Computational Arithmetic - Geometry for Algebraic Curves

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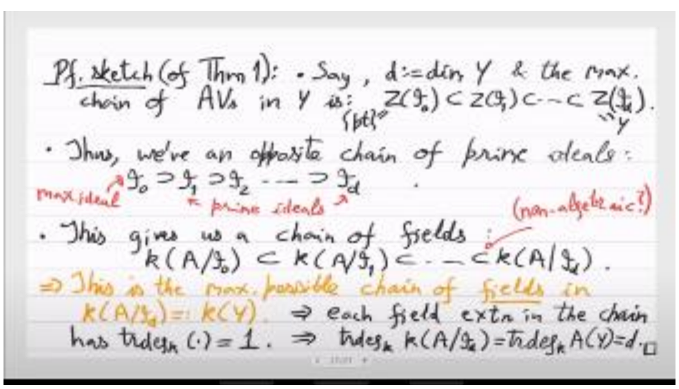
Week - 02

Lecture - 04

Projective Varieties

So, last class we defined the dimension of a closed set. So, the way we defined it was dimension of in fact an affine variety y is how many closed sets can you embed the longest chain of closed sets that you can embed. And then we showed, so this was the geometric definition, the algebraic one would be in terms of the, you look at the coordinate ring of that affine variety or you can look at the field of fractions of the coordinate ring, what is the transcendence of that over the base field k. So, the transcendence degree is we showed in theorem 1 that it is equal to the dimension definition. Any questions till now? So, already at this point with the new abstraction that we have developed you can ask the following computational questions. So there are many computational questions that you can immediately ask this will help you in appreciating the things we will do more abstract things which we will be doing.



So first is as stated in the overview if I give you a system of polynomials in the polynomial in n variables. So, this is the input. So, then the ideal defined by this system is the following ideal right. So, the ideal is basically the collection of all the polynomials generated by f 1 →f m which is ∑a i f i where a i is an element in the polynomial So, this is the relevant ideal and you can ask the question whether this ideal whether the 0 set is empty and which is equivalent to testing whether 1 is in the ideal and this was the content of Hilbert's neutral challenge arts.

So this is a very practical questions it appears in almost all the I mean any practical problem you can actually formulate as this. It is a very very expressive language any optimization problem or anything else you can convert into polynomial system and ask whether there is a 0 which is equivalent to saying I mean if there is no 0 then the fun is in the ideal. So it is a concrete computational question. I have already said before that it is NP-hard, but you can still think about the best algorithms and special cases. So, what you want in those special cases is a fast algorithm in time polynomial in the input parameters which is m. And the degree of Fi's and so this was finite field. So, let us say fp. So, log (p) and what else n is also the number of variables and finally, how were these Fi's represented right. So, there is some size bound So these are all the parameters in which you would want a fast algorithm. Now of course in general if you solve this you will be solving NP-hard problems, but may be in some special cases according to your input of interest you could actually find a fast algorithm.

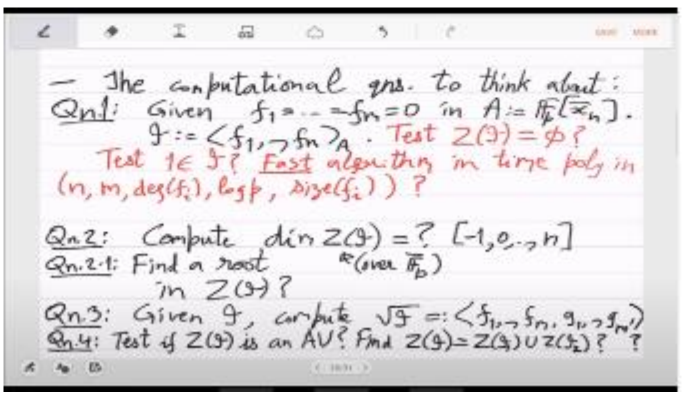
So obviously I will not solve this question here or even discuss it, but you can think about this and if you want you can present something as an extra talk. Second question is a slight generalization of this, so you can directly ask what is the dimension of this zero set. So, dimension will be a number between - 1, 0 and n. So, - 1 dimension means that 0 set is empty, there is no root, 0 will mean that this system has finitely many roots and 1 to n will mean that the system has infinitely many roots when we are talking about fp'. So, the 0 set we can just think of this over fp '.

So, positive dimension means that there are many many roots. So, what is it? So, it is it - 1 to n the specific number you have to output right. So, the output is very small the input is also small, but actually finding the algorithm here is hard a fast algorithm. And similarly you can also ask for a root in this. So, actually finding a root here.

So, those are questions of practical interest. What else did we do? So, we defined radical. So, you can ask if I give you an ideal, how fast can you compute the radical of that ideal as a generating set. So, if I was given via the generator set F 1 to then obviously F1 to Fm will still be in the radical, but there may be new polynomials which were not there before. So, you have to grow that generating set.

This is what you have to solve. So, given the ideal I via generators F 1 →F m find these extra generating polynomials G 1 → G m prime such that the overall thing is equal to the √ I that is another question. And the other thing you we have abstractly defined is irreducibility of a 0 set, a closed set, right. So, checking that. So, test if Z of i is an affine variety.

It is already a closed set. So, you just simply have to test whether it is irreducible. If it is reducible, then you decompose it. So Z (I1) UZ (I2). So these are the questions.



As soon as we have the abstract concepts you can immediately ask these questions. Generally these are hard questions except maybe question 4, but even that I mean when the number of variables is large even that will be hard. but you can convert this into an algebraic question which is testing whether z (i) is an affine varieties equivalent to testing whether i is a prime ideal. And decomposing the 0 set of i will be like, so i will not be prime and you have to write it as the intersection of two ideals, i1 ∩ i2. So that would be factoring of an ideal, it is a factoring question.

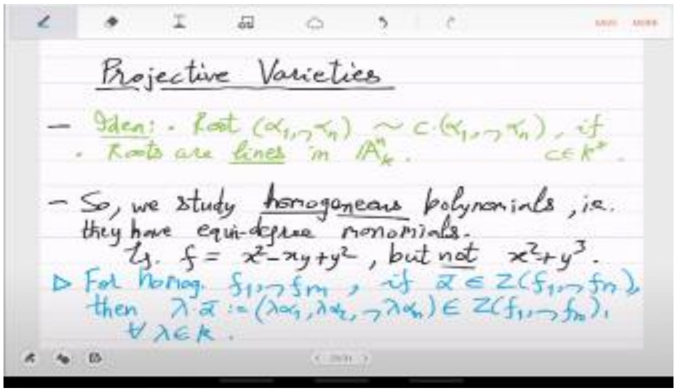
Ok, Any questions? So I leave you with these computational questions and move to a more abstract kind of varieties which is called projective varieties. So we have defined 0 set, close set and accordingly affine variety. Now since we are calling them affine, there has to be another term which is now projective. So we will also look at projective varieties. And then in the future whenever we will say variety, it can be affine variety or projective variety.

These are different things. So what is a projective variety? So in a projective variety basically. we do not distinguish a root from its multiple. So, the idea is √α1 → α n is considered the same as any multiple of it as long as c is in the base field and non−zero. So, up to non−zero constant multiples field constant multiples roots are considered equal. So, it is basically affine variety, but then you are modding it out by an equivalence relation. So, another way to say this is roots are now not points in the space, but lines in the space.

In the affine space and we will do a + 1 or may be not so soon let us keep it this. So, instead of these n coordinates we will be now we will basically lose a degree of freedom because we have we are only looking at lines in this space.

and so this is geometrically, algebraically we will be studying homogeneous polynomials that is. they have equi degree monomials. So, for example, $x^2 - x y + y^2$ right. So, this is a quadratic form that is a homogenous polynomial, but not $x^2 + y^3$. So, this is an inhomogeneous polynomial this is not a neither is it quadratic form nor a cubic form.

So, in affine varieties your system may have these things, but in projective varieties your system of polynomial equations will be like the first case homogeneous. So, in homogeneous the key roots are basically lines right, because any root can be multiplied by c it is again a root. So, we basically want to consider them equal up to this multiple c. So, for homogeneous f in fact you can take homogeneous polynomial set of homogeneous polynomials f 1 → f m. So, 0 set if α ‘ is a root then λ times α ‘ which is λ α 1, λ α 2, α n with every coordinate scaled by λ is again a root for all λ in the field including 0 right. So, in particular for a homogeneous system 0 is always a root and if α is a root then any multiple is also a root.

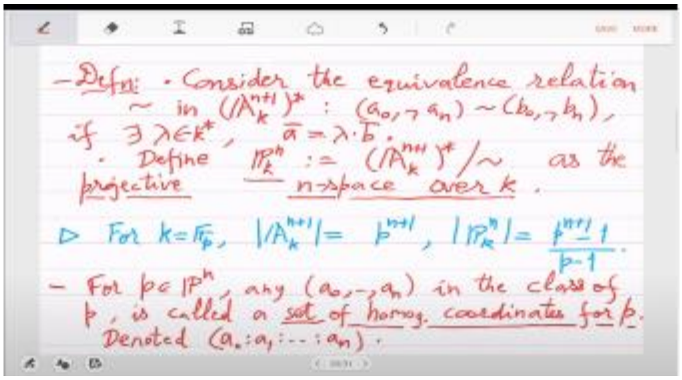


So, we just want to remove these extra roots you are getting from 1 and for that formally we define. So, consider the equivalence relation tilde in the affine space removing 0. So, we remove the 0 point and everything else is like a 0 … a n it will be considered equivalent to a point b 0 →b n if there exists a λ in k star such that a ' is λ times b '. So, the thing I said before and define the projective space, projective end space to be this.

So, this is the projective end space. So, note that projective n space you will need one more dimension in the affine space, because you want lines. So, you need one extra degree of freedom in the affine space and there you except 0 all the other roots you have this equivalence relation. So, equivalence mod equivalence relation simply means that one many points may have the same representation. or the same point may be representing many points, the other points being λ x b '.

So, b ' represents everything that is like λ times b '. So, these things form a equivalence class, they are in the same equivalence class. So, of course, you can count here. So, for k = fp, the affine space had how many points? p^{n+1} and the projective space will have. So, you have to remove the 0 from there.

So, p^{n+1} - 1. but amongst these points not everything is distinct right. So, you have to divide by the possibilities of λ which is p - 1. So, that is it. Yes, so projective n space in particular has more points than the affine n space and it is not a power of p sum of these powers. Is that clear? Another thing is for a point in the projective n space any a 0 to a n point in the equivalence class of P is called a set of homogeneous coordinates for p and we denote it by this notation.



So, instead of , we use the separator. just to signify that here what is important is the ratio I mean whatever is the let us say you can divide each of these Ai's by the same λ and it will be the same point. So, let us see the projective one space over complex. So, the points 2 , 1 - i and 1 + i , 1 are the same. You can see this because you can multiply the second one by 1 - i you get the first point.

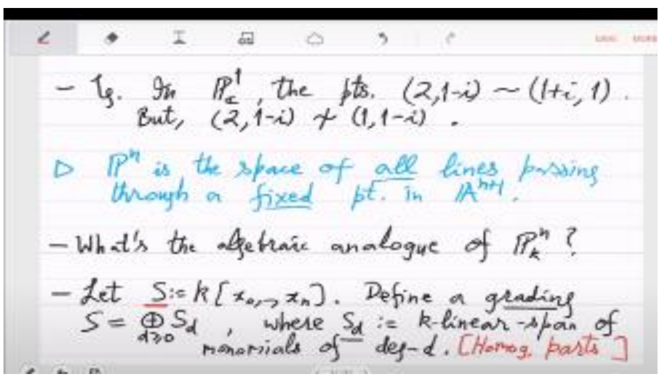
So, they are the same in the projective one space. 2 , 1 - i and 1 , 1 - i these are different these are different points because the ratio is for the first coordinate is 2 for the second it is 1. So, p^n is the space of all lines passing through a fixed point in the bigger affine space. So, in particular we took the point 0 and we took all the lines which are passing through 0, they give you distinct points in the affine n space, but you can do the same thing for any fixed point in e^n + 1 that correspondence is there for lines versus projective points. Yes, so that is the geometric picture. Now, what is the algebraic analogs of what we did in affine varieties? So, there we used ideal, prime ideal, radical ideal. So, we have to repeat all that for projective now. So, let us do that quickly. So, what is the algebraic analog? or algebraic description of projective and space and what are in particular projective varieties, what are the closed sets, open sets and so on.

So, as you can already guess we will here we will only focus on homogeneous polynomials right. So, ideals which are generated by homogeneous polynomials they will be the correct ideals. If there is any non inhomogeneous I mean if you have an ideal for which there is no homogeneous generating set then that will not be relevant in projective variety. So, ideals as you can see will always have I mean will usually have inhomogeneous elements you do not care about that what you care about is the generating set homogenous. So, let S be the polynomial ring in one more variable.

So, we call that variable X 0 define what is called a grading where S d is the degree d part of the polynomial ring. So, S d is the k linear span of monomials of degree d. So, grading is simply this decomposition of the vector space k vector space S which is the polynomial ring into these subspaces each is called S d for degree d. So, these are degree d homogeneous polynomials and their linear combination is what a polynomial is a general polynomial that is all we are saying here.

So, these are the homogeneous parts. to think in terms of that grading to give you the homogeneous parts. And one nice property is that if you multiply two homogeneous parts one of degree D other of degree E then you get D + E that is why it is called a grading. And with this grading now we can define all the algebraic analogs. that we had in affine variety in the affine case. So, an ideal I of this s which now remember has one extra variable n + 1 variables over the base field k is called homogeneous I will just shorten it to this is called homogeneous if I has a set of homogeneous generators.

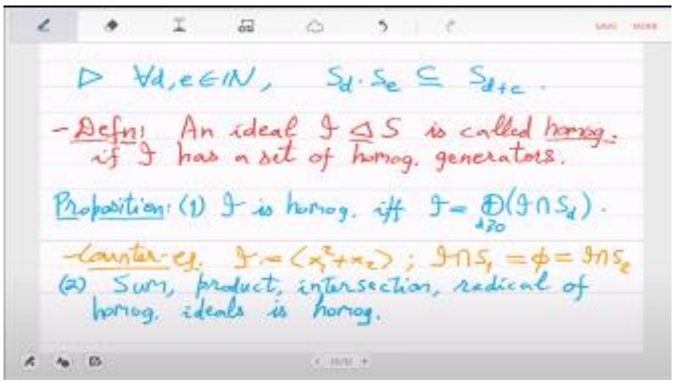
Now, recall again note here that homogeneous generators does not mean that every polynomial in the ideal is homogeneous. S is as before defined here. S is the polynomial ring with one more variable than the target. So, target will be projective n space, but we have to keep an extra variable to see things algebraically. So, ideal is homogeneous then simple proposition.



So, an ideal I is homogeneous if and only if I follows the grading by which I mean this. So, if you look at the degree D homogeneous polynomials in the ideal. for different d, then their linear span gives you the whole ideal. This is a property which is true if and only if you started with a homogeneous ideal.

For inhomogeneous ideal it is false. You can see an example may be right here. Counter example is if you started with the ideal $x_1^2 + x_2$, then what is $I \cap S_1$? What are the linear forms in this? There is none, right. I mean it is only, should I call Z well, may be I empty is the right thing. and same is with quadratic forms. So, homogeneous degree 2 and homogeneous degree 1 are totally absent in this ideal.

So, clearly the RHS is nothing. So, I is not the direct sum of or the linear span of these linear forms, quadratic forms because everything is empty. So, it happens when I is inhomogeneous. If you took homogeneous then you can see that it is clearly true. I mean that is a simple proof of the proposition. Second property is some product intersection and radical of homogeneous ideals is homogeneous.



So, homogeneous ideals behave miraculously well with all these things which we were using in the algebra for a fine varieties. So, ideal sum was needed for intersection of 0 sets, product was needed for the union of 0 sets. yeah intersection radical was required to have this one to one correspondence between varieties and ideals. So, all those operators behave well on homogeneous ideals they give they take homogeneous ideals to homogeneous ideals.

Yeah, that is a good question. I did not write I ∩ S3. I mistakenly said it is empty. I mean you have to check that it is empty. That also will be empty I ∩ S3 S4, but even if somehow it was non−empty it does not matter because the generator of ideal I is quadratic degree 2. So, if you cannot cover degree 2, then already this condition i equal to direct sum is false.

So, I do not need to see I ∩ S 3, because good things have to happen already in degree 2 and they are not happening. Yeah, but you can never generate $x_1^2 + x_2$, it is just linear plus quadratic part right. So, S 1 S 2 is enough, this shows Yeah, I think how did I define S d, I define it as a k linear span, so 0 is also there, yeah by definition 0 is there, that is fine. And example of property 2 you can see I will not prove this proposition, this is a simple exercise follows pretty easily from the property above S d x S e ϵ S (d + e) from the grading it follows, but still you can see this example. So, if you take x 0 x 1 ^{2} and x 0 x 1 these are the two homogeneous clearly homogeneous ideals.

Note that first ideal has homogeneous generators, but they are different degree which gives you inhomogeneous polynomials like x 0 + x 1 ^{2}, but still the ideal is called homogeneous because of the generating set. So, if you add them what do you get? you get x1 ^{2} x0 x1 ^{2} that I hope is the sum yes second one is contained in the first. So, this is the sum if you multiply them. So, product ideal you have to multiply all the generators in every possible way.

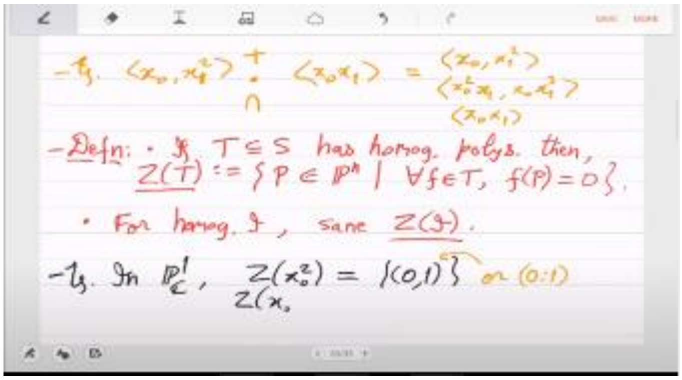
So, that will give you quadratic and cubic. now cubic and biquadratic. So, x 0, x ^{2}, x 1 and what is the intersection ? intersection is x 0, x 1. So, in this example you can see that all these operators they give you homogeneous as expected That is what you can show in general. Well, in this particular case x0, x1 is generated by x0.

So, multiple of x0, so it is a 0 thing, modulo the first ideal. So, this example is then non-trivial. At least one person is confused. So, work this out. Now, we can define the closed sets in this new space called the projective space, projective n space. So, what are the closed sets? So, you should look at subsets of this polynomial ring S that contains only homogenous polynomials.

then zeros of this given a system of homogeneous polynomials look at the zeros and then consider them equal up to the equivalence relation. So, basically points in the projective n space f p = 0, may be p is not a good choice in this course, may be big P. So, just like we had the 0 operator 0 functor before in the affine case, in the projective case it will be same thing except points will be considered equal up to multiple by constants. So, we write point p in the projective n space everything else is the same as you have seen before.

And, hence the same definition is true for ideals also. Ideal is also a subset. For homogeneous ideal I, same z i, use the same z i for ideals also. So, which means what? So, now, in the projective line this new z. So, now, we will be we will not say what I mean what z is will be clear from the space you are in. So, when we are in the projective space then it is the z corresponding to the projective space equality.

So, with that understanding z of x 0 ^{2} is 0 , 1 is this clear. So, the affine z would have given you what 0 , t for every t. right, but the difference here is that since there was a duplication there we are just putting different t's we have now compressed all of them into one, this is just one point. So, there is a difference sorry.



Yeah, yeah sure, sure yeah if you will prefer. So, I will abuse that because I will set the context in the very beginning it is a projective line. So, all those things are equal and here again similar thing will happen this is projective z. So, will you get 0 , 0. you cannot get 0 , 0, because projective line that does not have that point, 0 , 0 was removed was excluded.

So, all you will get is yeah. So, x 0 or x 1 one of these has to be 1 both cannot be 0 and when 1 is 1 the other is plus - 1. So, I can just write these two points by which I again mean 1 1 and 1 - 1. So, you can take x 0 x 1 equal except 0 value or you can take them opposite sign except the 0 value. So, there are two points in the projective line that satisfy this. So, there is a subtle thing that happens, there is a difference between affine z and projective z, but I hope this is natural given the homogeneous nature of the system.

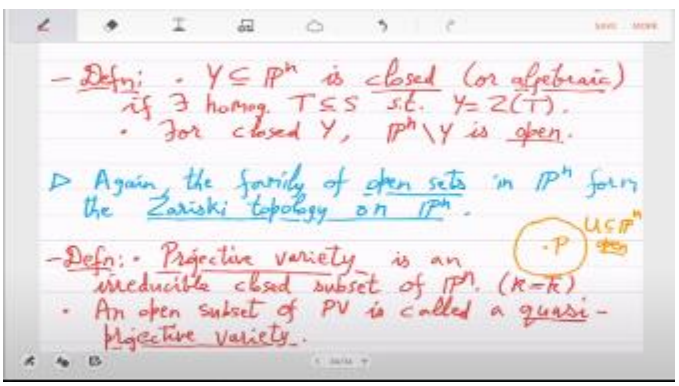
So, that is the definition of closed set as you can imagine. So, subsets y in the projective n space it is closed or algebraic if there exists a homogeneous T subset of the polynomial ring in n + 1 variables such that y is equal to zeros of T with the understanding of projective z. And for closed y The complement is called open. So, now we have a subtly different topology which we are going moving towards. So, we have defined z accordingly we have closed sets and open sets and now you can see all those properties that union intersecction of open is again open.

and so you get Zarisky topology on the projective n space. Those basic things have to be checked. So, if you want you can do it as a homework. So, family of open sets in P^n form what is called, what we will call the Zariski topology. So, the idea of a neighborhood around a point is now this abstract thing you for the point you have to contain it in an open set that is the neighborhood of a point. So, whenever we will draw pictures like this point P and this is the neighborhood this is the ball.

So, in reality the ball is actually a ball. but in the over a finite field what we are saying is that this is some open set U subset of p^n it is open, where open means this that you have to find a system of polynomials homogenous polynomials t zeros of t take the complement. that is what u is that contains the point P. So, these are the neighborhoods. So, whenever we will say in this course take a neighborhood of a point P it will be a very, very loaded term because all these definitions coincide and these pictures will always be wrong.

So, I will always trying to fool you by drawing these pictures. hopefully you do not get fooled or you do get fooled whatever works. Yes. No, no I continue with the same k just assume small k to be always f p ‘ for this course small k is f p ‘ well at least till. for the next 2 months I guess small k will just be fp ‘. In the last month we will when we will do when we will prove Riemann hypothesis there we might call small k to be fp, but theorems will always geometric ideas will always be for fp ‘.

If you do not take fp ‘ then the problem is that this neighborhood business will not because I mean as we saw in examples last week, previous weeks for a system of polynomials t the 0 set z of t may just be empty, if your field is very small there will be no root. So, you do not want that and so dimension etcetera is defined for algebraically closed field. So, now you have the natural definition of projective variety. So, projective variety is an irreducible closed subset of the affine projective n space for some n and may be I say here k is equal to k ‘. just assume that an open subset of a projective variety is called quasi projective variety.



Generally, whatever we will say about projective variety will be true for quasi projective variety. Only difference is between closed and open, both are irreducible. So, that will help and dimension is defined as before. So, dimension of quasi or projective variety is defined via the chain of closed subsets.

So, all the geometric and the algebraic perspectives we saw before will hold. with the understanding of homogenous systems. So, let us quickly see examples well as I said all the theorems will hold. So, in some cases you might want to study the open set instead of the closed Why is that? Why should it be open? Close may not be open. Oh, in pn you are shifting to pn then I see.

Yeah, but his question was even for quasi affine variety. Why do we define that? So, let us see examples of this again in the projective line. So, z x_0^2 that we saw this is a projective variety. What about the 0 set of $x_0^2 - x_1^2$? See Zx_0^2 is a projective variety because it was a single point dimension 0 and irreducible. Second one had two points dimension 0, but it cannot be irreducible then it factors into 2.

So, this is not a projective variety it is just a projective close set it is not a variety. Variety is a special property. Dimension of both these things is 0. These are finite objects.

Let us go to the projective plane or complex. So, now if you take the difference of these two. not these two sorry something else. Look at this. So, what you have to remember here is that this projective plane comes from affine three space. There are three variables x_0, x_1, x_2.

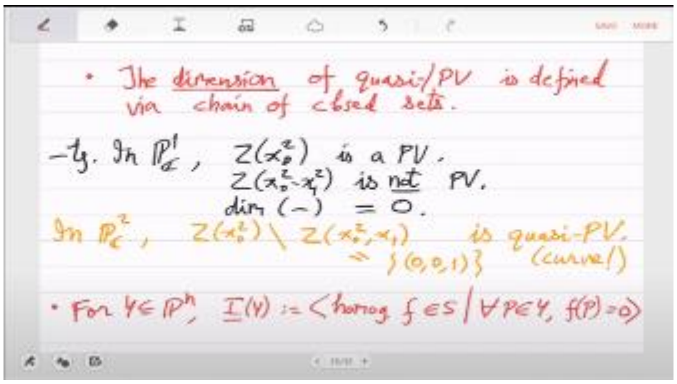
So, what is z of x_0^2? So, this x_1 and x_2 can be arbitrary. So, there are many points. In the second one, you have already set $x_0, x_1 \to 0$. So, what do you do with x_2? You can only set it to 1. This is a singleton. So, from the first one, we are subtracting the second and this is does not seem to be closed, but it is open.

So, this is a quasi-projective variety. In fact, it is a curve. It is a projective curve, because in two dimension you have set one constraint to 0. So, the remaining dimension is only one. So, it is a quasi-projective curve.

Yeah, you can define dimension for 0 set or yeah any closed set. Chain, just look at the chain. I see. So, you just look at the chain of varieties. So, that definition you have to change, you just look at varieties. these in this case there will be projective varieties, chain of projective varieties and the last element is then not the same.

Last element is the maximal projective variety sitting inside. That is an important point. That definition you have to change, but the thing is you want dimension to be geometrically intuitively correct. So, this is the only thing which is intuitively correct finitely many points it should be called dimension 0. If you are getting by your definition dimension 1 then the definition is wrong intuitively does not make sense. So, dimension positive should always mean that there are infinitely many points because you are in f p ‘ algebraic I mean if you are over complex you should have infinitely many points for positive dimension.

that is the basic intuition you have to keep in mind. So, now we can draw a nice picture of association recall the functors. So, I of y the ideal functor which will take. So, we are defining I which takes some subset of points of the projective space. So, maybe I should say that. So, this I functor will be the ideal generated by homogeneous F in S such that for all the points that you are given fp is 0.



So, now you the functor the ideal functor will take subsets of points to an ideal which just collects the maximal possible system which defines your subset y and then to come back. So, which is that we have already defined we already have defined the z of a functor given a system t z of t is the projective points and the last thing is the homogeneous coordinate ring. So, this will be denoted S y it is the polynomial ring modulo this ideal.

So, like we had in affines definition A of y. So, similarly we have S of y. So, S of S is a functor here which takes an open set which takes any subset of points to functions which are defined on top of these points. So, these are S mod ideal Iy. Is this clear? And this will give you the nice picture of associations. So, now you can look at the projective space, n space and the polynomial ring n + 1 So, these two we will think of them as associated by the dictionary of i z functor. So, a y here subset via the i functor maps to the ideal which is homogenous and a set of polynomials here will map via the z functor to close set.

So, closed sets are in one to one correspondence with what? So, not homogeneous ideals, but you have to take the radical. Again this is a version of Hilbert's, Dundald's and Lanzard's identical to what we proved before. So, radical homogeneous ideal. So in a way you can say that these radical homogeneous ideals they are completely characterized by the 0s which is the close set Y.

What? Yeah, yeah that was a proposition you have to prove. So, projective varieties will be in one to one association with homogeneous prime ideals and point single point will be in one to one association with maximal prime which is maximal ideal. So, these are one to one. So, these I and Z functors they behave based on the definitions we have given closed set and projective variety. One explicit way to study projective varieties is by reducing it to affine varieties.

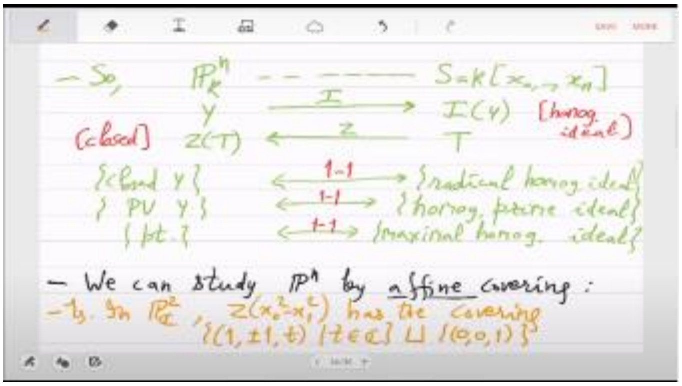
So, we can study projective or general projective n space by affine coverings. So, this happens because, so let us go to the projective plane over complex and the zero set that we had. So, we can I mean this is what we were doing also when we were trying to find the roots the 0s. So, what we think of is there are 3 variables x 0, x 1, x 2 in the affine 3 space and we cannot have all 3 0 = 0. So, we set x 0 to 1 then we set x 1 to 1 and then we set x 2 to 1 right.

So, basically those are the 3 affine parts of this projective 0 set. I mean you take a maximal ideal Yeah if it is inhomogeneous then you reject because here I need homogeneous. Yeah but I want both the conditions homogeneous and maximum. I thought this is a symmetric it is not symmetric.

Yeah I want an ideal which is both homogeneous and. I mean amongst homogenous ideals I want maximal. So, then I should I do that for the prime case also yeah. So, that is a bit asymmetric. amongst homogenous you get the maximal that is the point yes. So, here you just follow the algorithm you use to find the points right.

So, you set x 0 to 1 when you set x 0 to 1 then x 1 has 2 possibilities + – 1. third parameter is free, so any complex number. So, this you can see as an affine closed set that is the first part, but this does not give you everything. In fact, disjoint there are other things remaining, which is you are now allowed to set x 0 to Now, you can allow you can set x 0 → 0.

So, x 1 will also be 0. So, you can set both of them 0 because you have a third parameter which is free. So, that is then 1. So, you can. So, what we will do now is we will just look at this as two affines covering the projective close set. The first one is the first affine component and second is the second affine component the union of this we call an affine cover.



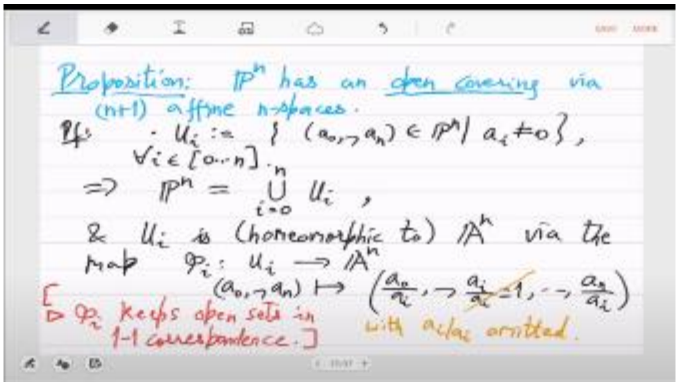
So, let us just formalize this simple thing is this clear example. So, that is a proposition So, p^n has an open covering via n + 1 affine n spaces. So, this is just you look at the following close set a 0 to a n such that a i is not equal to 0. and this you do for all i. So, I have defined u 0 u 1 ... u n, where for a particular I just say that a i is not equal to 0.

Now, since a i is not equal to 0, I can as well normalize this by a i, I make a i 1. So, these are the positions of 1 that you get n + 1 positions and that will cover all possible points in the projective n space. So, that is what I write P n is union of U i, i 0 → n and u i is a^n. Technically I should say homeomorphic, but that is something I have not defined.

So, u i is essentially a^n via the following map and that is a nice map. it is phi i hui to a^n. It does what? It sends a 0 to a n e 0 ... a n to as I said you just divide by a i. sorry this has to be omitted. So, from n + 1 coordinates you go to n coordinates, you just omit this a i by a i thing which is 1.

So, you go from a 0 to a i - 1 and then go from a i + 1 to a n. you just reduce one coordinate. So, that is a well defined map from u i to a^n and you can see that in the image you have the whole affine n space and in the domain of course, you have this u i which is which you can see what it is. This map is what is called homeomorphism because So, you can see, you can prove that pre-image of open set is open. In fact, the, no, it is stronger than that. Open sets of ui and open sets of a^n, they are in one to one correspondence.

So, when this happens then you are sure that the Zariski topology, the geometry is been completely preserved. So, in that sense u i we can say is a ^{n}, it is just a relabeling. it is the whole affine n space. So, this together with the cover union of Ui being p ^{n}, we have shown, we have basically decomposed p ^{n} into open subsets each of which is the affine n space. Are these Ui's disjoint? I mean there is no way to compare them right, because they are I mean the coordinates that you are dropping are coming from different places.



So, you cannot really compare, no you can sorry the embedding of u i is in the same n + 1 a 0 to a n coordinates. Yes, so you can ask actually then this question are these disjoint or not. So, this is not a disjoint covered. So, points actually overlap point may be present in many UIs, which is why you do not get the correct count.

You cannot just say that n + 1 times the size of UI is equal to the size of P ^{n}. There is a heavy overlap. These things are overlapping. So cover does not mean that it is a disjoint cover, it just means that I have a way to like look at neighborhoods which look like the whole affine space. So we have patched together different affine spaces. This patching is a very important geometric construction.

It can be taken to a insane level of abstraction which we will not do in this course. So, this property of p ^{n} is inherited by all projective varieties and quasi projective varieties. So, this inheritance means that what we did with p ^{n} we decompose into these affine n space patches we can do the same thing for a projective variety decomposing it into affine varieties and that is now straight forward. So, what you do is for projective variety y write y as union of same thing like before i equal to 0 to n intersect with u i. In that big projective n space you have these patches u i's.

So, you just intersect your projective variety with that patch you will get possibly a smaller patch which is a subset of y. yeah and that is an affine variety. So, n + 1 affine varieties now cover your projective variety. Is that clear? So, that is it next time what we will do is we will So now we have defined varieties of all types, affine varieties, projective varieties, quasi affine, quasi projective. What we need to do next is we want to compare two varieties. So we will do morphisms next time.

▷ This property of $\mathbb{P}^n$ is inherited by all PV & quasi-PV.

- Pf: · For PV Y, write Y as $\bigcup_{i=0}^{n} \underbrace{(Y \cap U_i)}_{AV}$. □

- We define morphisms next time.