Computational Arithmetic - Geometry for Algebraic Curves

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Week - 03

Lecture - 05

Morphisms and Rational Functions

So where we are in this big theory is currently we are at projective varieties. So we have shown these associations, projective n space over an algebraically closed field k, small k. So you can look at projective varieties Y, you can look at closed sets. and you can look at a point, so the associations are that in the algebraic word which is basically coming from the polynomial ring in n plus 1 variables, so x0 is the new variable, so x0, x1, x2, ... xn, so those are the, I mean this basically will be the location from where the functions will come from, this is the most basic ring So, variety Y actually I think I meant subset Y. So, subset Y you can apply the I functor and you can get an ideal of that subset and on this side if you started with a subset of polynomials T then you can apply the Z functor and you can get the zeros of that. that is called a closed set.

On this side, this ideal is a homogeneous ideal for the projective variety. And one to one correspondences are between closed subset wise and radical homogeneous ideal. and then it becomes more and more precise so project for projective variety you will have a homogeneous prime ideal for a point you will have a maximal homogeneous ideal is that clear and the projective space we will study via affine varieties via the affine space because there is this natural thing the proposition that projective n space has an open cover via n + 1 many affine n spaces which is just the obvious thing you have n + 1 coordinates a0 a1 ... an you set a0 to be ~ 0 or a1 to be ~ 0 or a2 to be ~ 0 and so on those are the UIs and then you can see the map there is a map from Ui to a ^{n} and then you can do that for all the eyes. So, you actually get a cover each of which is affine, each part of it is affine and the map is to the projective end space.

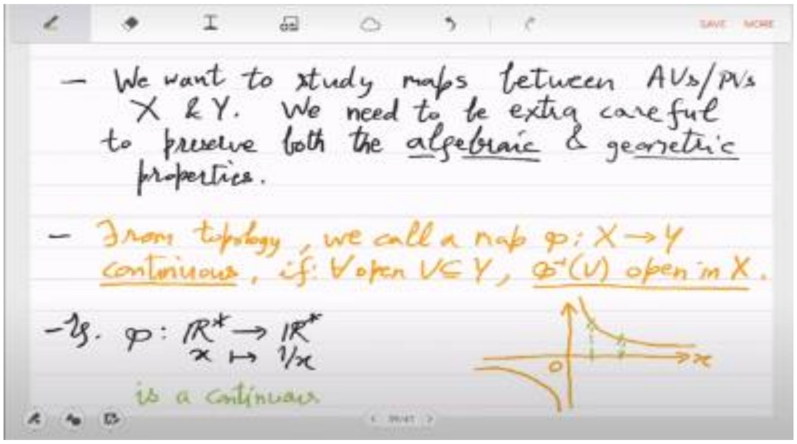
So, this is a I mean conceptually it is a new object because it is very different from whatever you have seen before it. So, it is not really, you cannot realize that space by any single affine variety that actually you can take it as an exercise that this is not an affine variety, but it is a union of affine varieties. and this property yeah this property of projective and space just naturally extends to all quasi projective varieties because you can if you have a general y then you just intersect y with ui to get the affine variety any questions okay so now we will compare varieties because we now have so many types we want a way compare them basically map one ^{n}ext and we'll do it in more and more precise ways. So first thing we will start with is an analog from complex analysis or real analysis. So we want to study between affine varieties X and Y which can also be projective.

So we need to be extra careful about continuity preserving that. Basically you have this Zariski topology on X and Y, so there is a notion of closed sets, open sets, you want to preserve them. So when that happens we call the map continuous. So at this point it is not clear what maps will preserve it. So we will make it precise and then give a definition.

preserve both the algebraic and geometric or topological properties. So algebraic property is that we want to stay close to the polynomial ring, our functions should emerge from there and geometric means that we should stay close, we should preserve the neighborhoods or the open neighborhood of a point. So, this is kind of a diversion. So, from topology or analysis we call a map φ continuous if for every open subset of Y. So, we are looking at the range.

So, there any open neighborhood what should happen with respect to φ? Look at the pre-image that should be open. So, φ^{-1} V is open in X. So, you want this property every open in the pre-image should be open. then we call a map continuous. Obviously this is quite intuitive and you can draw pictures when you are doing real analysis which is you can do the following.

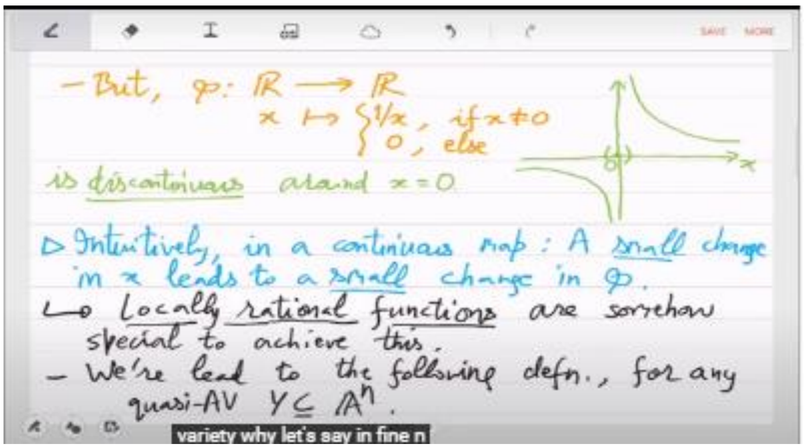
So look at map from reals to reals. removing 0, so that I can send $x \to 1 / x$ and you can show that this map is continuous in the real world because of this picture. So, if you plot it, so this is the x axis, so $1 / x \to 0$ is very large. and then at infinity it is very small so it's like this and the other part is the negative side so on negative it is in this quadrant so we have the whole x axis except the point 0 and you can see that if you pick any open set like from here to here, you can see obviously that in the domain from here to here, it is continuous. So, continuous intuitively means that when you draw it, you do not have to lift your pen.



So, you can see that any part of this curve, the pre-image is continuous. So we deduced that this is continuous and opposite example is if you looked at the whole real line you sent x to 1 / x when it is defined else you send it to 0. so what will be the picture now so now the same picture except that you will be forced to lift your pen because 0 here is also included right so the real line except 0 is the same picture but at 0 you have included it 0,0 is there in the diagram and you can see that what is the place where you have discontinuity in this map at 0. So, this is discontinuous around 0 because if you try to draw the part around 0 then you have to lift it from 0 to ∞. so that is the thing you want to avoid you don't want to study such functions at least not in the beginning of the course so we have to somehow avoid this in the definition when we give it algebraically so intuitively in a continuous map you do not have jumps, so a small change in x leads to a small change in Φ.

This is the property geometrically we want. but we do not know how to define any of this over finite fields. So, what we will do is that we will just lift good things from actual analysis or geometry which is polynomials and ratio of polynomials and we will call them forcefully continuous and we will show that it actually gels well with the definition of Zariski topology. So, in that word it is actually a continuous map. So, we will study locally rational maps.

So, these are our best guide in this business, rational functions. So, basically locally the function should look like G over H where G and H are polynomials. They are somehow special to achieve this property. So, we will not worry about the picture, we will just then lift this bottom line. that our maps should be rational functions if not globally at least on some neighborhood.



So, we are led to the following definition. for any quasi affine variety, y let us say in affine n space, subset of affine n space. So function f from y to the base field k is regular at p. P is a point on this quasi affine variety, if there exists some open neighborhood, so NBD will be the short form from the neighborhood, if there is an open neighborhood of this point U such that F = G by H on U for some G and H in the coordinate ring of Y. remember the coordinate ring was essentially these polynomials which are defined on Y in the affine n space full space it is all the polynomials otherwise it is polynomials modulo the ideal defining ideal.

So, G and H should come from there and their ratio is what is called also a rational function, but we will have a special meaning for that term here. So, we are only calling this F a regular function which is regular at a point, given point P with respect to some open neighborhood, it should not, it may not be true for all u's, it is true only for some u and we call a function regular on the whole quasi AV. So, f is regular on y. So we call it regular on Y if it is regular for each point. Is that clear? So we just wanted G by H but we define it point by point and instead of just at a point because a point you know is a closed subset.

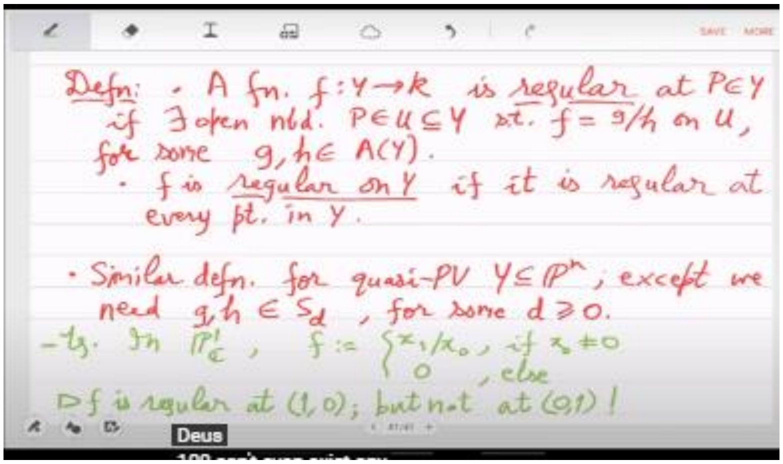
it is not really a neighborhood, so we actually pick a neighborhood around it. So, there are the data here is a point, a neighborhood and gh and similar definition for quasi-projective varieties. in the projective affine projective n space except we need GH to be what? Right, so we need GH to be in the same grade which is SD homogeneous degree D polynomial for some D. Why is that? Why do you need, so G is in SD but why do you need H also in SD? What happens if you take H to be S of degree ? Exactly, so in the space P^n the definition was that we only look at a point up to multiple. So if there is a degree difference between g and h then you cannot make the definition correct it would not be a function from these points up to this equivalence class or equivalence relation.

So 1 / x for example has no meaning because of 1 / x0 has no meaning because x0 itself is not. uniquely defined x0 is defined only up to multiples. So, you cannot say 1 / x0, but you can say x1 / x0 because that is that is a unique thing. So, in the projective line if you look at x1 / x0. So obviously then we have to say x0 is not 0, if it is 0 then we can just choose or fix it to be 0.

So this is certainly a map but does it have all these properties which we want. So this is show that this is regular at . So, at 1, 0 its value is what? It will come from the first part because x0 is not 0. So, its value will be 0, but more importantly in the definition you have to pick an open neighborhood of the point U. It will be what? You can simply pick it to be the first open patch which is this affine patch x0 ≠ 0.

Just take that, take this U0. u0 contains this point and you can see that on u0 f looks just like x1 / x0. So, everything in that definition is satisfied. So, it is a regular function at this point. What about the point ? So, at the second definition will apply.

So, its value will be 0. So, I mean it is well defined, but is it a regular function? Why not? How do you show this? So, the problem is that you will not be able to find the open neighborhood U which contains 0, 1 and where you can express f as a fraction of two polynomials. Is that clear? Well, no that is not the obstruction. No, you can also take, so constants are allowed, constants are valid functions. Yeah, you can allow degree − 1 also, that is not the problem. The problem is that there is no open neighborhood where the function is 0 everywhere.



So, whatever open neighborhood you will take of 0 , 1, it will have something which is coming from the first part of the definition. So, in some part it will be like x_1 / x_0, in some part it will be 0. So, there is no common function. you cannot find a common G / H that's the problem. So its definition is I mean there is this discontinuity essentially that in most of the part it is x_1 / x_0 whatever open patch you take most part it is x_1 / x_0 and then there is just one position where it's 0 so you can't have a common G / H.

So you can check this it's easy to check that's the obstruction in that. Let us look at 1 over $x_1^2 + 1$ on the affine line, let us say the real affine line, so is this regular, well everywhere on the affine real line this $x_1^2 + 1$ will be non-zero and it is also affine, so this is a regular function. but we prefer to see things on the affine complex line, what about that? So, here there is a value x_1 at which $x_1^2 + 1$ will vanish, so it is not regular. So, the thing about this definition is that it is highly algebraic, so it is very sensitive to fields. but since we always will take an algebraically closed field what you can learn from this example is that there should not be any denominator because if there is a denominator each then there will be some zero in the algebraically closed field and then it won't even be defined.

So if you want a regular function throughout the space then it has to be a polynomial. Is that clear? So functions regular on the whole affine space where K is K ‘, K is algebraically closed. So these are exactly the polynomial ring. which is A, x1 to xn variables. Okay, so this is one simple fact from the definition, you should prove this if interested.

Yeah, I think I remarked this before that we have now many kinds of varieties, FI, so or quasi AV, projective variety or quasi PV. So, these will just be referred as varieties We will subdue the special type out of these four types because usually I mean all the definitions and in the examples the context will be clear, in the definition whatever we will define will be the core thing, it can be simply I mean slightly changed if you want to go from affine to projective or projective to affine. So we will just talk about variety now. Right, so regular functions we have defined, but we still have to give a way to compare two different varieties, the concept of morphism. So, what should it satisfy? Sir, I have a question.

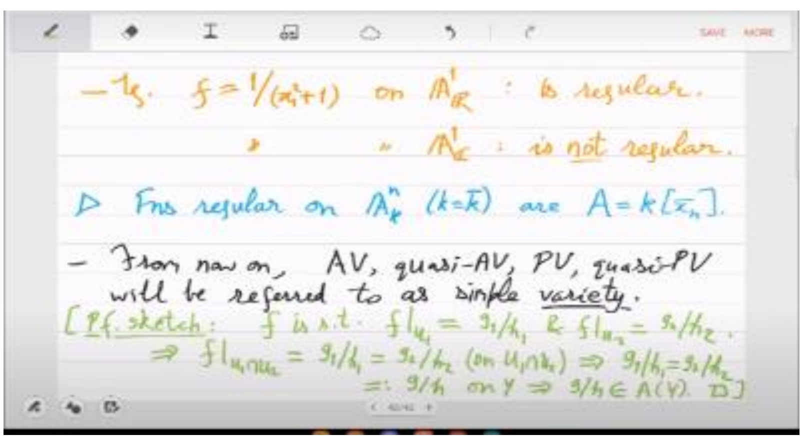
Yes. I have shown that functions regular or ANTR are only the polynomials, but isn't it possible that piecewise polynomials are there, that can also be regular. That's right, but then using industrial enzymes you can prove that this would happen. Okay, I see. You are saying that on different points this g / h value, the g and h are changing according to the point. Yeah, maybe I think we can prove that in the next class.

We will use Hilbert's Null Challenge Arts for that. So one of the key things in this proof in which I am skipping for now is that actually it can be done right here. So suppose you have a function which. on an open patch U1 looks like G1 / H1 and on a different patch it looks like G2 / H2. So, what I will show is that actually the GIs are equal and HIs are equal.

So, ultimately you can combine them into a single representation. Why is that? So this actually means that F on U1 ∩ U2 is what? It's both. And now since U1 and U2 are both open sets, their intersection is also open. Now open sets have the property that they are very large. because our field is algebraically closed right.

So, U1 ∩ U2 again has infinitely many points. So, you have these four polynomials such that G1 H2 = G2 H1 over infinitely many I mean over a open set from that you can deduce that they are actually. So, let me explicitly say here that this is equalities on intersection. but from that you can deduce an absolute equality on Y, on the whole variety Y. So, if you have an equality of fractions or rational functions on an open set, you can actually deduce it that they are equal.

almost in the polynomial ring you have this equality. So now you have a single G by H representation and then this H will have some zero somewhere and that will be a contradiction. So what this, I mean contradiction or you can just deduce that well it's, G1 H1 = G2 H2 on Y which will, yeah my space has reduced, so that is equal to some G, we can just call it G / H which will imply that actually H is a constant when it is in reduced form. G / H has to be in the polynomial ring, that is what you will get. G / H then ultimately has to be a polynomial because if it is not a polynomial, if there is a genuine H sitting there then it has a 0, that's a contradiction.



Yes, so this is a very nice example of how these open sets, how the Zariski topology will give you everything you want pictorially also. okay but to actually prove it you have to go through algebra which most of the time I will be skipping unless you interrupt me then you will see an algebraic proof but like this thing assumes that the intersection won't be null set no no these are open sets so k is algebraically say k is the complex field so open sets are way too big so when you intersect two of them it's again open It is open but like it might be empty as well, like the points are far apart.

Yeah, but... . Another thing you have to prove that it cannot be empty, it's actually very big. Or we can say that the entire space cannot be broken into like a union Yeah, it's also a variety. Yeah, exactly. It's a variety so it's irreducible.

The open sets are dense. Yeah, yeah. So, yeah, so the irreducibility is also important. It's a variety. Okay. So that is the proof sketch for the blue part.

I hope now you appreciate regular functions better. So, this is the point where we move to morphisms. So, the thing that we want, so a good map φ between varieties which may be completely different. X and Y are completely independent. We want to compare it through a map. So what should the map satisfy? Well first of all it should be continuous.

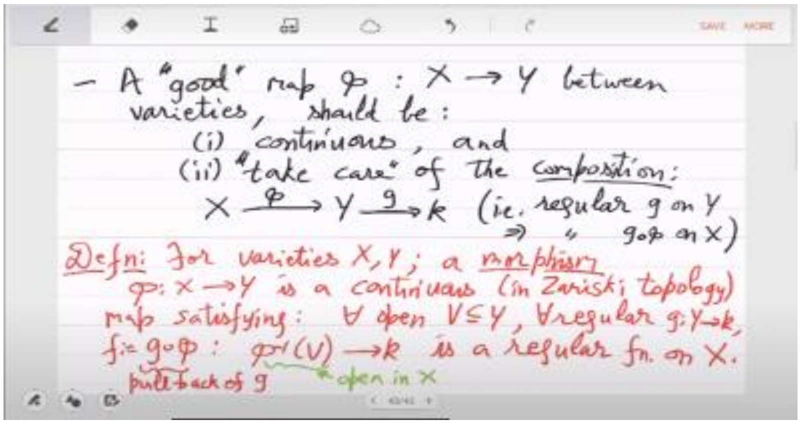
So it should carry over the topological property well. which means that the opens any open set V (Y) pre image should be open in X that is clear you want that property but you now also want the regular functions on top of X and the regular functions on top of Y they should somehow be connected through Y. So it should take care of the following composition let us say. So, see φ is taking x to y the points, but if you look at a function, regular function say on y that takes it to k. So, in other words there is a path to go from x to base field k. So what you want is that if g is a regular function then this composed map should also be a regular function on x.

So that is regular g on y should give you a regular g composed with φ on x. So this is a very important functor that you are now learning. that basically the geometry of x and y should be preserved under φ but also this thing which sits above y which is the ring of regular functions under φ it should map to something good to a subring of regular functions of x. G is any regular function or some regular function? No, no, no it's any because I mean any g takes elements of y to k. So, φ I mean if you want to define a good map φ to compare two varieties then it should be true for all φ's and for all g's.

So, these I hope you appreciate that these two properties are very different. they are not quite related because the first property is only talking about subsets and the second property is only talking about regular function ring okay so these two are needed together one will not imply another second so okay so with that philosophy your definition is ready how to compare two varieties So for varieties XY, a morphism φ from X → Y is a continuous map which is always in Zariski topology. Now we don't know any other topology in this course. is a continuous map satisfying the following conditions, so for all open sets v of y for all regular functions g Y → K, G composed with φ which we call map F which takes, so G is regular on the whole of Y which means that it is also regular on V and the pre-image of V is an open set So let us look at that, φ^{-1} v ← k, this is a regular function on x. Is that clear? So morphism is just a map between two varieties which preserves all the data that you can think of.

It preserves the topological information, it preserves regular functions and whatever is implied. No, we do not assume anything, because I mean other things you have to deduce. yeah I think you can map fi into projective and so on why not yeah you don't have to tell that it's too soon to make those big claims okay Yes, so this is also called a pullback. This thing that we did, this is called pullback of g. Because y is kind of forward, x is backward and g was defined on y, so you are pulling it back to x and making it a function there.

So this is the definition of pullback and just a note that this thing is, is open in X. So both the topological and the functional information is contained here in the definition of morphism and second thing is for variety Y, O Y is the ring of regular functions on Y. So I have already said that it is a ring but you can also deduce that from the definition of regular functions because it was defined to be I mean at a point we had picked the open neighborhood U and there it was written as G / H. So, if you have G1 / H1 and G2 / H2 then you can also add the two. So, clearly addition, multiplication all these things are defined constants are there because constants 0, 1 are treated as constant functions.



They are also regular at least on affine or projective variety. So this is a ring and we denote it by O (Y), this fancy O. Yeah, so if you are aware of the term sheaf, then this is the first example of sheaf. Okay, but you won't need that. But these are, so sheaf is the algebraic object which lives over the geometry.

So we are defining the simplest kind of sheaf here. So what is O of affine line? So regular functions on the line are, and since C is an algebraically closed field, in fact I should, it will be true for any algebraically closed field. So these are only polynomials and this is affine line so one variate. So this is just Cx. which is the coordinate ring of the line. So, E of that and this property is true for any algebraically closed field k.

So, second example is what Madhavan was saying. What do you think is the ring of regular functions of the projective line? what is this, so G by, you want these maps which at every point on the projective line should look like G / H, now I gave a proof sketch before that if they look in every open patch they are like G i / H i then there is a common G / H. From the common G / H now you also recall that in the projective case, actually we had deduced in the affine case that G / H then will be a polynomial. In this case what is the polynomial which is a ratio of 2 degree D homogeneous polynomials. just a constant right.

So, this is just the constant which is all the complex numbers. So, on the affine space you get the coordinate ring, on the projective n space you just get constants. Is that clear? Yeah, you may ask that if it is just if O is just constants then why have we defined it in such a complicated way and given it a name. The reason is that this O operator we can apply also on smaller things. So, we can look at just a part of the projective line and look at O of that, that would not be constant.

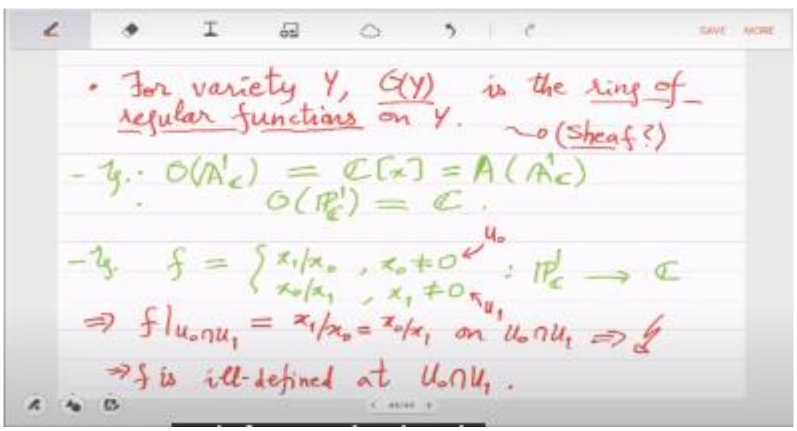
Okay, it's constant when you look at the global section. Locally it will not be constant. So we'll see that as we develop this more. Yeah, maybe I should give another example. just to test you so consider the following function okay it's a it's a map it's defined on the projective line so it is supposed to take the projective line to the base field which in this case we are taking to be complex. So, when x0 is not 0 then it is x1 / x0, when x1 ≠ 0 or in the case when x0 = 0 then it is x0 / x1.

So, we have defined this on these two open patches of the projective line. The question is, is this a regular function on P1C? because it seems that anywhere you see it is a ratio of two polynomials right that seems consistent with the definition of regular functions do you agree that this should also be in O (PC1) P1C what do you agree with this what is the problem in this So the test was in which you failed, so this is the definition of f on u0 and this is the definition of f on u1. So now which was actually also in the proof sketch that then for f to be a regular function it should have a nice representation on this also right, U0 intersection U1. So where both X0 and X1 are non-zero, so both the representations have to be equal. Now which clearly is not the case because you can just take X0 = 1 and X1 to be So you are getting 2 by 1 = 1 over 2.

So which means that that's a contradiction. So what is being contradicted here is that this is not a regular function. As a map it is well defined that's the thing. So I am just splitting hairs here. So there is a big difference between a map and a function and a regular function.

These three things are different. Yeah, but you can, okay. Sorry. Yeah, so it is, so what it is not is, it isn't even a function then. But I think it can still be called a map. No, because you can say that I decompose my line into two parts u0 and u1 and then I only look at these values for u0, these are the value, oh, but at the intersection there is a, yeah, okay, I see.

So this is, yeah. this the contradiction is that it is not a well-defined map. So, F is ill-defined at U0 ∩ U1. So, in particular it cannot be a function or a regular function. So, this is the content of that proof sketch I gave.



You may think that by giving different G / H in different local patches the definition will be satisfied but then the problem will be that when you look at the intersection of these patches the G / H will not match. So the definition of regular function is loaded with many invisible things that you don't see.

It takes care of many many issues and that thing is now even more serious in morphism. So, morphisms will be very special comparisons. But here if it had, if it was a function then you see it could be regular or if here it was not even a function so everything. Yeah, no, but the proof sketch already had these things, right. The proof sketch says that there has to be a common G / H and then the H also has to be 1 essentially.

because if H is a genuine polynomial then there will be roots. So, this if you prove this then I think you will understand all of this. Just prove that the regular functions on the projective line over an algebraically closed field is trivial, only constants are there. That contains all these ideas. But these are not the only kind of functions, there are many others. Like in the general theory we will be interested in functions which are not defined everywhere.

They are defined only locally. So in one local neighborhood or in one open neighborhood the function is defined and somewhere else it is not defined. So you also want to look at the set of locally defined functions and that needs another definition. So there are many other functions outside O that are regular on some open patch U. proper open patch U (Y). So basically this getting this common G / H is a problem, it's too strict, so we will now weaken it to only care about some open patch and not all, not everywhere.

So now on we will use u , f to denote a regular function f. So, denote a function f that is regular on open u. So, it is only defined there nowhere else. So, on an open patch only it is defined and there it is a regular function. So, it has this representation g by h.

So, to capture that data we have to also put in the domain. So, now instead of a function f on the whole variety y you should think of just u , f, f with the domain. they have a, there is an equivalence relation. So we define that equivalence relation as follows. So U, F and V, G will be treated as same, will be called same.

So formally we'll call them related by this equivalence relation, tilde. If, what should happen? They should be equal on the intersection. Right, so U is a big open set, V is a big open set. They have some intersection which is also big. So, if on this big common space f and g look the same then we should call them the same. It could be a bit counter intuitive because we are ignoring what happens in u - v and v - u, but somehow this will be good for theory.

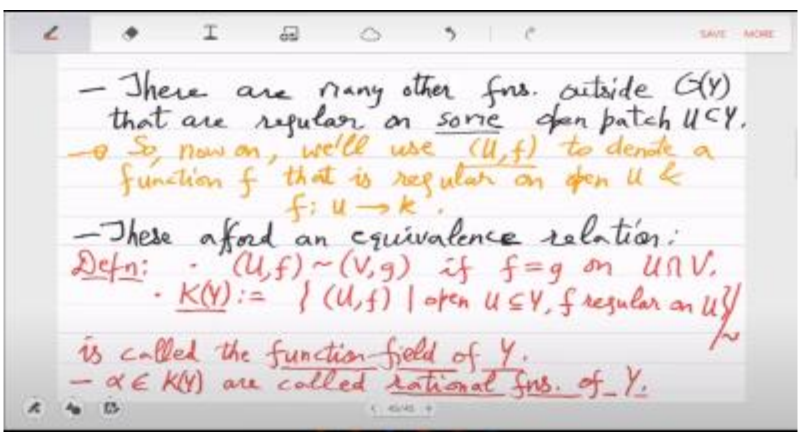
So, we will call f and g with their domains equal. if on the intersection domain they are equal. So, one intuition why this the theory will still work very well is that we never care about a single neighborhood, we care about smaller and smaller neighborhoods. Around the point we only care about neighborhoods which are as close as possible to the point. So, we always care about the intersection, we never care about bigger things. So with that intuition it's enough for f and g, I mean it's enough for us to call them equal if they are equal in a smaller space.

We don't care about the, we don't care about differences in the bigger space. And once you have this equivalence relation, you now have a brand new object which we call So ky is the set of all the uf's such that u is open and regular f and you mod out by the equivalence solution. So, you look at all the functions essentially with their domain where they are defined and with the understanding that there is a notion of equality amongst them. So, this k y is called the function field of y. of the variety y. Now, if I am calling it a field then it should be a field, right? Do you see that it is a field? What is the inverse of u, f? What is the inverse of f basically? Why should 1 / f be defined? So, either f = 0 then we do not care.

If f is non-zero then 1 over f is defined at least somewhere in the world, essentially in places where the zeros of f are absent. So, complement of Z , which is an open set. So, 1 over f is defined over some open set, there is some open domain and since f had a representation g by h, reciprocal has representation H by G on that open set. So, it is of the same type, it is in the set.

So, and then you can obviously see addition F1 + F2, F1 x F2 and so on. So, it was always a ring, but now it is also a field. The ring structure is coming from OY already, but the interesting thing that you now have because of the domain information is it is now a field. So this is called the function field of y and the elements of ky are called rational functions on y. So these are the true rational functions.

Obviously it is inspired by functions of the type 1 / x or x1 / x0 or so on. But now it is something which is inherent in the given variety y instead of coming from polynomials. So you have now regular functions on a variety. You have rational functions on a variety. and the first one is a ring, the second one is a field and obviously O is a subset, is a subring of Ky.



Is that clear? Okay. Yeah, so once we have this nice object or maybe I give an example first. So for the affine line, what is Ky? So you have to unpack that big definition with all these concepts. And in the end you will get something very simple, which is? Yeah, so first you look at Oy, so Oy we know is Cx. and from that you can guess Ky to be the, just change the bracket. So this is, so for example new things that you have included are things like, well 1 over x was not there and you also have $1 / x^2 + 1$ and so on. So, the reason why 1 / x, 1 / x 2 + 1 was not there before was because there are zeros, there are zeros where the denominator vanishes.

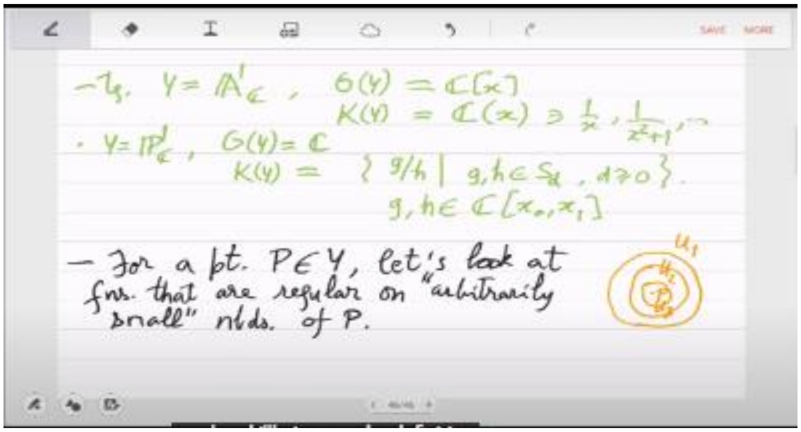
So, that was not a thing which was regular on the whole of the affine line, but now you do not care about their vanishing because you will just exclude the vanishing part, the remaining is open with the domain 1 over x is fine. So, you can make this into a proof, you can show that Ky is C . The other example you do not even need to write down, but still let us test you. So, what is Ky for the projective line? So, Oy was just constants.

What should be Ky? Will it again be constants? No, obviously not. So now if you go through the proof, we looked at g / h and we said that h will be 1, hence g will be 1. But now there is no need for h to be 1. So you can take now all the fractions. So g / h where g and h are in SD.

So these are basically the rational functions where, so this is, should write that. So g and h both are bivariates. So for example, x0 / x1 is there. and $x0^{2} / x0^{2} + x1^{2}$ is there and so on. These bivariates equal to degree you can take that fraction.

Is that fine? So, this is an interesting object and now is the time to make it even more interesting. So, what you can do now with this concept of equivalence of U, F with V, G. You can take a point, take a neighborhood and shrink it. So, basically you get closer and closer to the point P. And so in the limiting case, what are the functions which remain? You want to study that. As you get closer and closer to the point P, what are the functions amongst k y which will live and others will be excluded.

We want to understand that and now remember completely algebraically, there is no such So you have to do this purely algebraically. So for a point P let us look at functions that are regular arbitrarily small neighborhoods of P. That is the goal and I will give you the definition right away of this set of functions. So these are called the germs of or germs on y near P. So this is the set O sub py is the set of those functions which are defined at p and well basically p is contained in U.

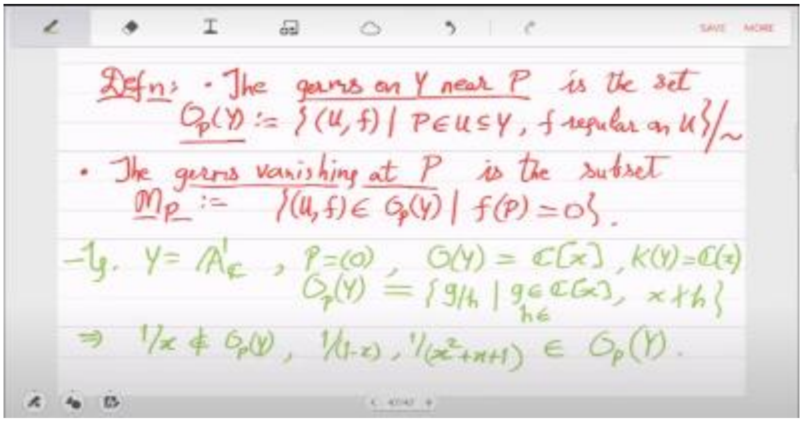


So p is contained in U, U is an open subset of Y and F is regular and then obviously you have this equivalence idea. So, now if you only look at, so once you fix P, point P in Y, you only look at those that part of Ky where P is covered by U. So, that is called germs on y near p. What this is defining geometrically is this functions which will remain defined at p no matter how small you get to p and in the end actually becoming p, but you can never become p because p is a closed set.

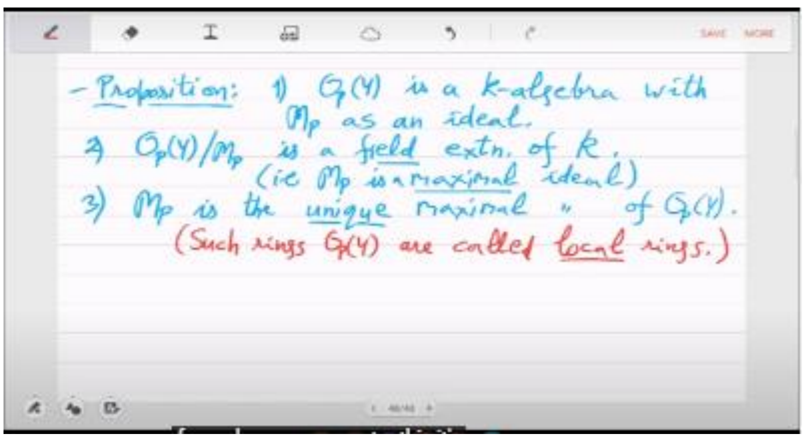
So, open sets will always be slightly bigger or much bigger than p. So, over all that you get the limit. but you get it now algebraically so we don't really have a genuine notion of limit here like you had in real analysis and the germs vanishing at P are somehow special so these are functions that are defined at P but when you evaluate them at P you get the trivial thing which is 0 Okay so these are the germs vanishing at P, it's the subset called MP of UF in OPY such that F at P is 0. Is that clear? Examples of this are actually trickier to write down. We will do this explicitly next time, but let us attempt on the affine line or complex and let us take the point P to be the zero point. So what was O ? This was just Cx.

Now can you unpack the definition to get O ? What is this? and actually ky also you know. So amongst all the rational functions in one variable over complex, which ones are defined around 0 in smaller and smaller neighborhoods trapping 0? You have to unpack this definition and do it in algebra. Yes, well g (p) is 0 only when g is a multiple of x, right, so this is basically all the functions g / h, so g is our arbitrary polynomial and h, well h is also a polynomial, but x does not divide Okay, so 1 / x is absent but 1 / 1 - x is fine or 1 / x + 1 is fine. Yeah, so we will define it in the next class obviously why this line was a simple case but I hope it shows you the algebraic trick which is happening. We will formalize it via localization of a ring next time. So in general we can write this, we can unpack this definition and we can just say that this is some localization of the coordinate ring.

and where localization means that you are allowed to divide polynomial G by certain polynomials not all.



In particular in this case you are allowed to divide by an H which is not a multiple of X. So these are the germs on the affine line near the point 0 and the germs which vanish, what is an example? So these will just be Gs which are multiple. X should be a factor of G, right? So XG / H, that kind of.



So X is there and X / 1 - X is there and so on. So these are the gems. What you can show immediately is that MP is a ring. you can add two vanishing terms and you can multiply because they will still, you will still get functions defined at P and vanishing at P. More importantly, this is a, it's an ideal of, of what, of OPY. Because functions which are defined in arbitrary small neighborhoods of P, those functions when you multiply with a function inside MP, then it becomes an element of MP.

So that also you can show as an exercise. It's an ideal. And we'll prove actually more properties of O P Y in the next class. Some of these you can already try as an exercise. There are tons of properties of this object.

So O P Y is a K algebra. with MP as an ideal, I already mentioned that. More interestingly OPY mod MP is a field. So, in particular or equivalently MP is a maximal ideal. And third is that not only MP a maximal ideal, there is no other maximal ideal present. It's a unique maximal ideal. So rings which have a unique maximal ideal are called, do you know the name? These are called local rings and the name local ring already comes from this picture that we drew.

Because these are rings which actually sit locally around a point on some variety. So such rings are called local rings. Where such is basically referring to the fact that your ring, let's say R, has a unique maximal ideal. So such things are called local rings.

They are used everywhere in cumulative algebra. But geometrically they have this origin. That's where the name comes from. Okay, so you can try this. Its proofs are simple. Just have to unpack the definition.