Computational Arithmetic - Geometry for Algebraic Curves

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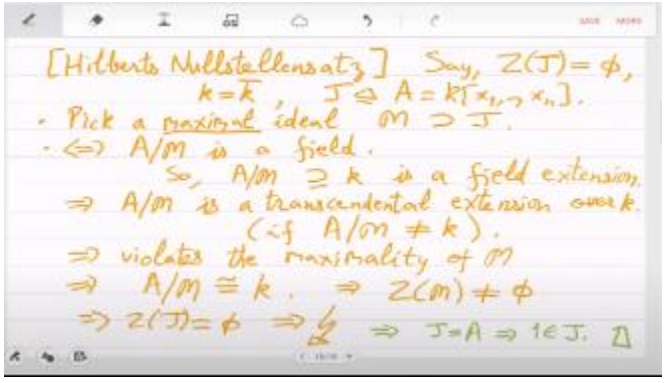
Week - 02

Lecture - 03

Dimension and Varieties

We are still trying to build this association between varieties and ideals. So, we defined a fine variety as irreducible close set and we took a detour and we defined radical of an ideal and we call an ideal i to be radical if square root of it is equal to itself, which means that if any power of a polynomial is in I, then the polynomial itself is in I. And what we have shown this proposition that for any ideal of the polynomial ring, if you look at the zeros, if you look at the zero functor and then you look at the ideal functor, then you come back to the √i. So, which means that these two functors are in a way inverses of each other, as long as your ideal is a radical ideal. So, if you take zeros and you take ideal then you get to this thing which is which may be slightly bigger than the ideal i, but it is very strongly related. Any questions till now? So, I have posted assignment 1.



You can check the new website and that assignment if you do then you will learn more things some of the things that we have been skipping the background of all this. So, where are we now? So, we have shown that. So, this is the geometry and this is the algebra the polynomial ring affine space versus polynomial ring. So, the following association we have shown that if you look at the collection of I mean basically if you look at a y which is algebraic it is an algebraic it is a subset of the affine n space which is algebraic it is a close set.

Then when you look when you apply the ideal functor you will get what you will get an ideal which will be radical. So, you will get here a radical ideal. And if you on the RHS if you start with a radical ideal I and you apply the zero functor you will get what? You will get a closed set, why? But more importantly if you now again apply I then you will come back to the same place where you started. So, this is one to one. So, we have shown a one to one correspondence between close sets or algebraic sets and radical ideals.

Is that clear? This proposition interpreted like this. Moreover, we will make it more precise and then that is the object we will study. The closed set we have to take it irreducible right, because we have said that if a closed set reduces or decomposes into two components $y_1 \cup y_2$, then it will suffice to study y_i's separately. So, we will only look at irreducible closed sets. and for that the ideal i will become even more special.

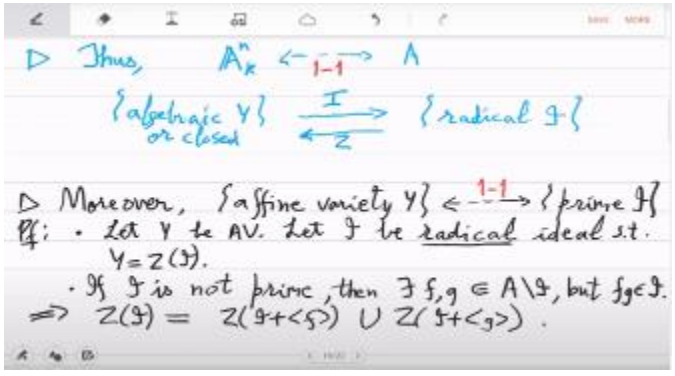
So, it will become what? So, you have if you start with an affine variety which is nothing but an irreducible algebraic or closed y. then this association is with prime ideals. So, if Y is irreducible and closed then the I functor will take you to a prime ideal and on the RHS if you start with a prime ideal then the Z functor will give you and a fine variety. So, geometrically affine varieties are nothing but algebraically prime ideals. So, what is a prime ideal? In case you do not know prime ideal is an ideal where if g x h is in the ideal then either g is in the ideal or h is in the ideal.

So, it behaves very similar to the notion of prime numbers if number if the product of numbers g times h it is divisible by a prime p then either d g is divisible by p or h is divisible by p it is the same thing, but now happening in algebra instead of numbers. So, how does that relate to affine variety? So, let us start with the an affine variety Let us start from the LHS, take an affine variety Y which is defined via Z (I). Maybe I should let Y be affine variety, let I be the radical ideal such that Y = Z(I). So, for any variety y from the previous association if you apply the i functor you will get the radical ideal which defines y. So, you will get y = Z(I).

Essentially your refined variety y is the zeros of this system of polynomials that generate the ideal i. Moreover, i is our radical ideal now. So, what we are claiming we have to now show that this radical ideal i is prime right. How do you show this? So, if i is not prime, then there will exist two polynomials f and g both of them are outside but their product is in I. So, this is the definition of not prime there will be a product such that the factors are outside the ideal, but the product is in the ideal.

So, what I will contradict. Now, using f and g is the fact that Z(I) which is the affine variety that you started with, it is actually factor, it is not irreducible. I will give you the two components. Can you guess the components? How will you decompose Z(I) into two closed sets that are non-trivial? and their union should be Z(I). So, you basically in one case you set f to be 0 in the other case you set g to be 0 you get two systems. So, z(i) + f and i + g.

So, you have to check this any point which annihilates i which is a 0 (i). can you see that it will be 0 of at least one of these either it is annihilating it is a 0 (f) or it is a 0 (g). So, it is true because f times g is in the ideal right. So, if you have a 0(i) then it means that f times g is 0. So, one of these has to be 0 it cannot be that both f and g at that point are non-zero. So, simply because of that any point on the left hand side is in one of these.



Now look at the other direction a point which is in one of these is clearly a 0(i) right. So, you have you have an exact equality. So, check this. And so, we have decomposed closed

set Y into two closed subsets.

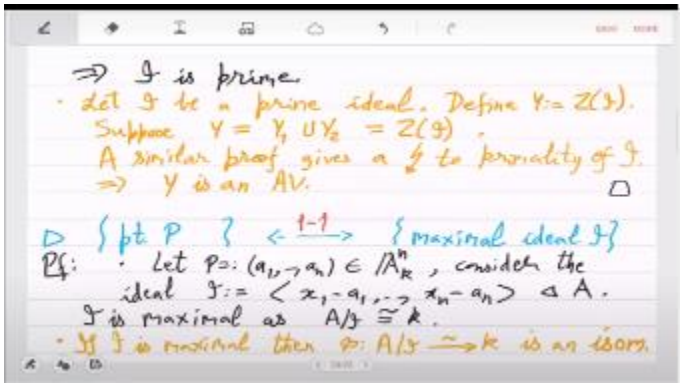
You just have to check that these are proper. So, why are they proper? Why cannot Z(I) + F be equal to Z(I)? And this is a proper decomposition. that contradicts. So, you have to show this why this is proper do this as an exercise. So, this would mean that you have you are contradicting the variety y it has to be reduced which means that i is prime.

So, if you start with an a fine variety y then you will go to prime ideal and we have to now start with the prime ideal and see what happens. So, let i be a prime ideal. Now, y we will take zeros of i. So, from RHS now we will move to LHS. So, Y will be zeros of I.

You have to show that this Y is an affine variety. So, it is clearly closed set. Why is it irreducible? So, if it factors then what happens? So, here you can again show that. you will get a contradiction to the primality of I. So, from this you can deduce that you will get that Y is an affine variety.

So, I have skipped some of these steps which you have to fill in. Is this clear? And yeah, so slowly we are making it this more and more precise the correspondence. So, close set corresponds to radical ideals, AVs correspond to prime ideals and what do points correspond to? That is the last thing. point P ϵ Y corresponds to maximal ideals, that would be 1 to 1. Yeah, this will be rather easy, one section will be very easy, so if you take a point a 1 … a $_n$ I should not say in y and k remember we are assuming it to be algebraically closed.

So, for a point p which has coordinates a $_1$ to a $_n$. the ideal will be. So, this will be simple you just set x_i to a_i that is the relation. This is an ideal of a. And you can clearly see that this ideal is a maximal ideal because if you quotient a |i| then you recover the base field k which means that this ideal is maximal as a | i| is k which is a field.



So, a point corresponds to that maximal ideal and the converse is what I had sketched last time, then E |i| will be isomorphic to k. here it will be simple. So, since i is maximal

almost by definition a |i| will be k and that gives you this homomorphism. So, let me write the homomorphism. So, then you have a isomorphism that takes a |i| to field elements which means that in particular it takes xi to a field value.

So, this is a field element and since phi is a homomorphism what you get is that this a 1 ... a n is a 0(i). So, since I mean any homomorphism from a |i| to k has to be I mean if you look at where x 1 → x n are being mapped is being mapped to a point which is a 0(i). you can I mean if this is in so the reason why you have to have this is because otherwise the homomorphism will would not be defined. The on the LHS a |i| any element in i is essentially treated as 0. So, all these elements in the image have to be 0 just because of the well defined nature of the homomorphism.

this image of X i's gives you the point, so that covers both the sides. So, points correspond to maximal ideals and affine varieties correspond to prime ideals and closed sets they correspond to radical ideals. Any questions? So, this is the dictionary that you have to remember throughout the course. There is one more exercise that you should try. we have defined closed sets and we have defined irreducible closed sets.

So, what is the relationship between the two? In particular algorithmically also you would be interested in the decomposition, right. Can you decompose any closed set into irreducible closed sets into affine varieties? So, show that every closed set y in the affine n space. can be expressed as a finite union of affine varieties y i. So, this is important, this shows that an arbitrary closed set given to you, it may when you start its decomposition, it is in the case that you will have to, you will require infinitely many steps in the decomposition. There will always be a finite number of irreducible closed set y i's, such that the union covers the y that you started with.

So, the idea here will be I mean this is related to the property that in this polynomial ring every ideal is finitely generated. So, y gives you this ideal So, you were given the closed set Y through some polynomial system which generates I or I mean even if the generators were in known what you know is that for any closed set Y when you look at the corresponding ideal whose 0 sets Y is there are finitely many polynomials in the generator. So, every ideal I is finitely generated. So, any for a Y the finiteness of the union comes from looking at the ideal identifying the finite set of generators and then

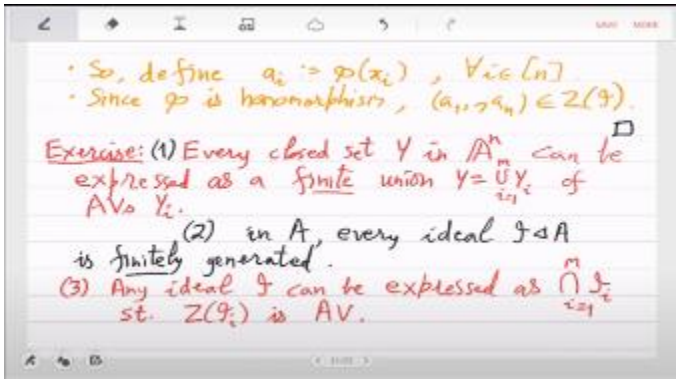
using each of these Fi's to define Vi's that is the idea.

Yes, that is true how do you get. But I thought this is related to the Newtonian property. So, this is called the Newtonian property. No. this is yeah yeah yeah right ok. So, then let me just state this as a different property.

So, that is one property that this is second property. So, what is the algebraic version of 1? You are claiming that it is factorization of an ideal. I guess yeah that should be immediate. So, let us also write that any ideal I can be expressed as intersection of ideals such that the 0 set of these ideals is an affine variety. So, the property 1 is basically equivalent to property 3.

So, you have to prove for one of these. So, this close set decomposition into affine varieties is essentially the question of factoring an ideal where by factorization we mean this intersection of i i's and this will be finite this is some i = 1 to m. the same m. So, prove this it is important because for algorithms you want these even these bounds you want. So, that you know how much you have to compute. So, that from a general given polynomial system I you can get irreducible ones the affine varieties.

So, here I can now actually delete this. It is just any ideal. Every ideal is finitely generated. Yes, so these are the properties I will not prove, but whenever you get into an algorithm you will have to ask these questions and you will have to give bounds. There are various ways to prove this.



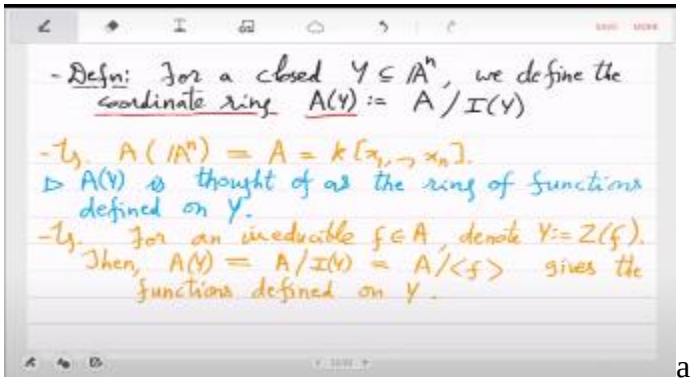
So, you can try this. So, now going back what do we do after we have this association. So, now we are interested in functions over this y. So, for a close set or for an affine variety what are the functions that we can study. So, in let us move in that direction now. So, for

a closed Y, we define the coordinate ring, we will write it Ay.

So, this coordinate ring will be, so remember that all the polynomials which are in the polynomial ring A, I mean the question is whether they are genuine functions also defined on y, they are right because those are actually functions defined everywhere on the affine space A^n. So, they are also defined on y, but we want to make them well defined we want somehow this relationship. So, the correct way to study those functions only on this closed y instead of the whole affine space would be to mod out by the functions which define the 0 set. So, these will be the functions we are interested in once you localize, once you specify a close subset y and this we call the coordinate ring. So, for example, what is the close the coordinate ring of affine and space.

So, for the whole space the coordinate ring is everything, but if you look at a smaller subset that is closed inside a^n, then what you will get is basically an extension of this polynomial ring, it will be a ring extension and it is exactly that a |i| (y). So, let me write that. So, A_y is thought of as the ring of functions defined on y, but we will call it the coordinate ring of y. So, let us see a proper example. So, take a polynomial f look at the 0s call that y and ask the question what is Ay.

So, for an irreducible f look at the 0s. So, we are looking at essentially this hyper surface of f, what do you think is E (y), the functions defined on y. So, this is by definition E |i| y and I y is the ideal f, so E | f|. So, all the polynomials let us say g mod f is the function that is defined over y. Is this clear? So, we do not look at absolute functions in A, but we have to now mod it out by the defining equations and the whole ideal we have to mod out by that gives you the correct functions and it is it matches the intuition for the whole affine space there it is the polynomial ring questions. If no questions then the next thing you want to do is you want to define dimension.



a

So, you intuitively know that a line a point has dimension 0, line has a dimension has dimension 1 and plane has dimension 2 and so on, but that you know because you can see it. Here we want to define it algebraically without drawing any pictures. If I just give you

this ideal I of y. from that what should the dimension be? So, algebraically what is the definition of dimension? So, intuitively we know that dimension of a point is 0, dimension of a line is 1.

and dimension of the plane is 2. So, can this be done algebraically? So, which we should then hold for any field basically including finite fields. Yes, so that for that now we have to define what is called tower of prime ideals. So, the bigger the tower of prime ideals you can embed in an affine variety the bigger the dimension is. So, I will just give the definition and then we will interpret it. So, the dimension will shorten it to dim of an affine variety y is the max positive integer such that there exists a chain.

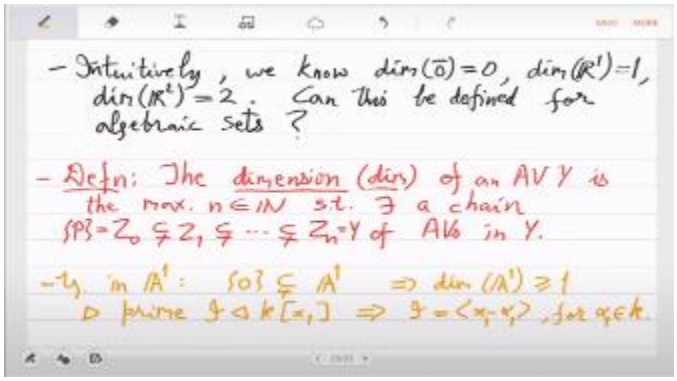
Zn of affine varieties inside Y. So, instead of defining it algebraically, I am actually defining it as a tower of closed sets, irreducible closed sets. So, what this is saying is that, so Zn here is the maximum fund. So, you can take z n to be y for example, and look for a affine variety which is smaller strictly smaller than that. So, call that z n - 1 and then in inside that you find something affine variety which is strictly smaller z n - 2 and keep doing this. So, I guess this is 0 and this is y right, sorry empty set is not an affine variety in this course, it should be a point yeah some point.

So, you start with the whole affine variety and then you keep making it smaller always remaining an affine variety and you reach all the way to a point. So, for example, if your y was a point then the dimension will just be 0 and if your affine variety y was the affine line in that case you will have z0 and z1. so the dimension will be 1 and so on. Degree of f no, no so we do not want to define degree we want to define dimension these two will be different degree also will be there yeah, but degree is a more complicated thing in this course dimension is far easier. So, yeah so those are the examples So now, you can fit in the intuition that you have of a point, a line, a plane into this format.

So, when you are in the affine line, what you have to do is you have to look at this. So, this is the chain of closed sets inside the line. But of course, you have to show why is this maximal, why cannot you have three things. So, something between the set 0 and the whole line, can you have a close set between a proper close set. So, this only tells you that the dimension of A 1 > = 1.

Right, but why is the dimension exactly 1, why cannot it be 2 for the affine line, how do you show this. So, you basically have to characterize the closed irreducible closed sets inside the line, which means you have to understand the prime ideals. If you have a prime ideal P So, affine line means that there is only one variable. So, what are the prime ideals inside K x 1? See by the basic algebra which you already know, you can show that every

ideal here is principle and since the field K is always algebraically closed. so the ideal I will have a generator just x 1 - α1, so these are the only prime ideals.



So, which means that essentially it is a point, so you the only close set that A 1 has a fine fun space has is this single element, which I have taken here to be 0. So, that means the dimension is actually 1, the tower cannot be extended, it is exactly 1. So, this is good for a point you know dimension is 0 and for a line now you know the dimension is 1 and this thing now you can check for other examples. So, the other example you should do is what happens in the affine plane right.

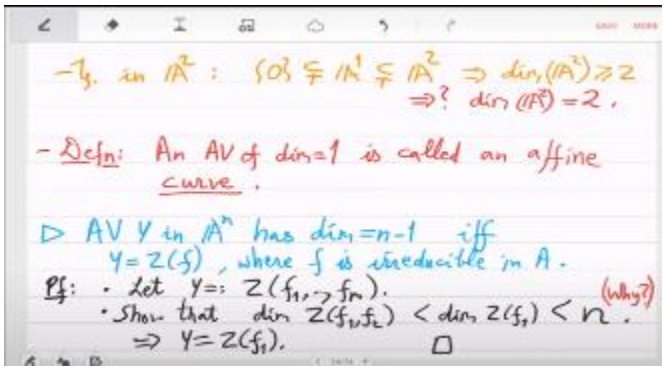
So, natural tower is 0 contained in line contained in plane. which would mean that the dimension of the plane affine to space is at least 2 and you have to show that it is not more, it is exactly 2. So, that I leave as an exercise. So, we will define dimension of this of this geometric object via the longest chain of affine varieties you can embed and you will see, you can see through the exercise that this will match beautifully with your geometric insight that you have from the real Euclidean real space, real spaces. But here now we will not need anything about the field K except algebraic closure.

So, it is true for arbitrary fields now. Any question? Yeah, and now in this course the promise is that we will only restrict to dimension 1, right. So, an affine variety of dimension 1 is called an affine curve. So, this course is about the study of dimension 1 affine varieties which we shorten to curve over algebraically closed fields and specially f p bar finite fields. Yes, one thing you can you should prove is the following property at this point that an affine variety $y \epsilon a^n$ when will it have dimension n - 1. So, affine varieties of dimension n will be the whole affine space and next question you can ask is what are the affine varieties whose dimension is 1 less n - 1.

So, what you can show is that these will be exactly of this type 0s of a single polynomial essentially hyper surfaces defined by a single polynomial, where F is irreducible in the polynomial ring over the algebraically closed field K. So, dimension n - 1 affine varieties are given by a single polynomial, single irreducible. So, studying them is essentially

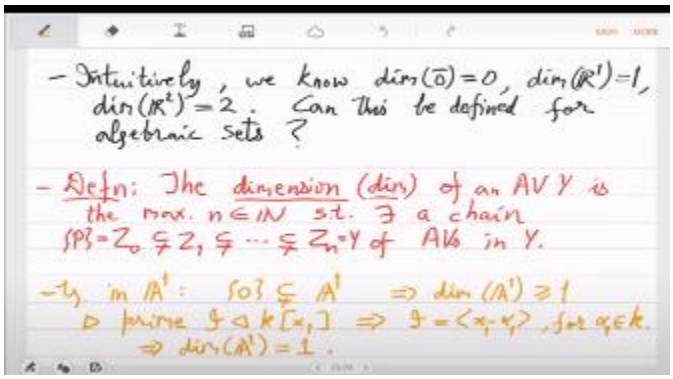
equivalent to studying the zeros of a irreducible polynomial. Proof of this is not very hard.

Basically, any y you can express it as zeros of some polynomial system. and then show that the dimension of f 1 f 2 will be smaller than the dimension of z f (1) which will itself be smaller than the whole dimension which is n. So, I mean you have the affine space intuition is that if you intersect it with an irreducible polynomial f then the dimension will fall and if you intersect it again with F 2, then the dimension will fall another step. So, if since you are assuming y to have dimension n - 1, you have to stop at the first step. So, there will be only one generator. This is the thing that you have to So, because of this you understand dimension n - 1 affine varieties and in particular there is this nice case of curves.



So, a curve y in the affine 2 space is just zeros of a bivariate polynomial. So, your polynomial ring in A 2 has only two variables and curve you are asking it to have dimension 1. So, then it will always be given by a single polynomial. So, curves in two variables it is just a single constraint which has to be irreducible because we are assuming it to be an irreducible closed set.

is that clear. So, again this course is just about studying f (x 1, x 2, 0)s of just this. This is the simplest case you can imagine above univariates. Yes, so let us take that example. So, I mean we actually already saw that example. So, A 1 we just checked know this was the example univariate means that you are in the affine line we have shown that dimension is when is 1 and we understand the chain inside A 2. Yeah, so then why do not you go to A 3?



So, in A 3 this is also a curve So, you have now two relations and they together reduce the dimension from 3 to 1. It is a curve and you cannot express this as a single constraint. This can never be written like this. Is that a question? Yeah, is that clear? So in higher affine spaces curves can be I mean at least their representation can be more complicated because there will be many constraints intersecting to reduce the dimension.

But in the affine plane this A2 curve is just a single constraint. When we study morphisms, we will actually relate the two. We will show that somehow in a cube and a ^{n}, when you are given a curve, we can actually reduce it to two variables and a single constraint, but that will take some time. So, let us now move forward to what Madhavan was saying about, essentially it is this algorithmic question that if I give you a polynomial system. in the affine n space, how do you compute dimension. So, the way we have defined dimension it is very abstract, because it seems to be going over all possible chains of affine varieties, which is which we cannot really compute with right.

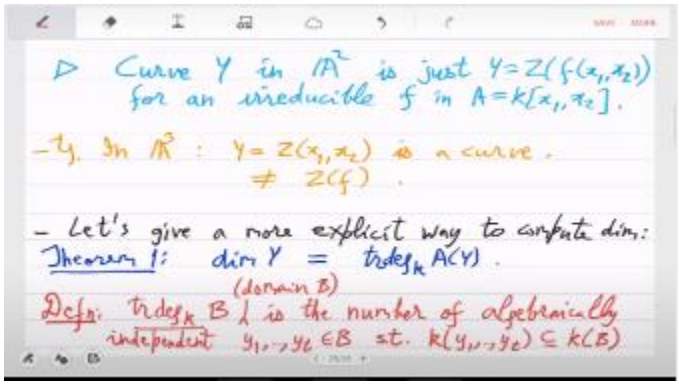
So, we have to give a more deterministic procedure, instead of this optimization over all chains of affine varieties. Let us transform this definition to something more interesting. So, let us give a more explicit way to compute dimension. So, what we will show is the following theorem. dimension of y is the same as transcendence degree of the coordinate ring over y with respect to the base field k.

If you do not know what is transcendence degree then of course, this does not seem very explicit. So, let me define this. So, transcendence degree is It is basically the maximum number of independent variables you have in your ring. So, independent variables means that elements which have no algebraic relationship whatsoever. So, for example x1 and x2 these are algebraically independent because they do not satisfy any relation in the polynomial ring.

In the polynomial ring they are both of them are free and independent and similarly x1, x2 and x3 they are independent. So, the maximum number of independent elements in the coordinate ring. So, remember this is something about the functions on top of y. So, the

functions which are independent their number this is exactly equal to the geometric this quantity which we have defined the dimension.

This is what we want to show. So, this is again we are setting up the dictionary between geometric and algebraic concepts. So, transcendence degree of a ring B is the number of algebraically independent elements y 1 to y t ϵ B such that if I attach them to K, And if I look at the, so I should call B to be, B is a domain. B is an integral domain. So, for an integral domain, you can look at the field of fractions.



So, for example, if B is integers, then the field of fractions is rationals. And if B is the polynomial ring in one variable, so f bracket x, then you can go to the field of rational functions over that. So, that is K (B). So, this extension is an algebraic extension. is a finite algebraic extension.

So, this is similar to what I had said, but a bit more concrete. So, y 1 to y t you should think of these as the maximum number of independent elements and once you have included them in your base field K then everything else that you see in B is dependent. So, everything in B is algebraically dependent on y 1 to y t in particular you can write I mean anything in B will essentially be a root of some polynomial over φ1 to φt.

There is an algebraic relationship. There is an annihilator. Yes. Yeah, I am defining this object transcendence degree of B. Let me discuss this in an example because then I think it will be clearer. So, what is transcendence degree of this object? So, k x y | $y^2 - x^3$| this is an integral domain there are no 0 divisors in this, k is some field we do not care what field it is, but with respect to this k now I am asking for the transcendence degree. So, how many in this domain how many independent elements can you find that is the question. So, for example, I can give you x right x just the first element x this is independent of constants in the base field k correct because there is no relationship between x and 1 right if there was then x would have become a field element because k is algebraically closed.

For example, you take k to be algebraically closed, so in that case you can see that x is

independent. Now the question is, is y independent of x? Clearly it is not because there is a relationship y^2 is = x^3, right. So, you can actually now show that once I take x, everything else depends on x in B, which is saying that transcendence degree is 1. and trivial example is the field base field itself. So, what is transcendence degree of k with respect to k that is 0, because any element you take in k it is algebraically dependent on 1.

So, there is a big difference between transcendence degree 0 and transcendence degree 1 extensions of the field k. And what is transcendence degree of k x? It is a question simpler than the first one. So, for k x once you include x again everything else is dependent on x.

So, this is one. What is the transcendence degree of k (x ,y)? Two variables. So I can take x and y. So for sure two things are independent. There is no relationship between x and y by the definition of the polynomial ring.

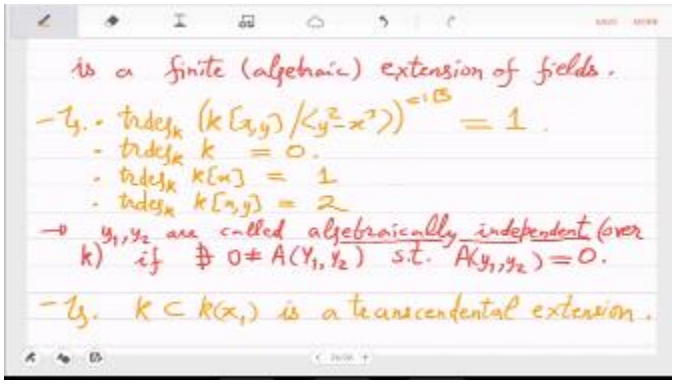
But there could have been a third element independent of x and y. You can show that it does not exist. So it is only two. So hopefully this gives you the picture of how the transcendence degree grows. So, base field it is 0, 1 variable it is 1, 2 variables it is 2 and when you cut it, when you intersect it with the relationship then it again falls down it becomes 1. So, you can already see that transcendence degree seems to be mimicking our definition of dimension.

That is what the proposition is saying, the theorem is saying. Theorem 1 says that dimension of this geometric object affine variety is the same as the transcendence degree of the functions defined on top of it. Is that clear now? So, formally I defined it here in red that is what it means. So, it is a property of field extensions. So, you look at the field k (y 1,y 2 ... y t).

and you look at everything which is in B, then this field you have

two fields right. So, it is a field extension. The field extension should be a finite extension of fields, but let me anyways also define independence. So, y 1, y 2 are called algebraically independent, let us say over a field K, if there does not exist a non-zero polynomial a big phi 1. we are still trying to build this association between varieties and ideals. So, we defined a fine variety as irreducible close set and we took a detour and we defined radical of an ideal and we call an ideal I to be radical if square root of it is equal to itself, which means that if any power of a polynomial is in I, then the polynomial itself is in I.

And what we have shown this proposition that for any ideal of the polynomial ring, if you look at the zeros, if you look at the zero functor and then you look at the ideal functor, then you come back to the √i. So, which means that these two functors are in a way inverses of each other, as long as your ideal is a radical ideal. So, if you take zeros and you take ideal then you get to this thing which is which may be slightly bigger than the ideal i, but it is very strongly related. Any questions till now? So, I have posted assignment 1. You can check the new website and that assignment if you do then you will learn more things some of the things that we have been skipping the background of all this.



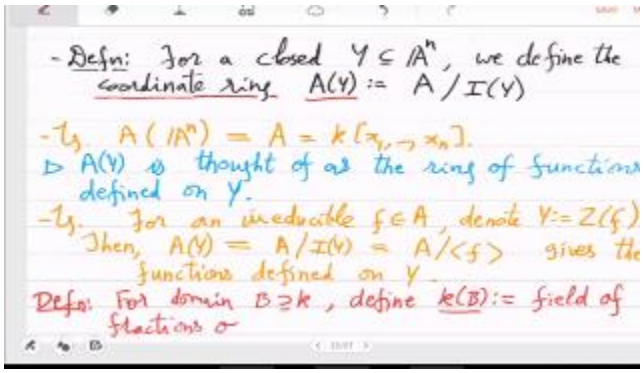
So, where are we now? So, we have shown that. So, this is the geometry and this is the algebra the polynomial ring affine space versus polynomial ring. So, the following association we have shown that if you look at the collection of, I mean basically if

you look at a y which is algebraic, it is an algebraic, it is a subset of the affine n space which is algebraic, it is a closed set. Then when you look, when you apply the ideal functor you will get what? You will get an ideal which will be radical So, you will get here a radical ideal. And if you on the RHS if you start with a radical ideal I and you apply the zero functor you will get what? You will get a closed set, why? But more importantly if you now again apply I then you will come back to the same place where you started.

So, this is one to one. So, we have shown a one to one correspondence between close sets or algebraic sets and radical ideals. Is that clear? This proposition interpreted like this. So, we will make it more precise and then that is the object we will study. The closed set we have to take it irreducible right, because we have said that if a closed set reduces or decomposes into two components y 1 ∪ φ2, then it will suffice to study y i's separately.

So, we will only look at irreducible closed sets. and for that the ideal i will become even more special. So, it will become what? So, you have if you start with an affine variety which is nothing but an irreducible algebraic or closed y. then this association is with prime ideals. So, if Y is irreducible and closed then the I functor will take you to a prime ideal and on the RHS if you start with a prime ideal then the Z functor will give you and a fine variety. So, geometrically affine varieties are nothing but algebraically prime ideals. So, what is a prime ideal? In case you do not know prime ideal is an ideal where if g x h is in the ideal then either g is in the ideal or h is in the ideal.

So, it behaves very similar to the notion of prime numbers if number if the product of numbers g times h it is divisible by a prime p then either d g is divisible by p or h is divisible by p it is the same thing, but now happening in algebra instead of numbers. So, how does that relate to affine variety?



So, let us start with the an affine variety Let us start from the LHS, take an affine variety Y which is defined via Z(I). Maybe I should let Y be affine variety, let I be the radical ideal such that Y is = Z(I). So, for any variety y from the previous association, if you apply the i functor you will get the radical ideal which defines y.

So, you will get y = Z(I). Essentially your refined variety y is the zeros of this system of polynomials that generate the ideal i. moreover I is our radical ideal now. So, what we are claiming we have to now show that this radical ideal I is prime right. How do you show this? So, if I is not prime then there will exist two polynomials f and g both of them are outside I. But their product is in I. So, this is the definition of not prime there will be a product such that the factors are outside the ideal, but the product is in the ideal. So, what I will contradict Now, using f and g is the fact that Z(I) which is the affine variety that you started with, it is actually factors, it is not irreducible. I will give you the two components. Can you guess the components? How will you decompose Z(I) into two closed sets that are non-trivial? and their union should be Z(I).

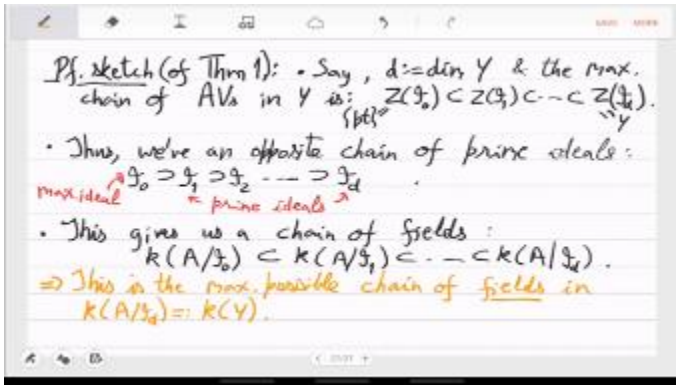
So, you basically in one case you set f to be 0, in the other case you set g to be 0, you get two systems. So, z i + f and i + g. So, you have to check this any point which annihilates i which is a 0(i). can you see that it will be 0 of at least one of these either it is annihilating it is a 0(f) or it is a 0(g).

So, it is true because f times g is in the ideal right. So, if you have a 0(i) then it means that f times g is 0. So, one of these has to be 0 it cannot be that both f and g at that point are non-zero. So, simply because of that any point on the left hand side is in one of these. Now look at the other direction a point which is in one of these is clearly a 0(i) right.

So, you have you have an exact equality. So, check this. And so, we have decomposed closed set y into two closed subsets, you just have to check that these are proper. So, why are they proper? Why cannot Z(I) + f = Z(I)? And this is a proper decomposition. that contradicts. So, you have to show this why this is proper do this as an exercise.

So, this would mean that you have you are contradicting the variety y it has to be reduced which means that i is prime. So, if you start with an a fine variety y then you will go to prime ideal and we have to now start with the prime ideal and see what happens.

So, let i be a prime ideal. Now, y we will take zeros of i. So, from RHS now we will move to LHS. So, Y will be zeros of I. You have to show that this Y is an affine variety. So, it is clearly closed set. Why is it irreducible? So, if it factors, then what happens? So, here you can again show that you will get a contradiction to the primality of I. So, from this you can deduce that you will get that Y is an affine variety. So, I have skipped some of these steps which we have to fill in, is this clear? And yeah, so slowly we are making it this more and more precise the correspondence.



So, close set corresponds to radical ideals, AVs correspond to prime ideals and what do points correspond to, that is the last thing. point P in Y corresponds to maximal ideals, that would be 1 to 1. Yeah, this will be rather easy, one section will be very easy, so if you take a point a 1 ... a n I should not say in y and k remember we are assuming it to be algebraically closed.

So, for a point p which has coordinates a 1 to a n. the ideal will be. So, this will be simple you just set xi to ai that is the relation. This is an ideal of a. And you can clearly see that this ideal is a maximal ideal because if you quotient a |i| then you recover the base field k which means that this ideal is maximal as a |i| is k. which is a field.

So, a point corresponds to that maximal ideal and the converse is what I had sketched last time, then E |i| will be isomorphic to k. here it will be simple. So, since i is maximal almost by definition a |i| will be k and that gives you this homomorphism. So, let me write the homomorphism. So, then you have a isomorphism that takes a |i| to field elements which means that in particular it takes xi to a field value. So, this is a field element and since phi is a homomorphism what you get is that