Lecture 02 : Ideals and Varieties

And so, for that we will assume field a small k to be usually f(p) or f(p) bar and the affine space will simply affine n space will simply be the space of all points. So, these are as a set it is basically n tuple with coordinates from the base field k. but we will have a more elaborate notation here this fancy a to the n sub k because it will not just be a set, but it will actually be a topological space which we will define in due course, but this would be a refine n space. So, this is where our points live and the polynomials or the functions are on this space So, function is basically it takes a point in the affine n space and it evaluates you should think of the function at simply a polynomial and the polynomial can be evaluated at a point and the polynomial also has coefficients from the same base field k. So, this picture is basically the diagram you should remember in all the things we do in this course that this green ball is the space, it has a point P and then the orange part are these functions or objects which are outside or on top of the space. So, in this case these are functions f and g and they will evaluate to possibly different things on the same point.

So, we want to study the green wall the set of points, but we will always studied by something else. So, for that we need to define the connection what is the connection between these two words. So, the first operator will be the z operator. So, this is zeros of a polynomial f which is just points p in the affine space such that f vanishes and the same thing you can do with a subset t.

So, z of t is points which vanish on all the elements of t. for all f, f of p is 0. You should ask the question, when is for a polynomial f, when is z of f empty? So, can you think of polynomials which have no roots, no zeros? So, of course, you can think of them, but usually we will assume small k to be an algebraically closed field like f p bar. So, any polynomial f you pick, there will always be a 0. So, Z of f will actually be non empty in our interesting cases of course, in finite fields and other fields which are not algebraically closed then Z of f can be empty, but it would not be the case in our the generic situation we will be in.

Can Z of f be everything? So, Z of f equal to a to the n whole the whole space, f vanishing on every point. This again is possible if k is too small, but in algebraically closed field it would not be possible. You can always find a point which is not a 0 of f. So, z of f will be somewhere in between it will not be empty, but it will also not be everything except of course, if you took f to be 1 a constant then there is no root. So, that would be the only case when it is extreme usually it will be in the middle.

So, example is so Z of x1 2. and x 1 + x 2 right. So, what are the 0s of this? So, remember you are interested in common 0s. So, first one implies that x 1 is 0 and second also implies x 2 is 0. So, the only solution is 0 comma 0, only 0 is this right, this will not even depend on what field.

So, it is true for any k. what is z of x1 2 assuming that we are looking at affine 2 space. So, here this is only setting x1 to 0, x2 is free. So, you will get 0 comma t for every element in k right. So, it is basically the affine line affine 1 space with 0 appended on top.

So, these things are called curves, we will formally define it, but the first 0 set is a single point and the second 0 set is a curve, it is parameterized, in fact it is even parameterized by a single variable here. So, with this z operator now we can define the closure or when is a set closed. So, basically in the affine space as I said we will not just look at it as a set that would have been boring. We will actually look at it with more geometric behavior. So, for that we will identify some special sets in the affine space and these are called closed sets.

So, a subset y in the affine space is called algebraic or closed if y is equal to z of t for some t So, simply put you will call a subset of the affine space closed if it is exactly the set of 0s of some set of polynomials. So, these are the closed sets and the complement of these we will call them open, but before that let us see an example of this. So the complex field is which is basically the affine space that this is closed why is it closed it is the 0s of what it is the 0 set of the 0 polynomial right. So this is closed because Because trivially if you take the zero polynomial then you get everything. So the whole set is the whole affine space is closed.

But a more interesting object is complex - zero. So if you puncture the complex line at zero, is it closed? So this is a proper subset of the affine space. can you express it as Z of t, why not, it is not completely trivial. So, this is not closed. So, you can show as an exercise that this is not closed.

So, some subsets are closed, the others are not closed. what this is complex - 0, the 0 point is closed and if you remove it then the remaining thing is open. So, we will define these also. So, complement of closed is called open. So these will be the cornerstone of our geometric perspective of varieties or zero sets.

So we will always think in terms of subsets which are closed and subsets which are open and then build things on top of that. Again we are doing this to build a connection between this green ball and the orange ball, we will slowly identify very strong connections between these two words. So towards that, so for a subset Y which is basically a set of points that you are looking at, what should be the corresponding polynomials? So, you are in this green word from this how can you go to the orange word. So, for that we should define what is called an ideal. So, for any subset y set of points define an ideal I of y to be those polynomials which annihilate all the points.

So, if you are not aware of the term ideal you can just take this as a definition also or maybe actually I should give the other definition as well. So, a set I subset of A is called an ideal. if so you need some axioms in particular you need I itself to be a ring you should be if you add two elements or you multiply them you still get an element within I so it should be a sub ring but there is more there is something extra which is a more important property that if you take any element outside the ideal and multiply with something inside you come inside.

you want I to be a sub ring and you want the product of any element for all a in R for all elements a small a in the big ring a, a times i is basically the set of elements of i each of them multiplied with a, this should again be in i. So, this is the key property.

So, you must have seen this object before in when you have a ring you can talk about ideals of a ring and we use the notation this one. So, whenever I will try this I will mean that I is a subset of A moreover it is an ideal, it is an ideal. Yeah. So, it is a yeah that is true. Let me not write it down.

So subring means that in here 1 may not be present, 0 is present. So 0 is there, 1 may not be. So, basically the usual axioms of ring that you can closed under addition, closed under multiplication and for addition there is an inverse, there is associativity, there is commutativity and all those things are there, distributivity is there. So, those list of axioms you have to go through or everything I satisfies in addition it satisfies this property that anything outside multiplied by inside is inside. So, I am giving this here because the object that I have defined has this property.

So, for a set of points y when you look at the polynomials which annihilate all these points y this set is an ideal. you can check that f1 + f2 if f1 is in Iy and f2 is in Iy then their sum is also there because the sum will also annihilate product will annihilate moreover if you take some g which is outside this set I and multiply it with something inside which is f, f times g will be inside So, the key axiom of ideal is also satisfied. So, this is the ideal corresponding to a set of points. On the points we are not assuming anything, y is an arbitrary subset, but Iy is always an ideal. So, that is a definition and a property.

So, now we have some kind of an association developed. which is so in the affine space if you take an object y a subset and this a is the polynomial ring functions defined on the affine space everywhere so y is associated with an ideal right so on the left hand side you have points and on the RHS you have polynomials so from set of points you can go to set of polynomials forming an ideal which is this IY definition and is there a converse to this if I give you a set of polynomials T then I have defined ZT before So, from points I can go to ideals and from set of polynomials t I can go to a 0 set. So, we have the first basic association and we will slowly make it stronger because we should now try to understand what where will these closed sets go. and is the association as strong as starting from a y going to iy and then when you come back, you come back to y. Can you take two steps and come back to the place where you started, in other words is it a one to one correspondence, there is an inverse to these maps.

So this object here is an ideal. and this object here is closed or algebraic. So, these are very different words this in fact if you have heard the term category these are two different categories and we are actually what these arrows are these are functors. So, from the category of affine varieties we are going to the category of algebras.

and back and forth right. So this is the thing which we will develop slowly into a well oiled machinery. So could we make these associations well behaved. example can we make them one to one. One to one means that if you start with two different, if you have two different closed sets then the associated ideal should also be different and vice versa. If you have two different ideals then the associated closed set should be different.

Is it a, is this an exact association of one to one correspondence. We will see that that is not true but we will come very close to making it correct. So, your time for a first proposition. So, this is trivial phi and the affine space both are open, union of open is open. and if these are finitely when you open then the intersection is open.

So I will prove this, proof is quite easy but before that any questions? Yeah, so well this is just a proposition from we have given the clear definitions, from the definitions we can prove this. Once we have proven this then we would have defined what is called the Zariski topological space inside an affine variety. But are the terms clear? Do you have any questions till now? Okay, so how do you show that empty set is open? You have to look at the complement which is A to the n and that is closed, right. And why is A raise to n open? Because the complement which is empty that you have to show is closed.

So let's see that. So A to the n is Z of what? Zero. And empty set is Z of what? One, right. So because of this you know that empty set and the complete set, full set, they are closed by the definition. So complements are open by definition.

Okay, so next is... you have yi equal to z of ti, we have assumed that these yi's are oh sorry yi complement. So, yi is z of ti complement. because we have assumed that phi i is open. So, it is complement of a closed set ti is a subset of polynomials. Now, what is their union? So, union is basically intersection of z ti complement.

This is what you want to show open In other words you have to show that intersection of Z TI is Z of something, can you guess what? Union of TI. Union of TI yes. So, which is equal to, so note that the intersection or let us just look at a simple case Z of T1 intersection Z of T2 is what? So this is Z of the union. Is this fine? Because, so points which are zeros of T1 and T2 on the LHS, you also get those points in the RHS.

and phi sub hrsa right. So, it is obviously an equality and you can repeat this infinitely many times also. So, hence in this union of yi you may have infinitely many phi i's we do not care it is always open is that clear. It can be uncountable as well. Yeah, I mean nothing stops us from the in the definition what did we say we just said that y is equal to zt for some t right so there is no count here so this is this follows from the definition third thing is now you have to look at intersection of yi complement so intersection of yi is what its intersection of ZTi which is equal to union of ZTi whole thing complement right. So, now you have to basically write down union of ZTi as Z of something what would that be.

So, let us see that. So, Z of T1 union Z of T2 what do you think this is. what should you fill in the blank. So, the point can be a 0 of either t 1 or t 2 one of these right. So, if t 1 and t 2 were singletons what you will do is just you would multiply the two polynomials. So, same thing you should do here also you should just multiply every possible way okay f1 from t1 and f2 from t2 you should take f1 times f2 for all possible choices is this fine except that the problem is that can you repeat this infinitely many times because if you have infinitely many ti's then you will have to multiply infinitely many polynomials and that multiplication is undefined okay so we need finiteness here so this means that finite union of finitely many, otherwise we cannot repeat this.

Okay so which proves all three properties. Any questions? Yeah so this is just a fast proof. You have to think about this as a homework. that we have missed anything or is this a complete proof. Now one question you may ask is in property C is this just a artifact of the proof or really can it happen that infinitely many YIs open sets you take, you take the intersection it becomes closed. Can you think of an example where you take infinitely many opens and the intersection is closed.

Which means that you take basically you take infinitely many closed sets and their union is not closed. Can you think of an example? Yeah, so you cover the entire space this would be the example. So we can do this as follows. So look at the union of points.

Point is just a complex number. So each of these sets it's obviously closed. And union of these infinitely many closed is. Yeah that's not a good example. Maybe I should remove 0.

Then what do you have? Then you have. the affine line - 0. So, union of open now that is not what I want, I want union of closed union of closed is open yeah is that fine. while this is not closed. It is open, but as an exercise you show that this is not closed, you cannot express it as Z of t. So, this covering the whole almost the whole space is the problem.

So, hence we have proved whatever can be proved these ABC properties are optimal. and this gives you what is called a topological space, it is a heavy word, but it just means this, this proposition to remember it we call it topological space. So, the family of open sets of A to the n is called the Zariski topology of the affine space. So, Sarisky was the first one who gave this definition. This is a very different definition from what was known before him because the way you define geometry or analysis, the way you do it in reals and complexes you define balls.

So, you around the point you take a ball and that essentially is the idea of your neighborhood and an open set. So now you can show that the union of opens in the sense of these balls is again open and the intersection is open and so on. And you have to pick an open ball. But that is something which is very metric dependent.

You cannot do it in a discrete space. So you want something else to be done in the case of finite fields. because your affine space here is coming from F(p). So, here instead what you do is that you look at these 0 sets. So, you look at the complement of 0 set. So, these are your open sets and believe it or not ultimately they will behave like these open balls you had.

in the real space okay but to actually see any use of it you have to wait for a long time but this will help you guide and compare many of the theorems that will prove and also the terms that we will define the inspiration is from this topological comparison so for example now you can talk about decomposition or irreducible spaces so let me define that so we will call a subset irreducible if there does not exist a proper closed, there does not exist proper closed y1, y2 such that. y is equal to y 1 union y 2. So, basically you have this set of points y we will call it irreducible if or we will call it reducible if it can be decomposed into two closed subsets and if you cannot do that we will call it irreducible. So, proper here is important because of course, trivially you can always decompose y as empty set union y that is not allowed, that is not of any interest and conventionally we will define phi to be reducible. So, the empty set is defined to be reducible every other set why we call it reducible if and only if you can express it as the union of two non-trivial proper closed sets.

So, the I mean the idea of this or the reason why we want to focus on irreducible sets is if it reduces then. kind of by induction we can reduce the properties or we can study the properties of the components. So, for reducible y we can instead study its components. y1 and y2. So, we can study them the smaller pieces and then from there we can derive the properties of y.

So, which is why we will only focus on the irreducible ones. So, example is again the line show that this is irreducible which is an interesting exercise. How do you show that? the line cannot be broken into two closed sets any ideas. So, say you write the line as the union of two closed right. So, it will be Z t 1 union Z t 2. So, union of two kinds of 0s, why is this not possible? Because it would be z t1 t2 and that would be 0.

Yeah, that is a good suggestion, yes. Yeah, so you have a set of polynomials. such that the 0s is everything which means that 1 has to be there, which means that it has to be 0 then, t1 times t2 should have nothing but 0. So, this as a set is just the 0 set. you cannot have a non-zero polynomial sitting there, because that would immediately reduce the number of roots, cannot get everything which.

One of them should be. Exactly, that means that one of them is 0, which means that one of them is improper. So this is a general proof the affine n space over any field that is irreducible. So now finally I can define what is an affine variety. So this was an affine, we defined affine n space but this is everything. usually will have 0 sets which will be much smaller, so we have to give them a name and that would be this, so an irreducible closed set of a raise to n is called an affine will shorten it to Av.

Yes, so note that we started with the full space which we called affine n space, then we defined 0 set or closed set or algebraic set, but those may or may not be affine varieties. So, affine varieties have to be more special, they have to be closed sets, but they also have to be irreducible. So, these smaller things we call them affine varieties, these are kind of the primes here okay any bigger thing can be decomposed into the smaller and we'll only be studying varieties so variety is a technical term it's very different from a zero set and one more thing I can define which is an affine varieties open set open subset is called a quasi affine variety. So, instead of closed it will be open. So, what are the examples? So, 0 of x 1 2 this is an affine variety in any affine n space that you are just setting x 1 to 0.

So, it is closed and it is a subset of that a raise to n and we have shown that it is also irreducible because it is a to the n - 1. So, it is an affine variety what about z of x 1 times x 2. is this an EV, this can be decomposed like this, so this is not an affine variety, right and what's a quasi affine variety. so open essentially means that we have to exclude some zeros some zero set so from zx1 2 that is the affine variety if you exclude something for example this then it is quasi affine.

x 1 has to be 0, but x 2 has to be not 0. So, when you introduce these conditions you get quasi affine variety. Any questions till now? So, now what we will do is we will make our associations more precise, in particular we will start with this y being an affine variety. and we will look at what is i of y and from there if we now look at the z operator so if you apply z on i of y you will come back to LHS but will you get y or will you get something else So you can either apply I and then Z or you can apply Z and then I. So how do these functors interact? So this is the I map and this is the Z map.

So you can apply I and then Z or you can apply Z and then I. You get the same answer. So we will now quantify that. so in this course i shouldnt use the term functor but i anyways will because its taking you from one word to another so there is no good word for this i mean you can always use association or map but functor will be something which is correct so lets study the functors z and i okay and to make this study simpler we will just assume K to be algebraically closed. In fact in this course you just assume it to be f(p) bar, it is this big field which sits above f(p), it contains I mean f(p) bar is essentially the field where if you define a polynomial then it will always have a root in some finite field. So it's the collection of all possible roots of polynomials over all possible finite fields of characteristic p.

This is f(p) bar. So think of k to be that. But the arguments will hold for any algebraically closed field, so in particular complex numbers. Or you can look at function fields and their algebraic closure and so on. And we'll need one more definition which is the radical of an ideal. So, this is denoted suggestively as 2 root of i. So, what is 2 root of an ideal? Radical of an ideal, this is the set of those polynomials such that for some exponent f to the e is in i, which means that so i may have f 2 it may not have f.

So, you introduce f inside further i may have f 3, but may not have f. So, introduce f inside. So, whatever power now the final object this 2 the radical or 2 root of i whatever g there is all the radicals of it if they are polynomial are also inside. For example, what is the radical of the ideal generated by x 1 2 x 2. what is it maybe I should ask the easier question first this one so in this since x1 2 is there you also have to put x1 right and that would be enough so you will just get this ideal now next is the radical of x1 2 x2 what is this yeah here I have wrong in my notes the correct answer is this will not change and yeah and so on.

Yeah, which power of x1, x2? That is true. Let us write that since x1, x2 is in the original ideal. is this clear yeah x 1 2 x 2 2 is in the original ideal. So, you have to then put x 1 x 2 that then would be enough you can show that these things are now equality, it was correct and yeah an ideal is called radical. if the radical is itself. So, now we can prove a nice proposition which is for any ideal if you look at the 0 set and then you look at the ideal of the zero set, then you get.

So, I mean ideally you would want the ideal I right, but what you will get instead is the radical of it. You also mentioned that the radical of any ideal is again. Oh, that is not clear. So, look at this definition. So, in this if you have F1 and F2, then you have to show that F1 + F2 is also there.

It is not very easy. So, you have to do this as a homework exercise. So, basically F1 to the E1 is in the ideal I and F2 to the E2 is in the ideal I, you have to show that F1 + F2 raised to something is also in the ideal. What you can show is you just take just look at F1 + F2 raised to some big E say E1 + E2 + 1 that you would be able to show. is also in the ideal. So, sum will also be in the ideal, product will also be in the ideal, product will be easy to see and if you multiply anything from the outside to f then you come inside the ideal. You can show that radical is again an ideal, but for general ideals either radical may be bigger right as you saw in the example.

So, what you are getting here x1 is a bigger ideal than what you started with, x1 2 is significantly smaller than the ideal x1. So, radical operator may take you to a bigger ideal, when it does not then we say that it is a radical ideal. okay and what this proposition is telling you is that if you start with a radical ideal then applying z functor and then the i functor brings you back so i and z they are kind of inverse functors for radical ideals okay which is a great one to one correspondence result Is this clear, okay. So, let us do this in 2 parts. So, first you show that the radical of i is contained in i of z of i, why is that? Since you have to just read this, look at the zeros of the polynomials in i, the common zeros.

So look at a root, for example, p. So since p is killing everything in i, you can see that p will also kill everything in radical of i, 2 root of i. right because the 2 root is nothing, but just powering yes which is why all these polynomials they are actually killed by z i that is all. Yeah but this i z i could have been bigger than 2 root of i.

So, now we have to show the other side. So, for that let us take an element g in i z i. So, what does that mean? that means, g vanishes on z i points z i. So, z of i are just points in the affine space and on these points g is vanishing. Yeah, but remember what we want to show. So, we want to show that from just this hypothesis, we want to show that some power of G is in I, because we want to prove the proposition.

So, there we have to show that any G which is here, its power is actually in I. So how do you show this? So this is a cute algebraic trick, it is called Rabinovich trick. So I will just give it, it is hard to introduce it. So you just write this in ideal terms as follows. So the 0 set of i and 1 - g times y is. So, the points which annihilate which kill i and they also kill 1 - g y, they do not exist because if some point is killing i then it also kills g.

So, the second part actually becomes 1. right so this this zero set is actually empty so here y is a new variable okay so i can introduce a new variable and then i can introduce a new polynomial which is this one - the product of g and y this new polynomial has the property that common zeros of i they cannot annihilate it because they make it 1 that is why the hypothesis right. So, this is the algebraic formulation of what the line you had before and this will be very useful. So, let us define the new polynomial ring which is a with y attached. A was the polynomial ring in n variables and now we have n + 1 variables this is a new variable and in that we have this ideal I + 1 - g y ideal which has no common zeros.

So, when do you think an ideal has no common zeros in an algebraically closed field. so as we had discussed many slides back intuitively it's only possible when this ideal already has one right one is the only obstruction to roots so this is what i want to claim that one has to be in the ideal okay so this is called weak Hilbert, Hilbert's null challenge arts. So, Hilbert null challenge arts is the, is a criterion for ideals to have a the polynomials in an ideal to have a common root over an algebraically closed field. So, we are invoking that I will sketch this proof after we have finished this line of argument it is not very difficult, but instead of getting diverted by that let us just continue. So, intuitively it seems right that if this ideal has no root then the ideal should have and let us just continue with that. So, what does that give you? So, this means that 1 can be written as some polynomial a 0 times 1 - g y and some polynomial a 1 times a generator of i dot dot where a 0 to a m are coming from a prime we do not know anything else about a 0 to a m 1 - g y is the distinguished polynomial we have introduced and f 1 to f m are generators.

So, using the generators of this ideal which has 1 by the definition of ideal we get that there will be these a 0 to a m in the new polynomial ring a prime. What do you get from here? What good is this? So, here you do not have any special information about other polynomials except 1 - g y. what you can do is that y is a free variable so you just substitute y to be 1 over g so let us substitute y to be 1 over g and what do you see so now 1 is equal to a 1 had x's and y so y becomes 1 over g. and f 1 was free of y right f 1 was only in x 1 to x n there was no y in f 1. So, you get this is this clear and now you can just clear away the denominator by multiplying by a high enough power of g.

So, g to the e is now equal to some polynomial a 1 prime, a m prime for e a natural number and a i prime in the polynomial ring a prime. In fact, in a there is no y now, so it is just x variables. So, which means that g to the e is in the ideal i and that is all we wanted to show right from the hypothesis g in i composed with z of ideal i, we wanted to deduce that g to the e is in the original ideal.

So, we have both ways. Is this clear? This implies that g is in the radical. Why do we need generator? Why did we need generators in the algorithm? You can do it without generators. If you directly use the element, you can write it as some. No, no, no, what do you mean? The module, this ring has changed.

From A, it's now A prime. The ideal definition has changed. Yeah. So... Multiply something in I with something else. No, no, but the out is not 1. There are A1 to AM, M things which are out.

You also mentioned that these rings are new to me. Because of that... Yeah, let us just keep that under the rug. Nobody will ever notice that. Everything is finite, don't worry. What is more important is what is this weak Hilbert's Nostral Insights that I have invoked.

That I can sketch because it's a useful thing. It appears also in other problems in computer science. So, let us just do that as a detour. Yeah, if you do not understand what this big word means, it means that this was a theorem proven by Hilbert about the zeros of a system. So, Nullstellens is basically positions of zero and Zarts is a theorem. So, it is Hilbert's theorem on zero positions. So, you start with the assumption that zeros of an ideal do not exist and you assume that k is algebraically closed.

So, k bar is the same as k which we are taking k to be f p bar or complex numbers. that is fine. So, over that base field you have a set of polynomials t in fact think of t as an ideal. So, he gave a criterion in fact he characterized how can it be that z of t is empty by showing that actually one has to be in yeah I need a really need an ideal where j is an ideal.

that one has to be in the ideal j. So, the line of argument is as follows. Let me specify the ring also. So, Zj is empty, k is k bar and j is an ideal of this polynomial ring say A, in fact you can think of it concretely to be k x1 to xn. So, where do we go from here? So, what you should do is first you pick a maximal ideal that sits above j. Okay so there will I mean you can basically you can intuitively see that for any ideal or for any set of polynomials there will always be an ideal which is maximal which means that between this maximal ideal M and A either M is equal to A or in between there is nowhere to grow further to an ideal. Okay so this is what this means is either a is equal to m, I want it to be proper.

The ideal containing m should be a or m itself. Sorry, the ideal containing m. The ideal containing the maximum ideal be there itself or the whole way. No, that is fine, I want to move to the next steps. So, I want to say here that, I want to say that a over m is a field. This you have to show as an exercise you take a maximal ideal above j then just by the maximality it follows that a mod m will be a field. basically if it wasn't a field then it means that there are zero divisors things which are kind of which are not invertible well yeah there are non-unit elements in a mod m so you can use those non-unit elements to grow the maximal ideal m further which will be a contradiction ok so the only place where you stop is an ideal that makes A mod M a field.

So this is a property of the maximal ideal, in fact you can take it as a definition also, this is what we are talking about and which then is a field extension of this K. Now, note that k is already algebraically closed right and you are going to a field extension. So, can that field extension be finite extension finite algebraic, it cannot be right. So, then the only field extensions which are exist for example, over complex are once that are transcendental.

So, you have to introduce something which cannot be a root of any polynomial. So, this means that E over m is a transcendental extension of over k. Yeah, if not k. So, if this extension is proper then it has to be transcendental, but can it be transcendental? Again I mean if it is transcendental you can show that this will violate the maximality of M. so which means that actually a mod m is equal to k which means that we have a root, so which means that m has a, so z of m is non-empty which will then contradict that z of j was empty.

So, from this contradiction what you learn is that to begin with this J should have been everything, M was A. So, overall this means that the only way to come out of this contradiction is that the j was equal to a which means that fun was in j that is the proof. So, there are these algebraic concepts if you are aware of them you will see this proof as easy if you are not aware of them then you have to do some homework. But as I said in our bigger picture this is not quite needed because when we said that the only way Z of something to be empty is when 1 is there in that ideal that is quite believable you can take it and move forward. If you want the details then the sketch is here.