially doing is it is defined on x because you take a point in x p apply φ u which will take you to y and in y you apply f and the answer you will get will be in the same field k. So, the you get a reverse arrow, but it is basically this composition which is doing all the work and this is then, this is how you get a morphism from key y → key x.

It is a field homomorphism. Okay so you have this field homomorphism and you have the converse of this so you have a field isomorphism. So by symmetry you get ky is actually isomorphic to kx. So that was the actually that is the most interesting case you go from here you go from variety to the function field. The converse of this is a bit easier.



Any questions here? I hope you can fill in that detail. So, let us go to the converse. So, let us take a field isomorphism ky to ky →kx. Now what this means is that remember k y is actually the field of fractions of a y which was the coordinate ring right. So, in particular the maximal ideals of the coordinate ring of a y respectively a x they are in bijection. So, this means that psi identifies the max ideals of Ay with those of Ax.

So, at the level of the coordinate ring you have an association between the max ideals. Now, what do the max ideals of the coordinate ring signify? max ideal M of a y signifies a point of on y. So, χ identifies points in y with those in x. So, this was easy the when once you have an isomorphism at the level of the field you get an association between max ideals and which means that you actually have association at the level of points. So, y → x you have this and I think you do not need isomorphism for homomorphism you will get similar thing, but the identification will only be one sided. which is why I said that if you have a rational map from x → y you get a field homomorphism in the reverse direction and when it is an isomorphism then you have algebra isomorphism.

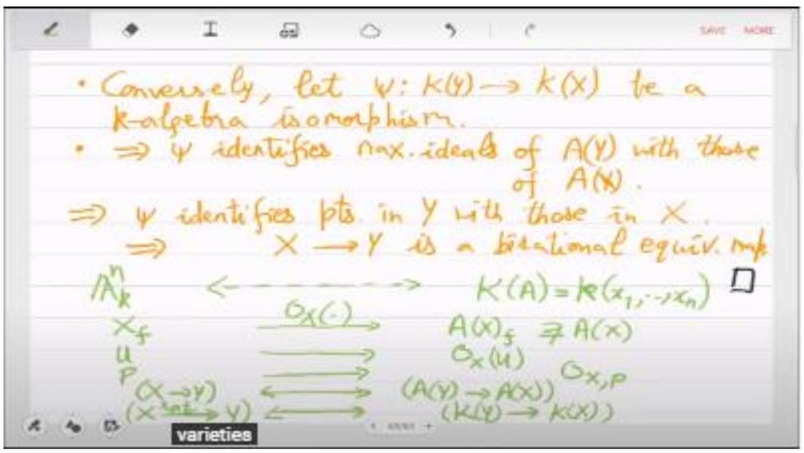
So this means that x → y is a birational map, so birational equivalence Is that clear? So, we have defined a lot of things today, but this proof ideas are all the same as we have seen before. None of these proofs are hard. So, what we should do is. Yeah let us take stock of the situation in terms of the geometry algebra correspondence, so where are we in that picture. So you have the affine n space and you have the function field now, I am sorry, big ka and this is small k.

So, in the geometry we think of the affine n space in the algebra we think of the either the polynomial ring in n variables or even better is the field of fractions which is the function field in n variables and here are the correspondences that we have shown or the at least the associations. So, the new things that we have shown are xf this distinguished open set or affine open where it has it map in the algebra it maps to. So, remember this will not map to polynomials it will actually map to fractions where in the denominator you are allowed to have anything except f right.

So, that was axf. the localized. So, this is a subset of all the fractions, but it is not it is much bigger than Ax. so we have gone beyond the polynomial ring. So, this is the ox functor, ox functor will take you there. For an open set u, ox functor will take you to ox u.

So, these are the functions defined, the regular functions defined on u. for a point you will go where? So, for a point you will go to the germs right that was $O_X(P)$. So, you go to the germs and what else is there? You can also actually look at arrows here. So a morphism, now you take two varieties in the affine n space x and y. A morphism corresponds to what? Morphism between the coordinate rings, reversed arrow.

So that is the association and it is both ways. and finally the rational map. So the rational map corresponds to field homomorphism, reversed arrow. Is that clear? So these are the associations between geometry algebra that we have seen in addition to the old picture. So, one nice one kind of powerful property that we will now get because of this concept bitrational equivalence or isomorphism of varieties is that every variety is in isomorphism to a single polynomial equation. Even if you are given the input it is through your algorithm the variety is given via a system of polynomials all those polynomials can be converted to a single polynomial. such that this hyper surface that you have and the variety you had are isomorphic which is quite a powerful result and that proof is really comes from algebra.



So we can show that that any affine variety of dimension R is isomorphic in the sense of bitrational map to a hyper surface. hyper surface in one more dimension, so ar + 1. Now, this isomorphism will not be, I mean as you have seen in the definition it will not be really point by point, it is not that the a fine variety and the hyper surface every solution corresponds to each other it will only be up to this more general field sense. So, field of fractions will be isomorphic right. So, at the level of point some points there may be a mismatch right because this notion of isomorphism of varieties that we have picked it is a bit weak, but for all the purposes of the course and also algorithms it is a useful one.

you do not lose too much. So how do you prove this? So let us try that. We will just see the basic idea of the proof. So consider affine variety X for a prime ideal I x of the polynomial ring which is in our variables. This I x will be prime because variety corresponds to prime ideals that we had seen before. So, what we defined was mainly this algebraic construction inspired by geometry localization. So, you can localize by a multiplicative set which means invert introduce fractions in the set big T and even more special you can take prime ideal P and then the big set T will be complement of this which is A - P that is a multiplicative set you invert all those elements that is called localization of a ring at a prime ideal.

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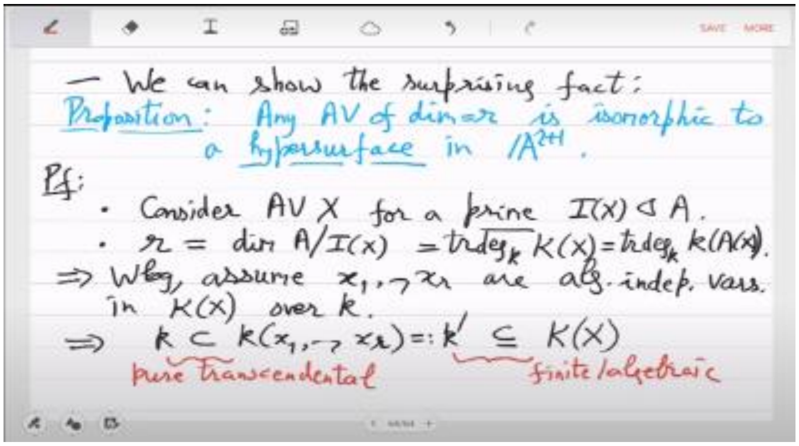
Then there are projective versions of this and we define this O x functor. So, this is specially good for morphisms. So, when you want to compare variety x with variety y, then instead you should compare O y with O x and the arrow will be reversed. So this is a contravariant functor, we will be using this all the time. We define distinguished open set x sub f for a polynomial, it is basically the zeros of f complement in the variety x and this can be used to cover any open set that we had seen.

and we also saw algebraic version of it. So, O x functor on this distinguished open set basically localizes your polynomial ring. It localizes it by multiplicative set powers of f. So, 1 / f gets introduced essentially in the ring. finally, the germs are germs of over the variety x at the point p is basically the polynomial ring localized at the maximal ideal, which means that anything outside the maximal ideal can be now inverted, which makes sense because these are nice functions defined around at the point p the fractions are defined. Okay so now what we will do is we will define a more I mean we will basically relax some of the axioms in a morphism.

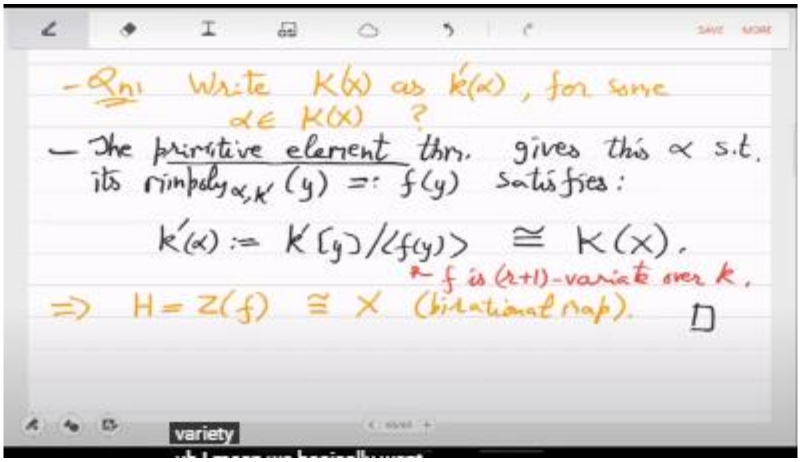
So this will be a new way to compare two varieties. So this arrow we will call the rational map arrow. The definition will be kind of the opposite that we gave for morphism. So, we said that morphism is a map which for any open set v of y gives in the pre-image an open set and so on. Instead now what we will do is we will take an open set u of x and start the definition from that. So, for varieties x y a rational map φ from x → y is given as take any set u open set u of x and a map defined on that.



so u is open in x and φ u is a morphism. So, what we want is we want now to start with the open neighborhood in inside x and we just want a morphism from that neighborhood to y. So, it is a more local property than what we had in morphisms before because morphism was kind of working with the whole of y instead of this now we have whatever the image of u is under φ u only that part is being used in this. yeah this set may be too big because there are I mean open set U may have a subset which is also open like U prime and so on. So, we want a notion to compare there. So, we mod it out by the equivalence relation with the understanding that if φ U and φ U prime are the same on U and U prime U intersection U prime.

then we call them the same. So, if in a small enough open set you have equal morphisms then you call them call these pairs as equal in the set something that we have been doing with rational functions all the time. So, the same thing where u φ u is considered equal to or related to formally v φ v if φ u = φ v on the intersection which is again an open set of x, this is happening in x. So, with that we have these, in fact if we took y to be x what will happen? what are these objects u comma φ u going from x to x. So, in that case I think you will recover the definition of rational functions, if you are mapping points of x to itself, but if you map from x → y where y is some arbitrary variety then you then we call it a.

That is true yes. Yeah, maybe I should write that. Field meaning that it's a line. A1. A1. Gives back the definition of. kx but now this allows us to work to actually compare two very different varieties y may not be a line may not be the fine line. And we will call this rational map to be dominant if it kind of covers everything in the image So rational map φ is called dominant if there exist an open U such that the corresponding map from U to Y. is dense in y. So, I have to define this term in case you do not know what dense set means in a variety.



I mean we basically want to say that it is almost everything. So, image of φ u inside y is almost everything in y. In particular if the image if φ u u is y then we will say that the map φ is a dominant No, there is no φ, what is the relationship between φ and φ u, yeah you can think of it sure it is a restriction, yes. but there is no common yeah so not for all no no it is for every u it is on every u but the thing is what name will you give φ it's not a morphism it may not be a morphism it's just a map Yes. Yes. No, but the definition is this, I am looking at all the pairs.

So, you want a morphism only on an open some open subset u in x. So, you are saying that it should not be this set. It should be all in the opens. Some collection of open u over where it is a morphism. But then just one may be enough. Yeah, just one is enough, but you still have the equivalence relation. So what should I say sum open U, yeah this I have to clarify it next time I see the problem, yeah I want this rational map to be much more relaxed than morphisms.

So I cannot take U to be X otherwise it will be the same thing as morphism. Yes so even if this map is not defined on the whole of x it will be enough it is defined on let us say even a single neighborhood if it is defined then we are fine and yeah so why does that make sense I will motivate that pretty soon. Essentially there is this idea of densities and even if we do not cover the whole variety we will be happy if we are able to cover most of it. So, what is the meaning of most? So, let us define that. So, W a subset of X is called dense if the smallest closed set containing W is everything. So, in other words geometrically So, in the real plane for example,

- Idea of P.E.T : Eg. $k \subset k(\alpha_1, \alpha_2) =: K$
  $k(\alpha) \overset{?}{=}$
- Eg. $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$
- So, use $c_1\alpha_1 + c_2\alpha_2$, for "random" $c_1, c_2 \in k$.
  $\Rightarrow k(\alpha_1, \alpha_2) = k(c_1\alpha_1 + c_2\alpha_2)$. [Why?]

Ex: Prove the existence of c_1, c_2 by considering a linear system.