Computational Arithmetic - Geometry for Algebraic Curves

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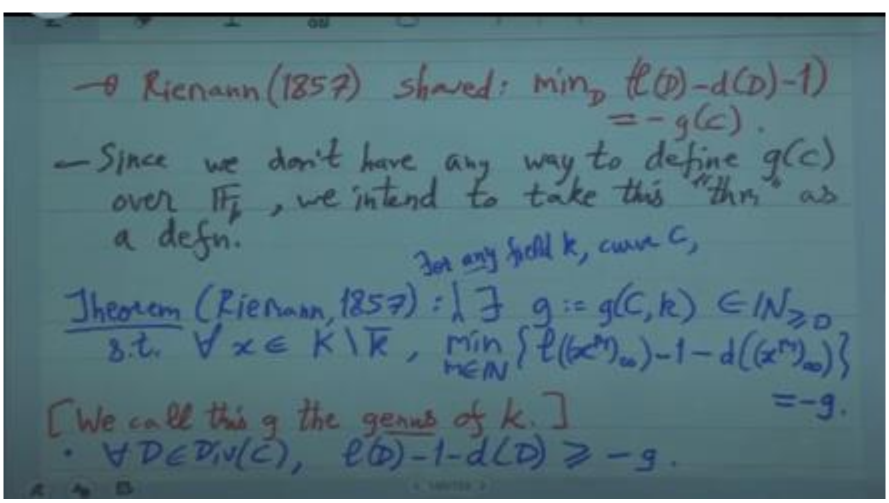
Week - 09

Lecture – 18

Rieman-Roch and Adeles

Any questions till now? So, we proved this theorem. Riemann's theorem which basically says that this l-d difference has a threshold as you go to bigger and bigger divisors. So here you are just doing x ^{m}. So this is a very simple divisor because x the pole of x is basically a single point and when you are increasing m you are just increasing its multiplicity. So what this is saying is actually something even stronger it is saying that you just pick a point on your curve and just keep increasing the multiplicity and that already l - d difference will give you a threshold and that threshold is what we define as genus. Now this is a bit different from what Riemann did because Riemann actually proved this as a theorem connecting L sheaf with the geometric genus.

So for him it was a property but for us it becomes a definition. So if you are interested you should read the complex analytic proof where genus is already well defined as wholes and then you see why these two things will be equal in that word. But in our it would not be important because we just take this as a definition of the number being called genus and it is independent of x. So that second part we have shown.



Yeah. Yeah. So, how can you inter the best way or the simplest way to see this is for the projective line. I do not think there is anything simpler than that. So, what is So for the projective line which means the function field is kx, so for k bar x what is the genus for the project, so the c is just p1, what is the genus of the curve in this case So you can take a point P and so what is LP? So how many independent functions can you write down over the line? so function will just be a univariate right and without the homogenizing variable is simply a univariate it should, so either you can look at the constant functions, so that is one type of functions that is always there, so in fact we should always consider Lp-1.

because constants are there other than constants what you the functions f will has to they have to satisfy f should have a pole at p of multiplicity 1 and not any worse right. So, if p is your point α you just look at 1 /x - α that function has a pole at α. So, that is just one kind of function. So, that so this is equal to 1 l p is equal right and what is the degree of the point, it is just 1 for algebraically closed field. So, you see that the difference Lp - 1 degree = 0 and that is the genus.

. Yeah, so that for that you have to look at the sheaf. So the functions, so this will be a k vector space. So clearly constant functions will be in the LP sheaf and also, yeah let us take P to be the point α. So you look at the function 1 /x - α, right. So 1 /x - α, principal divisor is what? Its pole is the point p, because x = 0 becomes ∞.

but it also should have a 0 right, every function has 0s and poles. So, what is the 0, the what sorry this is - p, what is the 0 that is ∞. So, it is ∞ - p. So, clearly this > = - p. So, 1 /x - α is in L p by definition because you want functions such that f + p > = 0 that is being satisfied and over the line you do not have any other option that is all.

Then multiplicity will increase for the pole. So, then what will happen is - p will become - 2p. then those functions will not be in LP, because you are only interested in functions whose poles are not any worse than - p. Sir, the poles will still be ordered one way. No, but when you give more 0s, you have to give more poles.

You give poles somewhere else. Yeah, but that is not allowed. No, none of the poles should be other than - p. So, that is the difference between this restricted L D sheaf and the whole L D sheaf right, because in this, so this is the whole L P sheaf. So, here you have to actually go over all the points on the projective line, you have to compare with everything.

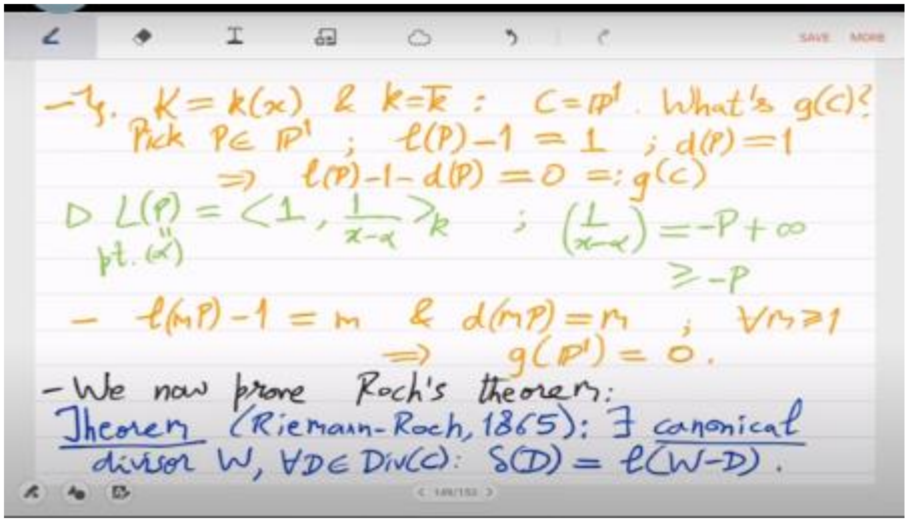
So, none of them should be pole except P or there is no pole in which case it is a constant function. So, this is what I was saying that L P - 1. is just this interesting function 1 /x - α and any linear combination of these will work also. So you get lp - 1 = 1 which is also the degree, so you get the best, actually in this case you get this equality 0. But this is not enough to complete the genus calculation because you have to also look at higher multiplicities.

So you also have to calculate Lmp-1 what is this and it is here that you have to see that this will be equal to m while the degree is equal to m also. it is actually true for all m here, which then means that genus is 0. That calculation I am skipping because it is the same calculation. So, that is why the projective line has genus 0. because even if you keep increasing the multiplicity of this point which is you are increasing the degree of the divisor.

Linearly the L functions are also increasing, the L mp sheaf is also growing and you can already see why it is just 1 /x - α and the other will be the new thing will be sorry 1 over exponential α square and the other thing will be, that is the third thing actually. So, you have 1, you have 1 over exponential α and you have 1 over exponential α whole square, that is the basis. So, for this answers your question. Yeah, yeah, but without jumping to bigger objects, it is actually telling you something much simpler. It is just saying that in your favorite curve is this property satisfied, what was being satisfied for the projective line.

That wherever you want you can put a pole and as you increase the multiplicity of the So, proportionately or linearly the functions also should grow. So, how far is your curve from this property? So, the as the curves grow more complicated they will you will see a loss and those are the holes. So, yes intuitively it gives you this idea of genus versus holes. but now purely algebraically so you do not need to see or define any loops or holes you can just measure it by the lmp ok so what we will do next is as promised we will now do Gustav Rock's extension of Riemann's theorem, which is to explain this degree of speciality for every divisor D what is the what is a good interpretation of delta D which is the, why is it that L D - 1 - degree + g is not 0, why is it positive. So, what is the explanation for that.

So, we now prove Rock's theorem. So this will require several lectures, but just to anchor you, here is the theorem. So Riemann-Roch theorem says that there exists something called a canonical divisor W, such that for all the divisors delta D = L of W - D. So, this is pretty nice because as the name suggest you have been able to identify this global or canonical divisor W which will work for every divisor D. So, delta D is actually another It is again L sheaf of some other divisor, but it is you have a - there.



So, you have - D here. So, it is kind of a dual of LD sheaf. So, it is I mean more advanced algebraic geometry this result is seen as a duality theorem. It is probably called Serre's duality theorem, but we will not do Serre, we will just do because you will see that even this is not, this will require some abstraction. So, the idea of this is the proof is this w - d is kind of a dual of d.

So for example can we realize it as linear maps from k | ld| to the base field. So, duality means that instead of this vector space LD, you should look at functions which vanish on the LD sheaf. right because you are interested in kind of - d. So, I mean the very vague idea is that you should look at the look at these maps. So, these are the linear maps which annihilate functions in L d, but not others right.

So, k | L d| is the subspace and dual of it is basically linear maps which evaluate a function to a field value which is the base field k. So you want to realize delta d as this object and we will try to achieve that. We will not be able to achieve that exactly, but we will kind of follow this template. Yeah, so that is one problem this, if you go over all the small k vector space of all the functions that will be even modulo LD will be infinite dimensional. So, that would not be good for our computational algorithms.

you have to somehow replace it by a finite object and then look at home of that to small k, homomorphisms to small k. Yeah, and then finally that thing has to be exactly equal to delta D, that is the goal. So, for that something completely new and mysterious is needed and Yes so that will be the following, so we will achieve this via by moving to more abstract functions. than the function field and we will call those things adults. So, this is also called the by this was the name given by Chevalier, but we will continue to call it Adele in some algebra literature it is called pre-Adele, but I do not think it will matter to you because you have not seen any of these things anyways.

So, I will just call it Adele. So, this will be I mean every function is an Adele, but Adeles will be more complicated looking. which is why I am calling it more abstract than the function field. So what is this? So let us directly give the definition because definition is not very hard, it is just that it is not clear why this thing will be useful. So for curve C again smooth projective curve given by a transcendence degree 1 function a tuple R of rational functions is called an Adele, I leave the accent, so it is called an Adele. the valuation of all these coordinates functions in the coordinates is non-negative except for finitely many.

Yeah, those things are equivalent. The old remark continues everywhere. If you are in a finite field, small k is a finite field then you have to talk about p as a cloud of points or a ‘ ideal. If you are in fp bar, algebraically closed field small k, then you can work with actual points. So, it is an infinite tuple first of all which is the problem, I mean which is a bit strange.

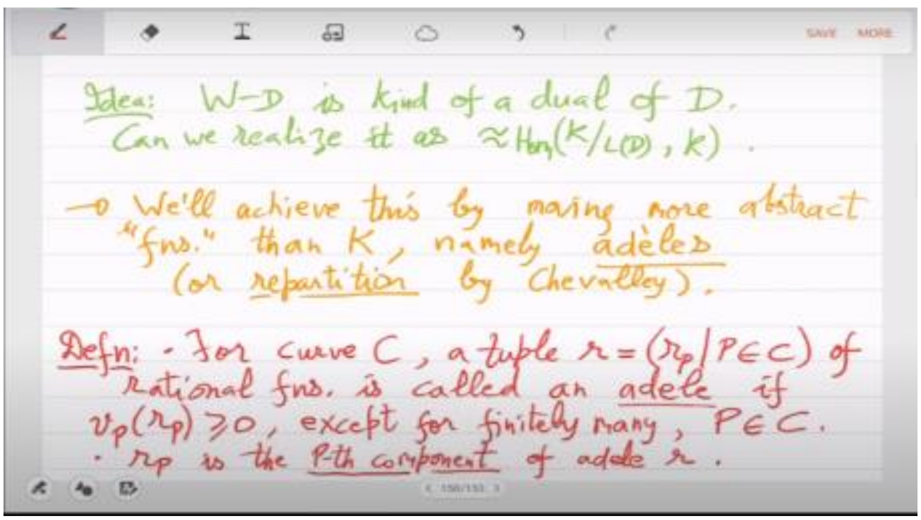
So, for every point you are basically assigning a rational function in a syntactic way. The only constraint is that except for finitely many points everywhere else the valuation at that stock should be non negative. So for finitely many places p valuation could be negative which means that the function that you have put at that place. So think of this as like these are infinitely many points you can even take the projective line. So on every point you put a function and you make sure that only in finitely many places the functions you have put actually have those poles.

Everywhere else it has to be non-poles. . Rational map that we defined, no, no, no, it is no, no. So, this will be completely different from whatever we have seen till now. Yeah but see so it is important to realize that this is very different because there is no global rational function which comes even close to this.

Because, the places where valuation of RP is non-negative, you essentially can put very different functions for point P1 and for point P2. So, it will be impossible to actually see what is there a function in the function field, which is close to this. So, this R can be very very different from whatever function you look in the function field. Yeah, because it is actually a tuple of infinitely many rational functions and this is truly infinitely many. Well, anyways we I mean continuity the we only talk about Zariski topology.

So, you can define continuity there, but I would not so it would not be needed. But yeah, the topological definitions then you have to refer to Zariski topology. I mean if you force it, it will not be very different from what you have already defined for Rp. Rp is a rational function, so you can and it is defined at that point. yeah but you do not know how it behaves on other points and you cannot you cannot definitely compare between the function you put at p1 and the function you put at p2.

So, they can be completely independent and this you are doing infinitely many ways. So, that is an Adele. Yeah right so that is true for functions in the function field if this small r you just pick a function from the function field say y or x then you know that it can be negative only at finitely many points, because negative means that you are talking about poles, now you already know that for rational functions poles are only finitely many, we have shown this in the first month. So, both poles and 0s are finitely many, so this is just taking that property of functions which is why functions are contained in an Adele, every function is an Adele, Adele element or an Adele actually, every function is an Adele, but Adele may be very, very far away from a function. So, Rp is well obviously called the pth component of the Adele R.



And the set of all adele will be fancy A and 0, the tuple 0 that also fits in right why do I want to say that separately, I think it is already included. Anyways, so the set of all adele is we will call that this a for the curve. yeah so the definition is simply I mean it is just at this point it is just algebraic or even syntactic you do not understand why this will actually help so let us now prove some properties so proposition 1 yeah from two propositions it will be clear that this object is necessary for what we want to do so proposition 1 shows that there exists an embedding of functions into adele. So, functions are adele's, A is a k algebra, but it is not a field. So, here already you see that the function field was a field, but adele is now not a field, but it contains the function field as a subset.

Valuations can be defined on adeles. So, for all point valuation extends from k to adele.

So we know what a valuation is with respect to a point P on functions. The same thing we can actually extend in a nice way to Adele also. And fourth is the analog of LD sheaf.

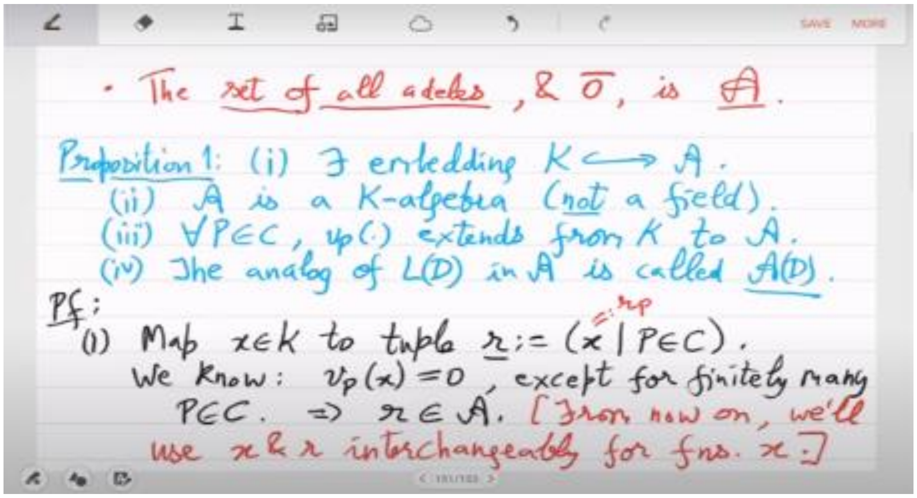
is called ad. So, this is our definition and it also will be a well defined object. So, by analog I mean just like you said that See once you have the valuation from property 3, now you can do the same thing which you did to define LD sheaf which is there you said that all the functions f such that f + d > = 0. You can now say the same thing all the adults are such that r + d > = 0. And what does > = 0 mean? Compare every valuation for every VP. So AD sheaf is actually what we are looking for, this is why we defined Adele's.

So AD sheaf will have much better properties than LD. So you remember LD when we were looking at the dimension of LD ' over LD, we did not get an equality, we only got our upper bound. So for Adele AD we will actually get our equality. So you can think of this as a way rock corrected the issues with LD sheaf. Any questions about these propositions? So I hope that this, the proofs will be very simple for you.

Some of them you can already see. But let us just go through them to warm you up with Adele's. So how do you show, so what is this embedding? Yeah, the same function you put in every coordinate and call that tuple R. So, this will just be x for every coordinate p in C. So, this will be the pth coordinate of R. And this is fine because you know about functions that it has finitely many poles and also finitely many zeros, but you only want the first half.

So, finitely many poles condition is satisfied. Everywhere else the valuation is non-negative. So, in fact we know that the valuation of x is 0. except for finitely many points on the curve. So, you actually know a stronger property. So, everywhere you will see 0 except in finitely many places where it is positive or negative.

So, that means it is an ideal. So, we will abuse the notation so from now on we will use x and r interchangeably for functions correct so i mean when i want to talk about other less coordinates i will have this infinite tuple everywhere x when i want to do not go into those coordinates I can simply say that x, function x is an Adele. There is a, obviously there is a difference because x is a single element while triple has infinitely many coordinates. But with this understanding or this embedding there should not be any confusion. So, I will just use x as an Adele also. Okay second property is about k algebra that is also is easy.



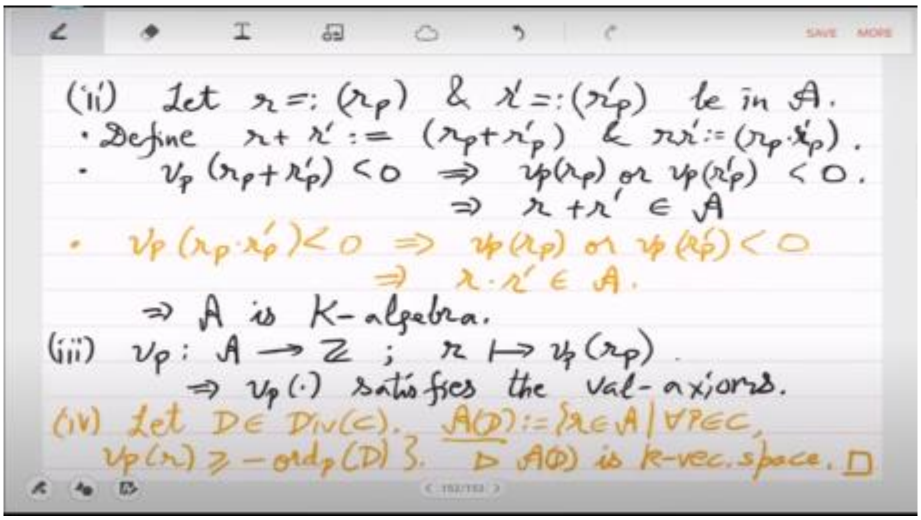
So, I have to show you how to add and multiply. So, let r be rps and r ' be rp 's, these are adels So how do you add R in R '? Well coordinate wise there is nothing else that you could have done. You have infinite coordinates so you can simply just add coordinate wise. Why is it an Adele again? Because when you look at the valuation how many times can it be negative? and let us also define in the same way rr ' and let us check whether these things are ideal. So, valuation of rp + rp ' negative means what? No, if it is negative then it means one of them is negative. But the possibility for that in rp's and in r ' p's is only finite.

So, it is only the Cartesian product of the finite locations everywhere else you will see non-negative, so that is done. So, this means that rp + rp ' is an Adele, sorry r + r ' is an Adele and what about valuation of rp times rp ' being negative. this again since it is a sum one has to be negative same thing. So, this again means that the product is an Adele. So, we have shown till now three things one is that function field is embedded So, in particular the base field small k is also embedded and you have addition and product.

So, you get a big K algebra, it is an algebra over the function field. Third property how to extend valuation. so valuation should send an adele to integers how will it do it so on r it should do what yeah i mean you just look at the pth coordinate you forget about the rest and show you can just check that VP satisfies valuation axioms, because you can check it on addition and product and it will be just it will just inherit the properties from the single function RP from that just pth location. So, that becomes a valuation. So, you have kind of this metric now on a del just like you had on functions and with this you have now the AD sheaf.

So, let D be a divisor which is just a sum of points formal sum of points on the curve so you can just define ad to be those adels such that for all the points on the curve the valuation on that Adele to be > = - order of P in D, right. So same syntactically the same thing that you did with LD sheaf. And the same way that you can add elements in an LD sheaf you can also add elements here. So, R + R ‘ both R and R ‘ being in AD sheaf you can see that the sum is also there. Because each of these inequalities that you are talking about this is only for a single valuation which boils down to just the pth location.

so the sum will also satisfy because the sum is coordinate wise. So, you get that ad is a small k vector space. So, that gives you all the properties you wanted. So now we will get more interesting things once you are warmed up with Adele.



So the first property is that Adele R has an associated positive divisor dr just like every function has a divisor it is at the same thing No so I think I should define this what I mean by this. So what I want to show is that R contained in E dr that's what I mean every ideal R is contained in some Ed sheaf for some divisor D.

So what was this for functions, if I take a function f then it is contained in which ld, yeah it was the this I think the ∞ part, f ∞ or f 0, yes. So that is the thing you basically you want f + d > = 0, which means that the negative part which is the poles that f ∞ part you should add, so that you get a positive divisor. So, in the exact same way the same proof will actually give you also dr, it is the ∞ part of dr del, but we will prove that for divisors. d ‘ > = d, again we will look at ad ‘ | ad|, which is a vector space and the dimension of it over the small k field. So, this was we showed an upper bound for ld ‘ over | ld|, what was the upper bound, degree difference.

So the amazing thing is that now with Adele this will be an equality. So you can already see the signs of where we are headed. Since this has become an equality, so the losses which we were incurring in Riemann's theorem, we will be able to analyze that because this equality will kind of give you other equalities. Wait what? l of d also in the beginning a priori was not finite dimensional that we have to prove. So, we actually first prove this dimension upper bound and from that we deduce that l d is finite.

For a d. And now for a d maybe you can do the same you look at degree d = 0 and you see. So, what is a 0? so you will want a of 0 will be those are such that in every coordinate you have valuation > = 0. You can just have different constants in different tuples here, yeah that is interesting. So, a 0 itself is actually Yeah infinite dimensional that is true still so it is yeah it is actually very mysterious then that the dimension of the quotient is finite. Yeah we will prove this and finally the reason why we defined all this is that the dimension of AD, sorry A | AD + K|.

So, what is this? A is the whole Adele algebra, AD are the I mean this based on the divisor the sheaf AD sheaf that is a small k vector space and big k is also a small k vector space. So, we can add them that is again a vector space and we are looking at A vector space mod this both of them are infinite vector spaces, but the dimension of this quotient vector space is delta d. Why is it a direct sum? We do not know that right because big D divisor is arbitrary. So, it usually will not be a direct sum actually.

Usually it would not be right. Usually it will be right because if your A of D contains constants then your D should No, but A D will contain functions. So, already there is an overlap. A D has functions let us say because in particular it has ld. So, ld is a bunch of functions that is shared with k, capital K.

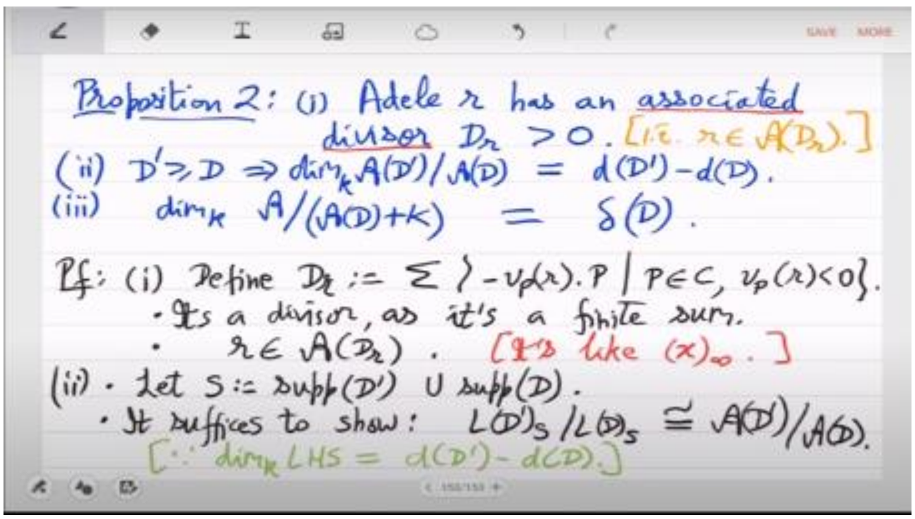
No, no small k will be is already there that is capital K. So, usually it is not a direct sum there is a huge overlap in particular as I said ld is an overlap. But when you mod out adels by this you get the degree of speciality. of the divisor D. So, we have characterized already and this would not be very hard, we can finish the proof right here. So, already you see that this A | AD + K| is the dual object of LD, this is the dual, it comes from Adele.

so and actually the technical reason is property 2 the property 2 is an equality so it actually gives us a better handle but you will see so first I think we already saw so just define dr to be kind of this poles like you did for functions f ∞. So, here what you will do is - valuation r sum over that for those points where valuation is negative. that's the sum. Now it's a divisor as it's a finite sum.

So, it's a divisor and you can talk about ADR. So when you look at ADR, clearly R is contained because when you do the sum R + DR, you will see that at every point the valuation is non-negative. So this is like F ∞. that is the associated, that is the meaning of associated divisor. It is like we could also, we could also call it R ∞ actually, right.

So, more interesting thing is the equality property, let us do that next. So, as before S will be the support of D ‘ and D. It suffices to show that, this we had already studied restricted L D We are already shown when we did LD sheaf that this restriction, quotient of the restricted LD sheafs has degree, has dimension difference of the degrees. So all we have to show is that this is actually isomorphic to Ad ‘ | Ad|. So, by the quotient of LD restricted result will be done.

So, I will just construct this isomorphism. So, this you can actually see as the place where rock might have caught in the idea of looking beyond functions. It is basically you want an object which is slightly bigger, in fact infinite dimensional, but in the quotient you get it equal to the isomorphic to the restricted LD sheaf. True, but cohomology word was not there, yeah. Since dimension of left hand side is degree difference.



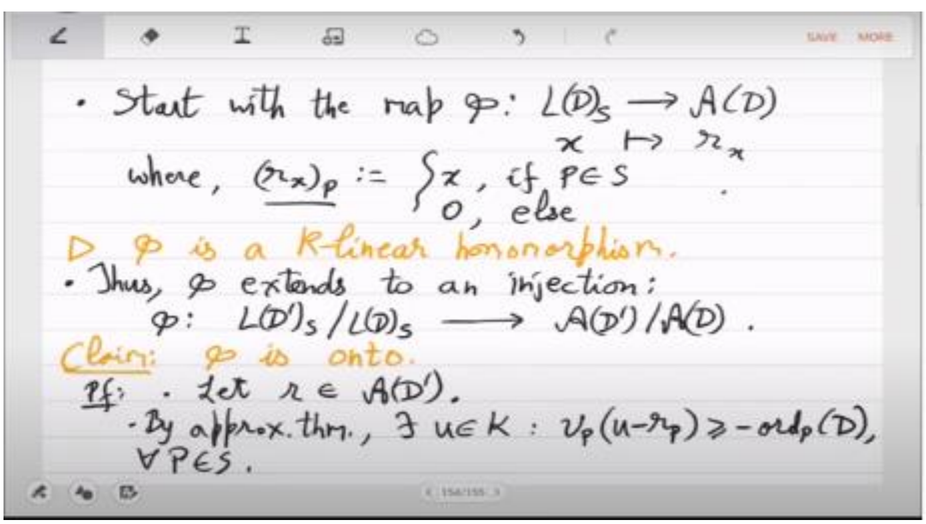
So, we have to start with the LDS, already LDS what is the map to AD, let us do that. So, a function x should be mapped to Rx which will be slightly different from what you expect, it is not that every coordinate will put x. So, where Rxp is defined like this. So you put x if p is in the support else 0. So this will be the correct map from the restricted LD sheaf to AD you only put x in the places coming from S other places you put 0. So this phi is clearly a k linear map of vector spaces, you add two things x + x ‘ and you can see that it respects the image, that you can see it simply by coordinate I mean looking at the action on the coordinates.

and right, so LDS is basically the modulus right, so LDS is going to AD, so 0 is going to 0, so hence we can now extend phi to the quotient subspace, so phi extends to an injection from LD ‘ sub s | LDS| to AD ‘ | AD|. claim is that this is onto as well, but is it clear that this is a injective homomorphism So if phi sends a function f to 0 which means that its image is in AD. So then in the interesting places you only focus on these finitely many interesting places, there you are saying that it is 0.

So which means that it is 0 in LD ‘ as well. ld ‘ s, in ld ‘ s. So, injection is actually easy. Why is it onto? That is the tricky part, because previously we only had an upper bound, we did not know whether it actually reaches the upper bound. So, this is the hard part. Why is phi onto? So, we have to show that every element on the RHS has a pre-image. So let us take an element let R be in Ad ‘ it is an Adele, so by the approximation theorem of functions that we saw in the early on in the first month or second month, there exists a function such that u - rp for all p in S.

So, note that S is a finite set. So, we just look at those points and we want the valuation of. So, we want u - rp to have I mean in a way approximating this poles that D is giving you. So, on the point is that R is an Adele, but it has I mean it has infinitely many coordinates, but if you only focus on these finitely many coordinates coming from S, we want to only approximate that part by a function u up to the orders provided in D. So, that will give us a function u by the approximation theorem and let us now see whether this u is a good preimage.

We will show that it is, that would finish the proof. So, let us just check the valuation of this function. So, v pu is what? It is the minimum of u - rp and rp. what is valuation of u - r p, this is at least - order of p in D and what is valuation of r p. So, by the assumption that r is in A D ‘, that also lower bound you know minus order D ‘.

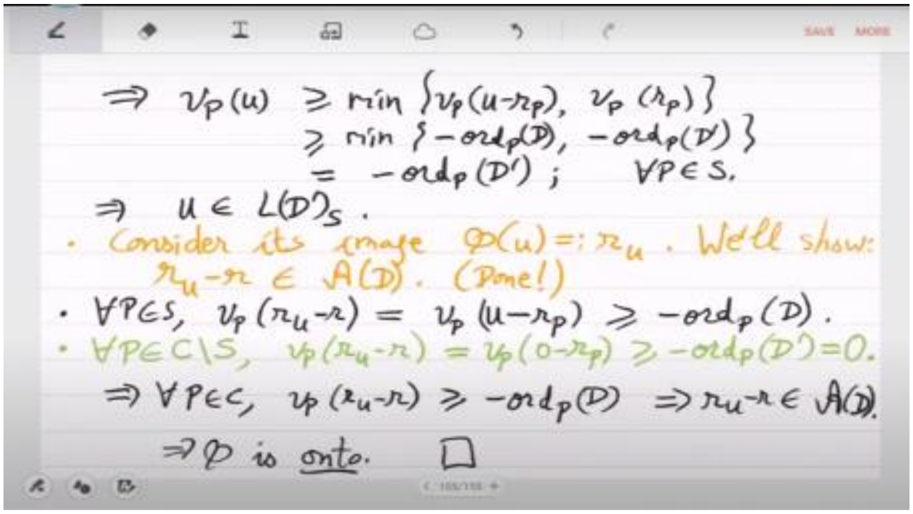


which gives you d ‘ > = d, so yeah. So, this means that, so this is true for all p in s, so which means that u is a function in l d ‘ s. So, that is what approximation theorem has given me. So, it is a good preimage we just take this as a preimage and let us look at its image I mean we have to check the image basically. So, consider lets call it ru its a function so phi sends it to an adel ru is the adel yeah all that is left to check is it r is ru r we will show that RU may not be R but the difference is in AD.

So, with that will be done. So, we just have to show that RU is actually R | AD ‘|, | AD| because in the image you only want to be correct up to mod AD. So, let us do that. So, yeah so go over the valuations which is what is the valuation of Ru. it is basically the, so it will be valuation of u, because at p it looks like u, so u - and in fact in r you just look at the pth coordinate this, yeah look at the pth coordinates that's ru at p is just u and r at p is rp and that we have already by construction we have this.

So for P in S by construction we have this, this is not surprising. What will be surprising or what you have to carefully analyze is all the other points because now you are in Adele world, so Adele world has infinitely many points. What is happening in S complement? In the curve - S what is the valuation? So, Ru at those points we have defined via phi to be 0, we put 0 there, so 0 - Rp. So, what is valuation of Rp this we, what is Vp Rp? That is by the definition, so r came from ad ‘, so you get d ‘. So, there is a difference, these good points, on good points the valuation is by - order d, on the bad points it is - order d ‘.

but anyways the bigger one of them is minus order d so you get that. So this means that for all the points on the curve Ru - R > = -, d ‘> = d, no will this be enough, wait, no, no, no, no, so there is something I missed. What is this order of d ‘ at p, what is this? see P is outside S right, so it is outside the support of D ‘, so it is actually 0. So in all cases it is at least order of D, which means that Ru - R is in fact in Ad, right that is it. So this means that phi is onto, so phi is an injection and it is subjective, so that makes these two isomorphic which means that we have the degree equality. Yeah, any questions? So, once you have seen the proof now the idea is you can go back and see that the idea is pretty simple.



It was just you took an Adele from Ad ‘ and in those finitely many places you approximated a function u and put that. So, basically from an Adele you are able to get a approximate function and once you have an approximate function you use that in Ld ‘ s and just finish the calculation that gives you an isomorphism. Yes, although left hand side you had restricted LD sheaf, on the right hand side there is no restriction, it is actual AD sheaf. This is why this Adele was created to go out of this support restriction.

So, with that we can prove property 3 also. be able to finish it today, let us just do 50% of that, yeah, so property 3. So, let us show that the dimension of this A | ad + k| is at most delta d. This is what we will show and next time we will show that it is at least delta d. So, suppose the there exist each elements, each adels R1 to Rh in A which are k linearly independent | ad + k|. yeah and for this basis we will work out some calculations which will imply that H has to be < = delta D.

So let us do that, so by one get the associated divisors get devices dr1 to drh all positive such that for all these ri is in adri. So, these are kind of the ri ∞ divisors. and we can take the LCM of these divisors. So LCM will be basically you look at the points which are appearing in these divisors and pick the maximum order.

So D ‘ is kind of a upper bound on all these positive divisors. So D ‘ is the biggest amongst which is bigger than all these and we will now work so in particular this means that Ri is in Ad ‘, this is the common one. So, we just put took the common one and this actually now helps us in deducing that R1 to Rh is a basis of ad ‘ | ad + k|. I mean we started with the basis. So since it was a basis before with ad ‘ also it's a basis and now we have this quotient space.

size h of ad ‘ + k | ad + k| is that clear. So it's only some pre-processing we haven't really done anything great. What we have to now do is we have to actually calculate the dimension of this and show that it is somehow related to the degree of speciality. So here is the calculation at the level of subspace.

So ad ‘ + k let me ignore the bracket. So, this quotient subspace is isomorphic to this. Actually bracket somehow will be necessary, let us continue with that. So the first is this, I mean it is simply a property of vector spaces. So you have v | u|, so what I am saying is I can do v | v ∩u|. So you can prove this by just pick up basis of ad + k the modulus and extend it to the bigger basis and then what.

So this actually just is. extend the basis of the denominator to the numerator. Do you see that will prove this? I mean this is not a property to do with Adels, it is just a property of vector spaces. This v | u| you are writing as to be precise u + v | u| is isomorphic to v | v ∩u| it is just this that we are using. Well because of this basis argument you can see that the dimension of left hand side vector space is the same as the RHS space, because it's the, so when you are extending the denominator basis to numerator suppose you added R basis elements that R is the dimension of both left hand side and RHS. So the dimensions are equal that's one thing.

Now why is it an isomorphism? I mean the isomorphism you can just use the obvious thing. The vectors in U are considered 0. So, essentially in LHS you only have to consider vectors in V and that has a obvious image in the RHS. So, using just basic linear algebra you can see that the two vector spaces are isomorphic even if they are infinite dimensional. You can turn this into a proof and yeah let us finish that next time because such things I have to repeat.