Computational Arithmetic - Geometry for Algebraic Curves

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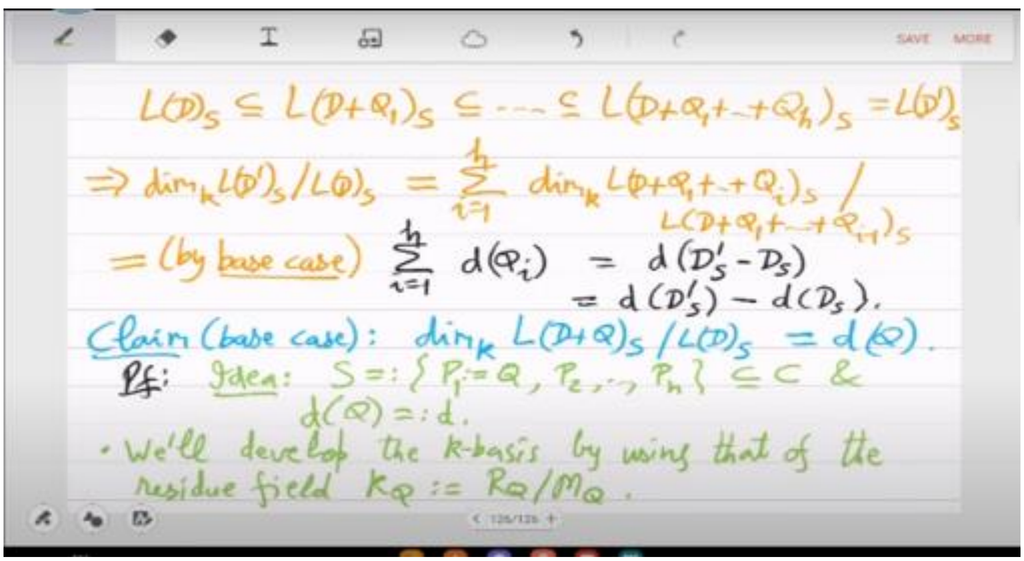
Week - 08

Lecture – 15

Riemann-Roch Spaces II

Okay any questions? There is no geometry in that, that is just an algebraic fact. . It is, I mean, in some sense those are infinite things. So, infinite things, they could be in bijection even if one is a subset of the other. So, you have already seen this with even numbers and integers. So integers are as many as there are even numbers is the same thing.

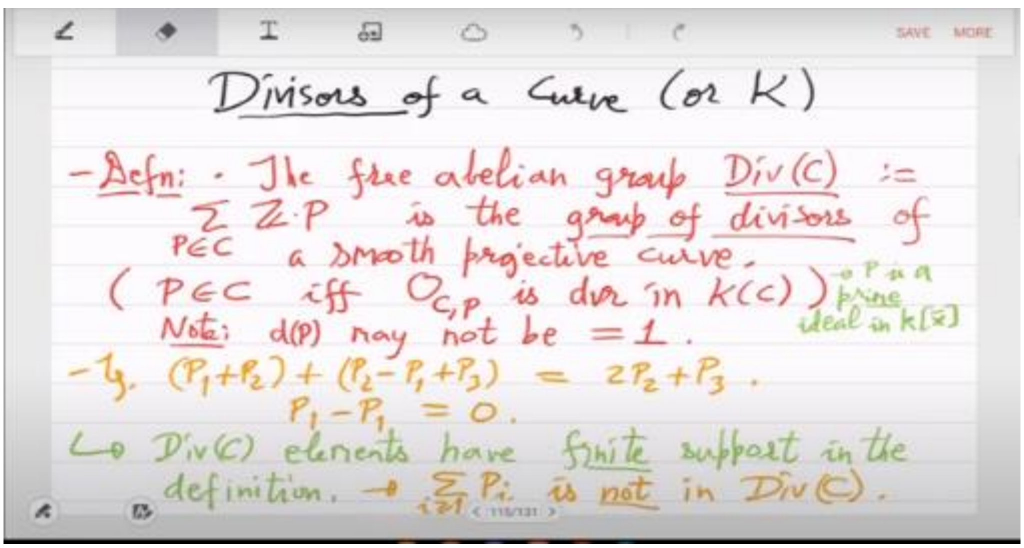


No, no. . So you have kx and √x and the map which we define is, yeah that is the only thing you can do, so you can map x to √ x. and then this is a map only on defined only on one element, but you can generalize it to I mean you can extend this to all the elements right, because now all fractions like this become this.

So, you can see that this is a φ is a field homomorphism it is a field homomorphism and you can see that the it is not the 0 map of course. Now, you know that field homomorphisms or isomorphism are embeddings right. so you are saying that why don't you take this the polynomial ring so where is $t^2 - x$ mapping to so this is actually mapping to this that is the mistake So irreducible will go to irreducible. So this x that you are thinking, there is actually no common x. This is x and this is some y.

It has, there is, I mean, sure they have a common word which is kx, but I don't need to think about that. I can just think of this as ky. And what we are showing here is just that kx and ky are isomorphic. Now y you can take to be any function in x. Same proof will work.

As long as you don't take y to be, I mean y has to be transcendental. It should not be some algebra, it should not be in k bar. So this proof is actually true for any y not in k bar. that is the thing it's yeah but definitely it's confusing that's why we saw this example geometrically. Any other questions? So, what we have been doing two weeks ago is we started divisors of a curve which for us is now just a function field big K, transcendence degree 1.



So the free, so divisors of C is just formal combinations, integral combinations of points, where point should always be thought of as a prime ideal. So in the polynomial ring, it's just a prime ideal that we are looking at. So this prime ideal actually may not represent just a single point. I mean, it could be a single point in the algebraic closure or it could be many points. When it's many points, then we have seen examples where it's basically the conjugates that are clustered in a cloud.

So there is this concept of degree of a point. So it's not really a point P, but it's a cloud of points. And then we define degree and div 0, support. Every point P actually defines a valuation, which is, if you remember the theory that we developed before, you basically have to look at the germs and the maximal ideal. And since the maximal ideal will be principal, its generator gives you the valuation.

that is by uniformizer. Also small k in this divisor theory small k may not be fp bar, it can be any finite field as well. Fine then we saw that for a rational function, I mean we saw this example basically. This $x2^{2}$ $x0 - x1^{3}$ example was instructive. So function here is x1 / x2 and we saw all the zeros and poles.

So zeros are p1 and p2 and poles are p3 and p4. So the principal divisor of this x1 /x2 is p1 + p2 - p3 - p4. That is the, it's an important example to remember because here the degree comes out to be 0. We'll show this actually now for all rational functions. And so these divisors are called principal divisors div A, which correspond to rational function poles and zeros with multiplicity.

This is a subgroup of div C. We showed that, so div A is a subgroup, div 0 is a subgroup. In the future we will show that div A is actually a subgroup of div 0. It's a tower. Yeah, then we define this LD sheaf which is all those rational functions whose principal divisor is, whose divisor is at least - D.

So this is how we'll quantify now. We'll actually study approximation theorem in greater detail via this sheaf. I mean this single definition will produce large amount of theory which is very important in the whole of maths. Yeah, so L-D sheaf, remember it collects, what it intends to do is it collects functions x the functions x whose poles are not any worse than d or - d. So the poles cannot have multiplicity worse than that in d, but it can be greater.

So what first thing you should wonder is whether this I mean you can show that this LD sheaf is a vector space that is easy. It is easy because of the valuation property. I think that is the. Yeah, that's the proposition. LDS actually, let's recall what is LDS.

LDS is similar to LDSheave, but it's restricted to a subset of points. So, all the rational functions whose poles are not any worse than S points in D. S are some points, so we are not looking at all the points in D, but only those points which are contained in the subset S. We will generally take S to be finite subset. So, divisor may have, yeah let us look at this distinction.

So, there are these two inequalities x >= - D, D is a finite sum of points. But this inequality is making a statement about infinitely many points because you are interested in X whose zeros and poles although they will be finite, >= - D means that it is not allowed to have any pole outside of D. and it is allowed to have as many zeros as it wants. They can be from anywhere but the poles are restricted by what you see in D. Actually it is not completely true.

D may have both positive and negative order points. So for positive, it puts a constraint on the poles of X and for negative order in D, it puts a constraint on the zeros of X. But it is a statement about all the zeros and poles of x while this second statement is not. This is not a statement about everything. So, this is a statement only for points in S.

How do they behave? It is not a statement about points which are outside S. on that it is silent so yeah it's not immediately you may not immediately read it that way but it's actually what is meant because this when you write xx on the left look at the definition of ds so this so you are only looking at zeros and poles which are contained in s and you are comparing them only with only for those you are looking at - t. So only about zeros and poles in S. So that is an important distinction. outside s x may have zeros and poles that is they are free there is no constraint on them.

So, then clearly LDS can be bigger than LD, LD is contained in LDS because LDS puts lesser constraints. So, there will be more functions and we have this property that LDS is a k vector space This is true also for LD and you have these comparisons. If you pick a bigger divisor then the LD sheaf is bigger. If you pick a bigger subset then the LD sheaf can reduce and yeah. So, these things actually follow from valuation.

okay so now let us come to this theorem that we had started which is a which is a highly non-trivial quantitative statement about the LD sheaf so what this says is that if you pick a bigger divisor then the as you know LD sheaf gets bigger and the amount by which it gets bigger is exactly given by difference of degree okay and The way we will prove it, this we already discussed, we will basically look at the difference of ds ‘ and ds. So, it is a sum of qi's, qi's are points on the curve. It is so because we have assumed ds ‘ - ds is non negative. So, every point in the difference divisor has order non negative. So, we can write them as qi, qi may be repeated, they may not be distinct.

And then if you look at this tower, how is D growing to D ‘ one point at a time. These are all, this is a chain of sub vector, I mean subspaces. So you can actually count the dimension step by step. So the dimension of L D ‘ over L D is sum of the middle quotients, their dimension. And what we will show is the base case that for the middle quotient, the degree is just of a single point, dQI.

Okay, and then you'll get the result. So let's look at this base case. When you add a single point, we want to show that the dimension of the quotient is just the degree of Q. In particular, if your Q refers to only, it is a cloud of points where there is only one point, then the degree is just 1. So, L D + Q grows by only one dimension, exactly one.

It is a exact result that we will now prove. So, how will we prove it? it's well unsurprisingly we will actually look at the this neighborhood around Q right because Q is the new thing so we want to understand what happens when you add Q to the divisor D so you look in the neighborhood of Q which means you are looking at the germs which is this yeah I guess RQ, so you are looking at this ring RQ, the germs around Q and you mod it out by the maximal ideal which are the germs that vanish at Q, so that gives you the residue Now this residue field, since we are in the general field case, this residue field may be slightly bigger than k, small k. So there will be a basis, it's a field extension of small k. So it's a vector space over small k, so you look at a basis of this.

Oh, yeah, yeah, sorry. This S is just... In the claim, S is just a finite subset of points. Yeah, yeah, and it has to contain Q because that was in the theorem.

In the theorem... well the way we are looking this ds ‘ - ds this qi is that we are looking at right these are all in s so q1 q2 everything is in s yeah so in the claim the point q that has been added that is in s so we call that p1 and then there are some other points which are unrelated, they may be n of them, n - 1 more points and degree of Q happens to be D, let us assume that, denote that. So let that basis be oh yeah so this degree will degree by definition is actually the degree of this field extension so you get d many so let x1 ‘ to xd ‘ be the k basis of KQ the residue field when seen modulo MQ because RQ is just the ring that is just the DVR or germs the field you get only when you mod out by MQ maximal ideal. So, x1 ‘ to xd ‘ | MQ| they give you a basis. okay we start with that d is the degree of the point so so all action is now happening in the germs around this q and we'll try to understand l of d + q so so here we invoke the approximation theorem yeah so some some things here will be tricky hopefully you will understand better once i have given the proof but remember that these this x1 ‘ to xd ‘ that I have, these are the functions, right? Now we don't yet know whether they'll actually live in ld + q. So we'll actually identify elements inside ld + q.

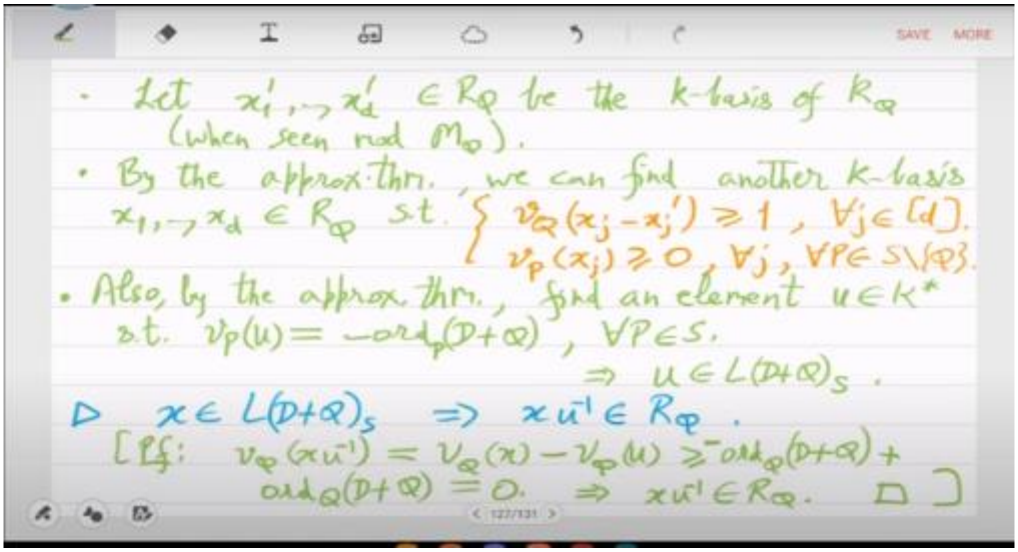
So for that we have to work a bit. So let's start that work. So we'll get it from approximation theorem. So by the approximation theorem, we can find another k basis x1 dot dot xd more functions such that so what we want is xj and xj ‘ should be roughly the same so by that I mean that the valuation should be at least one which means that xj and xj ‘ are the same |mq| right. So it is just xj is just xj ‘ + some maximal ideal elements, so they are approximately the same and for other points p the valuation is at least non-negative. Yeah, so this is a simultaneous set of inequalities that you have to satisfy.

It's given by the valuations of all the points in S. So with respect to this Q, distinguish point Q, xj, xj ‘ are the same, approximately the same. And for other points, the valuation is non-negative. okay this is simultaneous system can be solved by approximation theorem that we have seen two to three weeks ago so this will give you x1 to xd these are just the ‘ slightly modified is that clear and also by the approximation theorem we can find an element u in k star such that u is in, u's order is d + q. So, this we are doing because we are interested in the L d + q space.

So, we want this condition for all p. yes this in particular means that U is in L D + Q S but it is actually bit stronger because being in the L D + Q sheaf you just wanted >= 0 but here we have set it to 0 but this also this U also you can find by the approximation theorem and the advantage of this U is that Now you get any rational function in the L D + Q sheaf satisfies this condition. So, this U actually helps you to map the functions which were in the sheaf of interest, this L D + Q, it maps them into R Q. This ring which the ring of the DVR or the germs which for which we have a basis found. This exactly is the connection between the LD + Q sheaf and the ring.

Why is that true? Just look at the order. So, to show that it is in the ring RQ you have to look at the valuation with respect to Q of XU^{-1} and what is that? Valuation of X - valuation of U which is. So, valuation of X is because it is in LD + Q valuation will be at least. - d and - 1 right. So, - of this and the valuation of U is by definition this so you get 0 it was designed to satisfy this condition. so valuation of Q valuation of U and valuation of X are the same actually sorry yeah greater than equal so valuation of X you only have an inequality lower bound but for valuation of U you have exact matching thing so they cancel and you get 0 so which means that $X U^{-1}$ valuation 0 so it's in the actually since in the it is a unit then greater than equal to 0 sorry yes correct so it is in the DVR right so it is just a scaling function which allows you to connect now to bring everything in L, D + Q sheaf to the ring of germs and so now you can already imagine how the proof will work.

We will now try to show that the basis that we have found in RQ, it gives you a basis for L, D + Q. right. So, this basis here will become basis there and that will be your proof of the claim. So, for that let us now use the x j ‘s to make it work ok. So, this means what we have just shown in blue this means that $x u^{-1}$ has a unique expression in the basis up to the maximal ideal, so there is an x ‘ So ajs are in the base field and x ‘ is in the maximal ideal.



So you can write xu^{-1} | m q| in the basis because it is an element in the ring and you have a basis found already x1 to xd and ajs are field elements. you have something for about x so right so now we have to understand where are xju and x ‘ u i want to show that they are in the that would give you a basis in the LD sheaf. So, let us check that. So, where is x ‘ u for example? So, the valuation of x ‘ u is what? now what is the valuation of x ‘ x ‘ is the max is in the maximal ideal right so it is at least one and u is by construction this which = - order (d) because order of q in q is 1, 1 - 1 is 0. So, you get - order of q d, which means where is x ‘ u? It is in the LD sheaf.

Sorry, not yet. We have to prove one more property for other points. So, for points what happens for the other points for q you have understood for other points vp x ‘ u yeah what is this yeah so i guess we have to look at this equation now the green equation x = ∑+ x ‘ u so from there yeah this will not actually help we don't know those values but the equation can still help so valuation of x ‘ u has to be Can I say at least x? I can, right? right for x j u yeah if I can get to this then I will get then I will get this. How do I do the middle step? v p of x ‘ is greater than equals to 0 and what is v p (u)? v p (x ‘). No, we have not said anything that is the problem I do not know v p (x) ‘.

Yeah, it's only VP (xj) that is at least zero.

And VP of u, I have exact control. So... And VP (x)... Yes, I think I have missed this calculation. Something here is missing. so this part is coming from the fact that x is in l d + q so this is fine where is this part coming from should come from the equation because No, from the second equation x u^{-1} equals to summation of A i x j. Yes, yes. If you apply valuation on both side you get this quantity has to be strictly positive A i summation of A j x j.

No, if you pick a term a 1 x 1, valuation of a 1 x 1 is valuation of a 1 time + valuation of x 1. But a1 is a unit. No, vp (x+ y) is greater than vp (x). No, valuation has to be at least the minimum.

vp(x + y) > vp(x). So, vp (x ‘ u) is vp (x) - summation that is greater than vp (x). Yes.

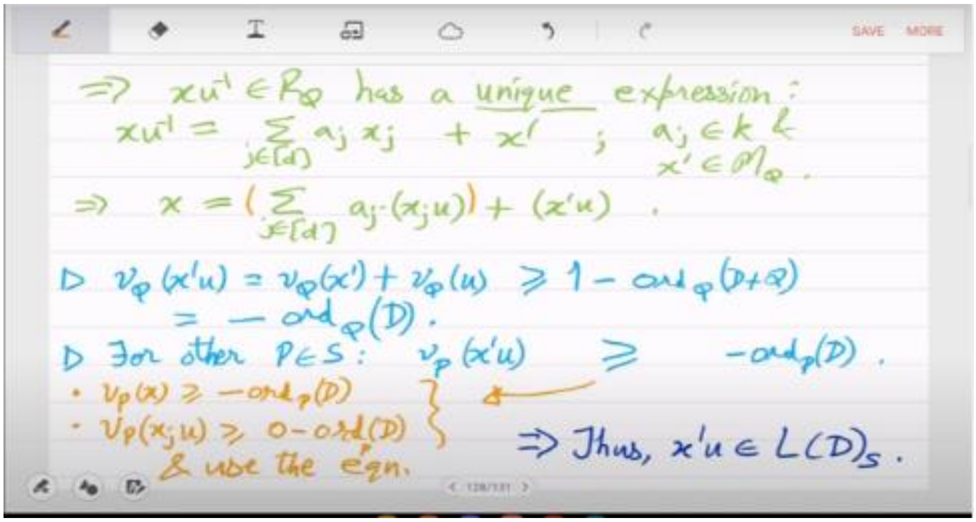
Get that. No, but how do we know that the valuation of x is... No, that part is fine, but how do I compare the valuation of this with x? It has to be greater than both, right? No, it is not greater than both, it is from greater than the, at least the minimum. So I have to compare valuation of X with XJU. Yeah, I do not think I can do it right now.

Maybe Deepthujit will tell this afterwards. Let me move forward. So, this will fill the detail. Because right now I cannot immediately think about... see what I know is okay let's do it so valuation of X with respect to P I only know from this L D + Q sheaf right that it is at least that in D and what do I know about valuation of XJU.

So, valuation of XJ is at least 0. and valuation of U is - order D. Oh, I see, I see. Okay, the middle term is not needed. I'm making the wrong, I'm making a stronger claim, which is not required.

It is just this. Yeah, that was the mistake. Yeah, is it clear? So, actually valuation of x, valuation of xj u all these I can compute for v for p. So, Vpx is at least order of d and Vpxju is at least also - order of d. So, then x ‘ u is forced to be at least that much by the equation. So, now I know about whole of s, I know about q, I know about all the p's.

So, I can now say that x ‘ u is in the LD sheaf. Is that clear? Yeah, so the zeros of this x ‘ u function is, they are not any worse than then - d and so that is good. So, the error term kind of the error term coming from the maximal ideal it actually maps into the LD sheaf which anyways we wanted to do mod right. So, that is so this is going in the correct direction.

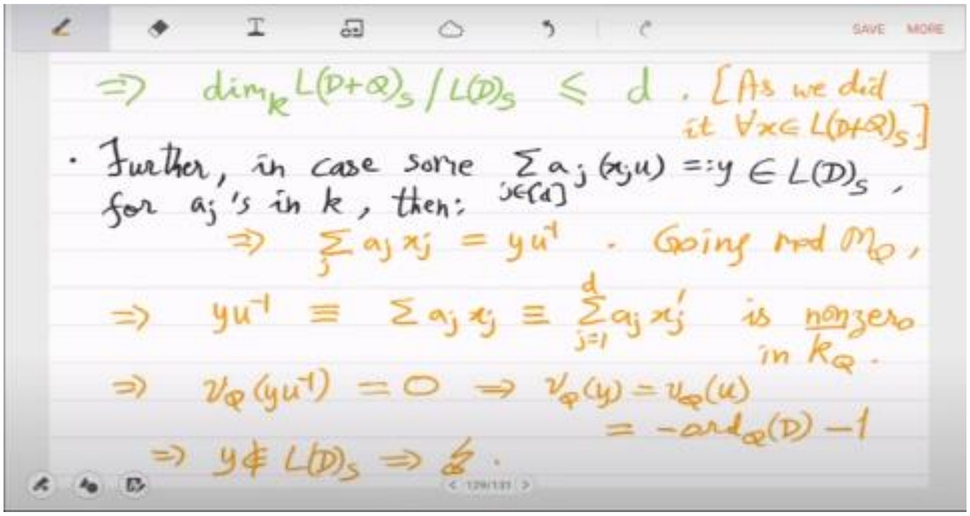


So, what we have deduced is that the dimension of L d + Q s | L d S| over the base field cannot be more than that equation right, that equation a j x j u you have identified this basis, you take any combination all these x's get produced in the L d + Q sheaf, modulo L d sheaf. so the dimension is at most d right that is what we have shown so we have shown the upper bound we want to show we want to make it equal now so let us do that oh less than dy let us as So do it for all because of this.

We actually went over all the x's in the L D + Q sheaf. Every x actually is generated by the same xju which are only D many. So the dimension is at most d. Now all we have to show is that these d elements, they are actually independent, they form a basis. So let's do that next. So assume that in case this y some y a j x j u becomes dependent which is it is in LDS.

Then what can happen? So then we want to show a contradiction, right? If a combination of xj use the map to zero model ds, it should be a contradiction. So what is the contradiction? So let's take out u. So ∑aj xj = yu $^{-1}$. and so which means that yu $^{-1}$ modulo the maximal ideal xj and xj ‘ are similar we will now use that property so this let xj and xj ‘ were equal |mq| so i get this and oh yeah so xj ‘ was also a basis yes so this x1 to xd was a basis |mq| so is x1 ‘ to xd ‘ that also is a basis so this is in then the is non-zero in KQ. It is actually a unit in the residue field because we are looking |mq|, so we are in the residue field, there is a, it is a unit, it is a unit, I mean it is a non-zero field element.

Sir, how did this give submission Ag xj = Ag xj ‘? Yeah, so that means that xj and xj ‘ are the same |mq|. MQ has this uniformizer, you are saying that its valuation is at least one. So xj - xj ‘ is divisible by the uniformizer. So yu $^{-1}$ is actually non-zero element and yeah I am not sure why this x i ‘ is needed for this ah no sorry no no we started with some arbitrary x i ‘ then we made it better by using these x ones ok ok sure yeah this is fine.



What this tells you is that yu $^{-1}$ is actually, valuation is 0. It is a unit in the residue field. So the valuation is 0. So this means that valuation of y and valuation of u has to be equal. and that is a contradiction why is that a contradiction well because valuation of u we know it is - order of q d + q right and if you assume that y is in LDS then y should have order at least - d but here it's one lower right so which means that y is not in LDS that's the contradiction so this contradiction means that any combination of xju is non-zero modulo the LD sheaf fine So this means that dimension of Ld + q |ld| >= d as well that is all.

That is the proof. So we have found a basis and it it is the full basis so we have found the exact dimension so this proves the claim base case and from base case you can go to the main theorem so we have proved the main theorem that dimension of ld ‘ |ld| is degree difference one point at a time you can do this Okay, any questions? Yeah, so this gives us a very powerful handle on the LD sheaf. I mean, now what we want to do is we want to remove the S. I mean, because we want to actually understand the LD sheaf, not the LDS sheaf.

But that can be done easily. So let's do that. in the following corollary. So, for any D ‘ >= D divisors dimension of L D ‘ mod L D, remember that L D is the subspace of L D ‘ over the base field k, this is at most degree of D ‘ - degree of D. So, this is weaker because it is not an equality anymore, it is only an upper bound. while in the theorem we had an equality. So why do we have this difference? The difference is because that s was actually a finite set.

We took a finite s. So finite s allowed us in the proof to do induction. But then when you work with LD ‘ without the S, without the restriction, the conditions are actually not for finitely many points, but for infinitely many points. So we don't know how to handle that. So we don't get the exact equality, but let us anyways get something. So We will use the theorem right.

So, let us define S to be the support of D and D ‘. Obviously, D ‘ and D are divisors. So, it is a finite sum of points take all these points that you see. that is S. And then what should you do? Then you should invoke the theorem on LD S and LD ‘ S.

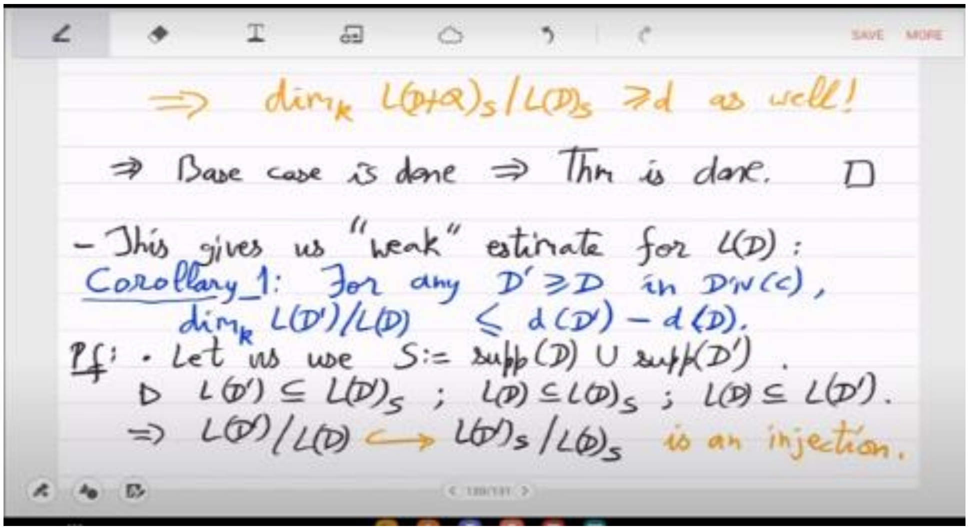
That will work. So let's just check that. So LD ‘, well you know that LD ‘ is a subgroup of LD ‘ s, you also know that LD is a subgroup of LDS and you know Ld is a subgroup of Ld ‘, you know these easy conditions of subgroup containment. So, from this actually you can make sense of Ld ‘ by Ld. So, Ld ‘ |ld| can be seen as a subgroup of this. Why is that? See the only non-triviality here is that on the left-hand side, LD you are treating as 0.

We are going |ld|. So this containment can only be achieved if LD is a subspace of LDS because on the RHS, LDS is considered 0. So when you consider LDS 0, you have to show that LD is subsumed in that, but that is obviously correct because you know that LD is a subgroup, subspace of LDS. So, can be seen as it's injection it is an injection so, is that clear? So, nothing else other than LD is in the kernel that is what you have to show here. Yeah, but because of these containments I think you can complete this, this is almost obvious and once you complete this you get the upper bound by the theorem.

No, I am not sure about the injection because if you take an element outside L D. Yes. You are considering that its class in L D ‘, modular L D.

Yes. Now. And then it is also an element in L D ‘ S. So, you can go to the image. Yeah. It is the same element.

And you have to take now modulo L D S. Yes. But L D S is a bigger space. So, it can be 0 also. That element can map to 0. Yeah. So, that is why maybe it is not as trivial as I imagined. but still you can do this it just comparison of subspaces I mean no other property of curve is used here it is just this these 4, 5 subspaces that you have to study okay let us try something so Ld ‘ |ld| = Do we agree with this? I think that is the key thing which will stop the problem that you were talking about. If you go back to your earlier statement, do you need an injection? Yeah, because I want an upper mono dimension.



If your map itself is killing half of your basis elements, then you cannot get the corollary statement, right? Yeah, but its dimension would be strictly less than- No, no, no, no, no. Why? Your map is already, I mean, you're already setting basis elements to zero. So then when you go to the image, its dimension doesn't upper bound because you used the wrong map. no it has to be an injection I think the problem that you are saying is resolved here so this quotient vector space you basically have to observe that it's the same as this and yeah which is isomorphic to |LDS|.

These are just basic properties from subspaces. So, this quotient space is isomorphic to this sum which then is contained inside LD ‘ S |lds|. this gives the injection okay yeah that will be the formal sequence of steps you have to do and then you just invoke the previous theorem So this step, basically the last step is where the problem is. In the last step it's only a containment. So this last quotient space may be actually bigger, which is exactly given by the theorem statement.

the left hand side may be smaller, ld ‘ |ld| may be smaller. So we do not really know how much smaller it is. So that is corollary 1, you just get an upper bound. I do not know if this is taken in a qualitative way. Right now we do not know, but later on as the course progresses you will see that in reality these are different.

Yeah, but at this point, probably you can construct examples. I don't have example ready. But in reality, it won't be an equality. Actually, these questions that you are asking, these are interesting because this is what inspired Riemann to define genus. So, as the genus of the curve will increase, this inequality will get worse.

it will become, it will kind of fluctuate. Sometimes it will be equal, sometimes it will be not equal. It will really depend on the genus and the divisors D and D ‘. So, exactly measuring the fluctuation is the source of this algebraic definition of genus, which we will see tomorrow and later. with that motivation let us define an important function this dimension of the LD sheaf but when I write this you should ask why is this finite because till now I have given you no reason to believe that LD sheaf is a finite vector space. It was just collection of rational functions which have poles no worse than D. So these could also be infinite, I mean the, definitely the functions are infinitely many but even the vector space can be infinite dimensional.

Everything can be infinite. So corollary 2 says that actually this is finite. it's a finite dimensional vector space so LD is finite for all D in the divisor okay that is what is so impressive about the theorem the theorem actually implies this that LD is a finite dimensional vector space so do you see why Yeah, we'll just go to the theorem setting. Just compare D with some other divisor. And that divisor will be. So take a positive divisor, strictly positive.

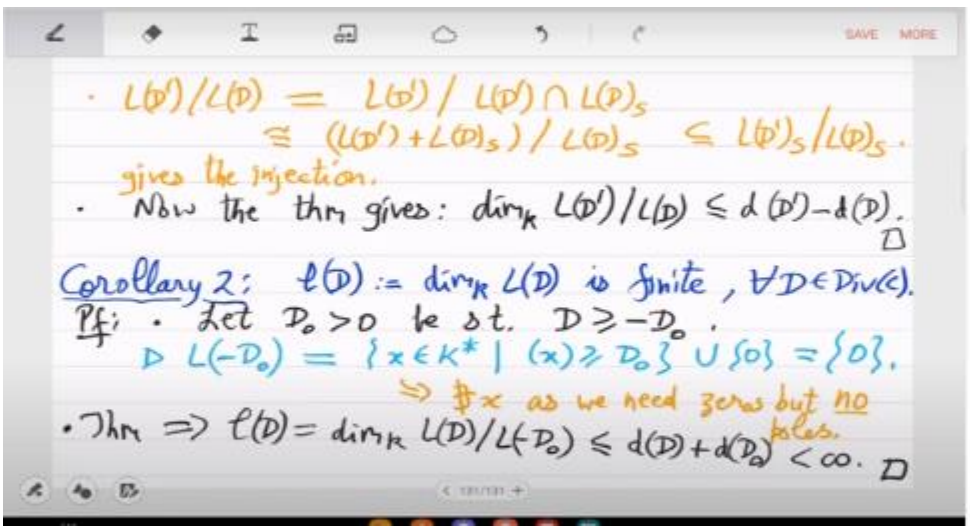
This exists because remember every divisor D is a sum of finitely many points over your finitely many integers. You just pick the minimum integer and use that negative number to define - D0. So D0 is a positive divisor such that none of the multiplies orders in D are worse than - of that. Now what can you say about L of d0 - d0? What is this space? See these are functions such that this is greater than equal to D0 or 0, right that is the L - L sheaf. See D0 is positive, right. So, is there a rational function whose principal divisor has only positive, has only I mean Whatever there is, it's all positive.

Can that happen? In particular, there is no pole. That can't happen, right? There is no such rational function which has no pole. Yeah, but we didn't see the proof of that.

No, no, no, no. A rational function is g by h. Yes, h can be unit, but then you only have g x bar. So, g x bar has a kind of pole at infinity. For example, look at x. So, this is x over 1. It seems that there is no actual pole, but still in the projective space, there is this pole at infinity. you see it by homogenization, you introduce z, so it is actually x / z and z = 0 is the pole now, that is the point at infinity.

So yeah, so this actually then has to be a constant but then if you take a constant it also has no 0 so a constant has neither 0s nor poles but we have said that d0 is actually positive right so it has to have a 0 but no pole so nothing is possible here there does not exist an x as d0 is positive as we need zeros but no poles. So the only option left is zero. So that's all. That's L - D zero. okay so l - d0 is actually 0 there is an easy property for l sheaf we could have seen this as an example when we define the l sheaf right so now you on this situation you apply the theorem so you will get dimension so basically L d = dimension (L - d 0) is anyway 0, but in the theorem hypothesis we will use L d mod L - d 0 quotient space which will be by corollary 1.

degree of no exactly we will get upper bound so degree of d minor + degree of d0 so that is finite is that clear. So that is the, ultimately the beautiful machinery which tells you that the LD sheaf is finite. For all curves and for all divisors. Okay, so. Yes, any questions? So, corollary 1 is telling you that L d ‘ - L d <= degree of d ‘ - degree of d right that is how we will read it.



So that's exactly corollary 1. So what this is saying is that if you make your divisor bigger then the L dimension is better approximated sorry have I made a mistake yeah that's a mistake it's less than equal to yeah it was an upper bound. L D ‘ is upper bounded by L D. Yeah, so you should read this as following, as the following this as you increase your divisor the difference gets smaller. In other words L is better approximated by the degree. so as d ‘ increases ld ‘ gets closer to the degree it's a better approximation or at least this is saying that it's not very worse so you working with bigger divisors will be helpful in the analysis you can just understand it like this so the question here is in the limit will l become equal to d.

So, do large enough d satisfy ld = dd. Right. So if you take a very big divisor then in terms of degree I mean, ah, will the L space be exactly equal to the degree? If not, how much is the difference? Is this difference going all the way to - infinity and so on? Right. So the- so these are the questions. because it could keep on decreasing. So if it decreases will it reach - infinity or will it reach zero or will it be positive and so on.

So we will now be interested in understanding this difference and so answers to these questions will lead to genus. genus of a curve. So, the new term genus which is an important invariant of a curve that appears because of these issues or you want to understand these issues and actually a lot more analytic algebraic structure will come because of or attempt to understand this difference. So, the thing is that degree is something which you easily understand in terms of points, it is basically just counting the points in your cloud of the prime ideal, but l is something more mysterious because this is to do with the. rational functions which satisfy certain number of zeros and poles. So that's a much more mysterious thing, it's more algebraic analytic and the two things will actually come together to give you genus.

so we will I think do that after the recess tomorrow what we will do is we will work with degree of principle divisors we will prove some theorems about that. We will try to understand these principal divisors, sorry not 0, A. We will try to understand the proper, what kind of zeros and poles and how many do they appear in rational functions.