Computational Arithmetic - Geometry for Algebraic Curves

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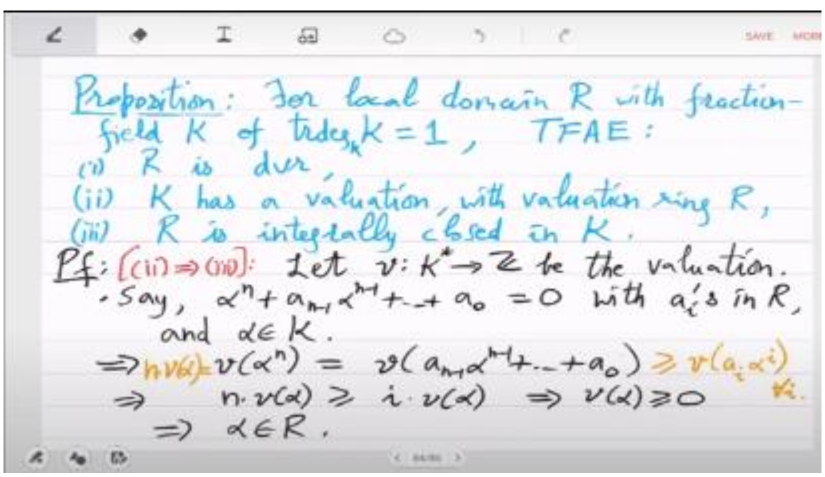
Week - 06

Lecture – 11

Existence of Non Singular Model

So, last time we did more about valuations. So, we proved this proposition that I mean essentially when you look at the germs at a non-singular point at a smooth point on a curve then that germs ring call it R. these three things are equivalent. It is a discrete valuation ring, it defines a valuation on the function field and it is integrally closed inside the function field. Integral closure being the main defining concept I would say because you can think of it as there are field elements in K for which there is a monic defining polynomial over the ring R. So, you just introduce that element in the ring and make it bigger and you keep doing this till you cover all the integral elements in K inside R.



So, at that point you will have a DVR that is the thing you do when you have a singular point and we characterize the DVRs. of the affine line which is this theorem in blue. So, the distinct DVRs for the function field essentially you have one variable attached k (x), kx. So, for that function field we showed that the DVRs they essentially correspond to valuations where you either you take a irreducible polynomial f.

So, for example x - 1 and then look at the valuation with respect to x - 1. So, that corresponds to the DVR R sub x - 1. So, you can do this with x - 1, x - 2, x - 3 so on or you can do it with quadratic irreducibles in case your base field k was not algebraically closed. Otherwise you will only have x - α. So that type and the second type is 1 / x.

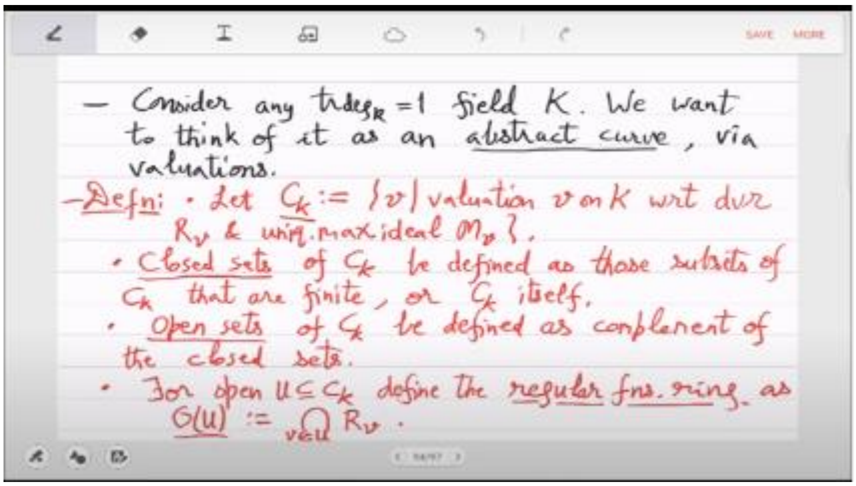
You can also define valuation with respect to 1 /x. So that is the new unusual valuation. and that is all. So, we prove that and then we can extend this to curves from k x we can go to K that is only a finite algebraic extension and there is a way to do this. It is basically again by this you have a local domain you take an integral closure inside K that will give you the DVR.

It is easy to see that a curve has only finitely many singular points. because a curve is defined essentially by a single constraint in two variables and singularity is another constraint. So, if you intersect your curve with that then you get dimension 0. So, essentially you only get finitely many singular points. You want to remove all of them and get a equivalent non singular curve.

So, a curve which should have no singular point. So, in that direction we have started. So, we defined this abstract curve. So, our thing of interest is this function field K which is given. I mean it is basically of any field it has transcendence degrees 1 over the base field k.

okay so you get this basis so somebody gives you a curve which has singular points so you look at the function field of that curve call that k of c so that that is this K and now from here you want to define a different model of a curve which has no singular points, every point there should be non-singular. So, we will make basically we will now give a model which is non-singular for this big field K. So, in that direction we first define a very abstract curve which is C sub K. So, this C sub K is what are the points here, the points are actually valuations. okay so we call it abstract curve because it is very abstract the points are actually not real points they are not in the affine space they are actually functions so you are looking at the set of all valuations of the field so the valuation will define will have an equivalent DVR RV and its maximal ideal will be unique we are calling it MV okay so look at the valuations as the new kind of abstract points, but this itself would not be interesting unless you define a topological structure on this.

So, we will define open sets, closed sets. Closed sets will simply be the finite subsets of this or Ck itself, the whole of Ck and open sets will be complement of this. So, open sets are simply complement of finite sets or empty set. yeah even here we don't want to stop because actually we want to define functions on open sets right to get proper geometry which we had seen in the case of curve before so for that we have to define what are the regular functions on an open set u so u essentially has infinitely many valuations inside as points which we are calling points abstract points and on that you want to define a function right so functions will come from a big field the field K in particular they will be this an element of the field which is present in all these RVs for all the valuations that is the definition of regular functions on U now why does it make sense so the physical interpretation is as follows so each function in OU function f defines, each f defines a distinct function which maps u to the base field k.



How? So, it will take a valuation v contained in u and it will map it to So, the valuation corresponds to a maximal ideal and you should just look at f mod that maximal ideal and this you can read as an element in the base field k right because what was k.

So, k was its isomorphic to rv |mv|. So, if you take an element in RV remember that F ϵRV right because F ϵ OU which is the intersection of all the RVs. So, F is in this particular RV also in this DVR and when you look at it |mv| you get a field element base field element. So, if K was FP actually F |mv| is just a number from 0 to P - 1. So, this is a natural way to evaluate a valuation at a function.

So, a function can be evaluated at a valuation. So, this is why we can call it an abstract curve and abstract functions because actually the function is being evaluated at a point. This is what we are simulating here and functions. No, no f is an element, f is a field element in K. K has these DVRs which are sub which are rings and you have taken intersection of the rings.

So, f is simply a element in the function field you can think of it as x1 /x2. So, you have to define how to evaluate x1 /x2 at a valuation. That's a complicated thing. But the only way to define it is by taking |mv| because you want to get a field element. So the valuation will actually give you a field element for a fixed f.

So think of this as the new meaning of f at a point. This is what we have defined. How to evaluate a regular function at a point? which happens to be a valuation here and functions fg are the same functions, we will call them the same if and only if they are the same |mv| for all v. So, if you take two element two functions two regular functions which are regular on u of course by the definition they are the same if and only if for every valuation they give the same answer which is the same as saying f - g ϵ f - g is in every m v intersection over all the v's. So, do you see that? So, f - g is present in each of the m v's which is what.

So, you are saying that valuation of f - g is positive for infinitely many valuations. But they could perhaps still differ on the complement of the, but if this holds true for a cover u i. No, see f and g you know that these are actually fractions. in the function field. So, it is a finite fraction, f - g is a finite fraction and this is saying that it has infinitely many, your v of this is positive for infinitely many v's.

I think it means that it has to be 0, f has to be equal to g in k, but I do not want to jump to that, let us see this later, no actually I need it right here. Yeah, okay, it will follow from MV being principle, that's the trick. Since MV is principle, you are actually getting an element dividing f - g for every v. So, you are getting an infinite product dividing a finite object. So since MV is principle, this means that actually f - g is 0 in K.

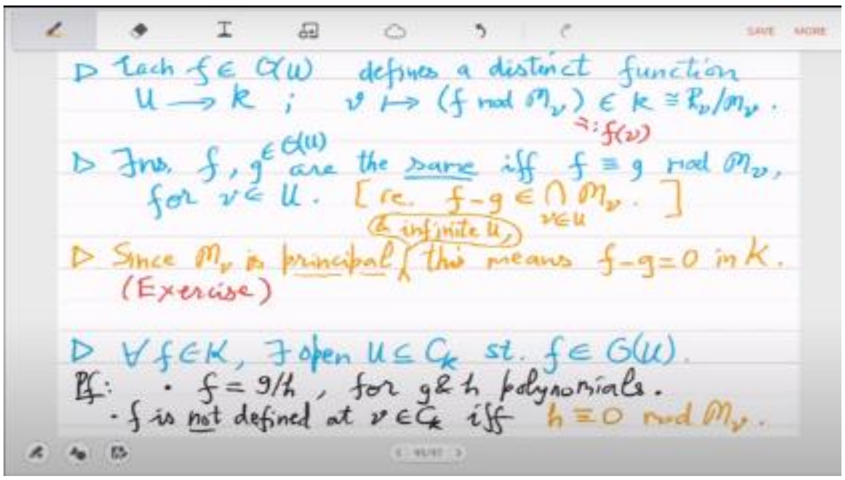
Is that clear? Yeah, so this is being used. It is not just by valuation positive, it is actually by the division. You take every uniformizer and their product divides f - g. That's an infinite product.

So yeah, this can be done. This probably should be done more formally. I will not do it. You can prove it as an exercise. Yeah, but most of the important ideas now you already have. But still, you should try to do it in proper notation.

So since mv is principal and u is infinite, that also we used. Let every open set is infinite. So, you are taking an infinite product. So, which is perfect you have defined regular functions on open sets containing valuations and the notion of equality exactly matches that of the function field. different elements in the function field actually are giving you different functions on any open set.

So this means that you are on the right track. This abstract curve is a natural object. And similarly, you can also prove that any function field f that you take, it's defined on some open patch. so there exist an open U on the abstract curve such that f ϵ OU why is that so this proof actually you have seen the ideas the main idea is that f = g / h for gh polynomials in the coordinate ring no sorry in what I guess in the polynomial ring let me not say where. So, f is just a fraction of comprising two polynomials g and h is the denominator where is it defined.

So, it is defined whenever h is non-zero, but that defines an open set. So F is not defined at a valuation if and only if H is 0 which means that H ϵ MV or in terms of congruence this. because you will be going f |mv|. So, if h vanishes then this function is not defined. So, you have to basically now check, you have to exclude these valuations, these so called points in your abstract curve.



So, how many are they? This again is I mean since h is a finite object, there will be only finitely many v's. So, that is a closed set. So, complement is open. This implies that these bad points v form a closed set which means that f = g /h defined on open u. okay so every rational function is defined on some open patch and two different functions will give you will actually be looked different on some open patch so we have a good interpretation of every element of the function field of this field K yes yeah yeah that's an equivalent thing but you don't need to think in terms of that no you just see that then H ϵ the intersection of MV for all bad bad V's no no what is the meaning of H at V it is this that is the definition right I mean f at v is f |mv|.

Say that again. That we are identifying every polynomial just by f |mv| like a sequence or infinite sequence. Yeah, so these are the values of f at places v. And since these places are infinitely many, in a way you can say that they define f. If you have all the information you define f. It makes sense because we have shown here that only way f and g can be the same on an open patch if they are indeed the same in the function field.

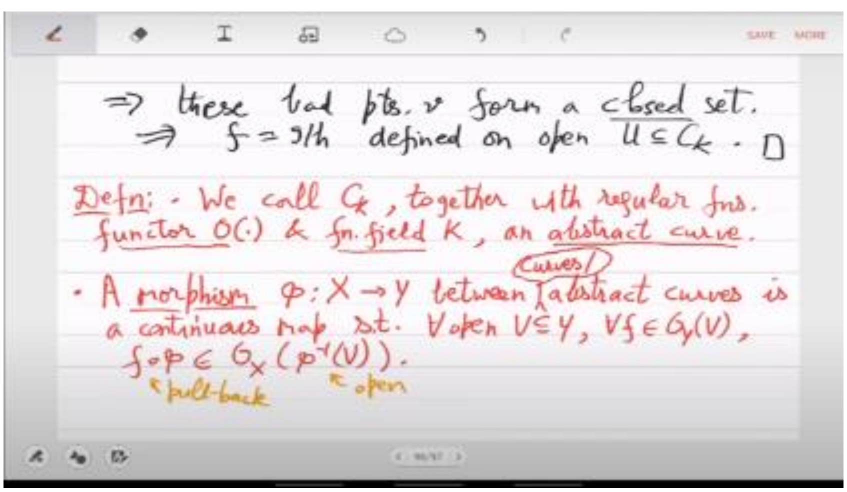
So if you have value at all the places then you have in a way the function uniquely defined. there is just some h (x) is 0 like does it also relate to that definition here. Which one? It is not defined on this set if there is just a point h (x) is 0. Yeah, but the point is v right, now the meaning of point in the abstract curve is.

I am doing the actual field elements. Yeah, so what you are asking is the abstract curve real, so we will show that it is, that is not trivial. that is yeah that involves beginning of modern algebraic geometry, we will skip some of that part, but we will give the main proof. Now, for now let us just develop one more thing for the abstract curve which is morphisms. So, we call Ck together with the regular functions and k. So, in fact this is the regular functions functor and function field k.

So, we are defining the functor and we are defining its function field and ck as a set is just set of valuations, but it has all this data attached to it, it is not just a set, it is a data which has open set, close set concepts, it has this regular function functor and it has the function field. This everything we together we call it an abstract curve. and we can define now morphisms between two abstract curves. or they can even be variety, let me not go to variety. So, between abstract curves it is a continuous map such that the same thing which we had I mean we just copy paste the definition you had before of morphism.

The definition you had for variety is the same thing you will use for abstract curves that for every open set v (y) and function there. f composed with φ is a function pulled back to x. So two things are happening here. This is open. So for every open set of y, when you look at the pre-image or the fiber, it is open.

Φ^{-1} v is open. and any function there can be pulled back like this. So, such maps between abstract curves are now called morphisms between abstract curves. This is not surprising, it is the same thing as you saw for real curves. So, let me add curves slash abstract curves, you can mix and match. You can take X to be a curve and Y to be an abstract curve and so on.



And with this definition, you can now see easily that every non-singular curve is isomorphic to an abstract curve what is the proof of this so let x be a non singular curve so you want a morphism two way morphism from x to an abstract curve, the option is, what is the candidate for y? Well, you take, you look at the function field of this non-singular curve and take the abstract curve for that. okay and you show that this there is a morphism which also has an inverse right so what will you do where should a point go point should be mapped to the valuation corresponding to the point So the forward map will be a real point. You map it to the valuation that it defines. And it defines a valuation because it's a smooth point.

So its ring of germs is a DVR. As OXP is DVR. if and only if P in X is smooth, is non-singular. So, because of the non-singularity you immediately get DVR which immediately gives you a valuation, use that valuation on the big field, the function field. So, all the valuations of the function field correspond to points. Well, but there was also this 1 /x thing, which point will that correspond to? There seems to be one extra valuation, right? It does not correspond to a point.

Yeah, so that will be more complicated now. So, you are saying that we should look at projective curve. We cannot do this with a fine curve. No, you are now looking at all the valuations of the field Kx, the function field of the curve. In that function field there is an extra valuation which is when we characterize we got that, this one. I mean here we took the kx field, but then what it shows is that for any field of transcendence degree 1 either you take valuations by a polynomial.

So, which will basically be x - α in the algebraically closed case, but there is one extra one which is 1 /x. So, 1 /x does not really correspond to a point, x - α corresponds to x = α point. but 1 /x corresponds to as he said point at infinity. So, you have to now embed point at infinity. So, should I correct that to non-singular projective curve.

But then it is not clear what is the point in field of projective curve. So I think the forward map is clear. For the converse, you have to do something with this extra valuation, which I don't remember it off the top of my head. So let's leave that to homework. So what is the converse of this? What's the inverse of φ? So it's clear on infinitely many on everything except the extra valuation 1 /x that you get.

That's what to do with the point at infinity. R 1 /x. That's such a thing you have to handle in this setting. V of 0, you mean if 0 is a point, no any point like if you have two coordinates x1, x2, so you are saying x1 and x2 both are 0. What does 0 mean in the curve? In VP what does that mean? I guess order dividing that.

Yeah, yeah. 0. No, no. So the point 0 , 0, if it is a point on the curve, then the maximal ideal you are looking at is x1 , x2. So corresponding to that, you get the germs that are vanishing. That's the unique maximal ideal. So modulo that ideal, and since this is a non-singular curve, it's actually a principal ideal.

So you'll find a y which generates both x1 and x2. So you have to define valuation with respect to y, the uniformizer. So that's the same thing we have done till now. You just copy that, put it here. But in the inverse, yeah, I forgot that there is an extra 1.

extra valuation is there. So, that has to be handled carefully. Anyways, now we will do the more important thing which is existence of non-singular models. I also want to know what you like, what does it mean like two curves are like have a morphism between them. Like when we had rational glass, it was a way of comparing their function fields. So, what does morphism between two curves even imply? Oh, it's an abstract morphism. What do you mean? What does it imply? As long as it is well defined, it's there.

No, no. So, if you had y to be a curve instead of an abstract curve, then whatever property you had there, the same property you are trying to study in abstract curve. The same thing. So, you look at the function field of X and Y. I forgot when we define morphism what is related to coordinate ring or coordinate ring that was related to coordinate ring. So, just the single morphism that is related to the coordinate ring, which actually we have not defined for an abstract curve.

So, that connection is missing. But if this morphism is invertible, so if it's an isomorphism, then you can just directly talk about the function field. What it is doing is that k (x) = k (y). And I mean all this has been done for a purpose. So what we will do is we will actually realize now the abstract curve via a non-singular actual curve, which I am calling non-singular model.

So that is the major theorem of this geometry part. So, let base field be k and this general field be K of transcendence degree 1. So, think of this as k and k x, that is the case of the affine line. but that is non singular. So, maybe you were given some singular curve defining the function field K and from there now you want to resolve the singularities. So, what the theorem says is then as we have defined there is an abstract curve Ck, the unique abstract curve Ck, this is isomorphic to a non singular projective curve.

Yes, you see that this projective will be needed. So I think in the previous claim also you will need a projective curve. So ultimately you will have an equivalence between non-singular projective curves, actual curves and abstract curves. Yes, as Madhavan pointed out, for a projective curve, if you look at the function field on the whole curve, you'll only get the base field constants. So we can't make a statement about the full function field because in the projective curve case, it's trivial.

Of course, it is not equal to K. But there is still this regular function functor. So you can talk about that. And you can talk about the functions. Or what did we call that? Yeah, I mean, basically you can look at functions with their open patch.

and also the equivalence relation we had. So we say that two functions with their open patches are equal if they are equal on the intersection. So keep those things in mind because now we have moved to projective from affine. So I will try to quickly sketch the proof of this without going into the real algebraic geometry details. So the idea will be that we will glue the finitely many smooth curves one for each singular point. So we are thinking of the function field currently given as via some model which was not good enough.

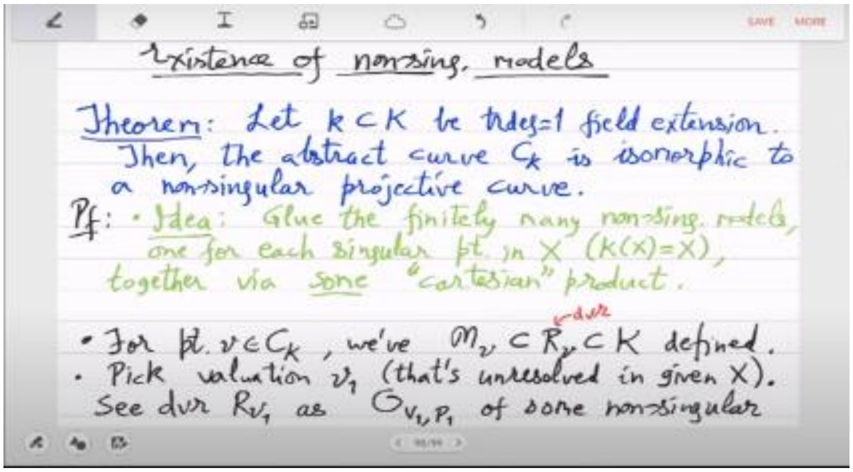
X was the model which was defining the function field K. If X was already non-singular, then we were already done. We didn't have to construct anything. But the real problem is that X was a curve which had singular points. We know that the singular points are finitely many.

And we know how to resolve each one of them. So the problem is that resolutions will be giving you different models. So you will have finitely many non-singular models, one for each singular point in this input model x. So you have to glue them by via some Cartesian product, let us say. So, you essentially want to take these non-singular models each resolving a unique singular point, look at the Cartesian product and may be that gets you closer to a actual non-singular model which works for the hole. So, how do we implement this? So, first is let us recall how do we resolve a single singular point.

So, well for a point v in the abstract curve we have the maximal ideal, the unique principle maximal ideal or the unique max ideal which is principle this MV and we have the DVR RV and we have the big field K that is the DVR. So that is defined as before. So we will do this, I have to I think switch between x and ck. So I mean everything except finitely many valuations are already modeled in x.

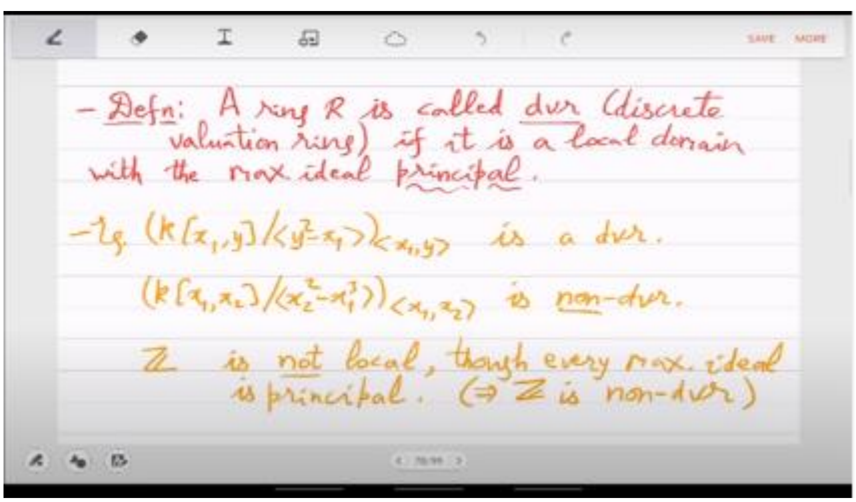
The only things which are not modeled, they are because of the singular points in X. So let us go one by one through these singular points. So let us consider the first one, call it V1. that is unresolved in curve X and for that you have the RV1 DVR. See this DVR as the germs for some curve V1 at some point of some non-singular point P1 in a quasi-affine P1. So, how will you do that? What I am saying is that for a valuation, this is… it was singular in the input model it was singular.

So, somehow we have to resolve it. So, what you do is you take this first singular point, consider the valuation of it, it is defining RV1 DVR. So, you should look at this basically construct a curve V1 and a point there which corresponds to RV1. O should be isomorphic to R . So, I leave it as an exercise to realize this V1 and the point P1.

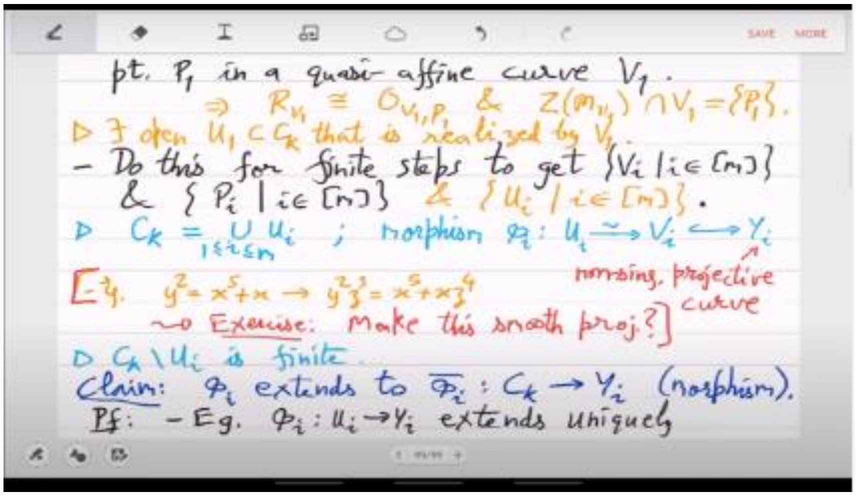


This basically, I mean we have done it many times in examples. You basically have to just reverse what we saw in those examples. So, for example, let us just go back quickly. Yeah, the standard example that we had, when we define germs I guess, this one. So RV1 is a DVR.

So it's the first case, kx1y modulo y ^{2} - x1 localized at this principal ideal. So what I'm saying is that if somebody gives you this as RV1, then the curve that you should consider is y ^{2} = x1. So the polynomial ring modulo the ideal, that ideal will actually give you the curve definition. and the point there will be 0, 0. So as soon as you are presented RV1, you will see what the model is.



That's the curve V1 I'm talking about. So I'll not go into the intricacies of this because it's pretty obvious in a computational wave of what's happening. Just from the presentation of the DVR, you will be able to get the defining ideal of the curve V1, and also the point P1 such that RV1 is isomorphic to OV1 P1. So, that is basically resolving the first singular point. So, how does this help? So, now what are the properties that we have we have got R V1 isomorphic to O V1 P1 and if you look at the zeros that this max ideal defines inside V1 that is just one point P1.



So, the max ideal since I mean RV1 is the ring of germs at P1. So, the 0 set of the max ideal inside V1 is a unique point only P1, there cannot be two points. So, we just record that. So do this for finite steps to get the vi's let us say m many and the points. So get the curves and the points on the respective curves with all these properties in orange satisfied.

So that's the individual realization but this wasn't the problem. The problem was that you have to find a global curve which works for all these simultaneously. So now we will do that. Now we will glue this. I think I need one more thing.

there exists an open UI in X that is realized by U1 that is realized by V1. Since P1 is contained here and the way we will do this the connection between X and V1 will be that on an open patch, they agree. So basically you also have UIs here. So I need all this data for the gluing step.

So you have these patches. Actually do I want x? No, I think I should rather work with ck here. Yeah, so I have these vi's which have realized parts of the abstract curve like v1 has realized u1 and each vi has been defined via the point pi that it contains. So, vi is the resolution of that point. In fact, the point you can also see as a valuation now, the valuation v1. So, the immediate properties are that Ck is being covered by these.

Okay, the union of all these, I mean we did this for finitely many steps so that we get a cover. So, these M open sets have covered the abstract curve and So you're choosing ui so that it covers the finitely many ui's? Yeah, you can cover it in finitely many steps because they basically correspond to the singular points on the given model, input model x. So just those singular points you have to resolve them into vi's. The other things you do not care because they were already non-singular to begin with.

So, the connection in this data is actually a morphism. The morphism is from Ui to Vi. That is actually an isomorphism. Okay so we have till now what we have done let's say algorithmically is we have covered CK into finitely many open patches and each patch we have realized by completely different curves, VIs, isomorphic on that patch and for some strange reason I want this embedded in a projective curve. so this is projective in a non singular projective curve so by vi here you mean valuation corresponding to it or ui is an open set oh vi is no vi is a realized curve Like the example which I showed you, in there the curve was $y^2 = x1$.

It's just that. And on $y^2 = x1$, the point P1 is 0, 0. So that's all. You just look at our V definition or the presentation and from there recover the curve and the point as an actual curve. Yeah, so UIs were abstract points. VI set of actual points.

It's a curve. And since it was affine, we have embedded it into a non-singular projective curve. Because VI was a non-singular affine curve. Are you fine with this or do you want to see how you can embed an affine curve into a projective curve? It's the trivial embedding, you just introduce a new variable x0. So, you can see an affine curve always as a projective curve and it will also be smooth.

If you started with smooth, you end with smooth. So, just embed it into this projective curve. Can you explain that line once more? Which one? The first line, V1 is in a quasi-affine curve V1. Yeah, the explanation was by the example. I am not going to the details. Because in general, if you go into the details, it will be tricky.

Basically, what you have to see is how is this RV1 presented to you. So only sensible presentation is it's a polynomial ring modulo ideal localized at something. So that ideal already gives you the curve definition. Yeah, so I think that example if you remember it's fine. I don't want to set up that whole machinery here. The issue is that you actually RV is coming from, I mean the root to RV is pretty long, right? You are given a model curve X and from there you have picked a singular point.

For the singular point, you take integral closure of the germs that gives you RV. and from this integral closure the DVR RV you get the model. So, that part we have done in the previous classes. So, I do not want to repeat that because it is too much. Computationally is it I mean can you describe this integral closure fast.

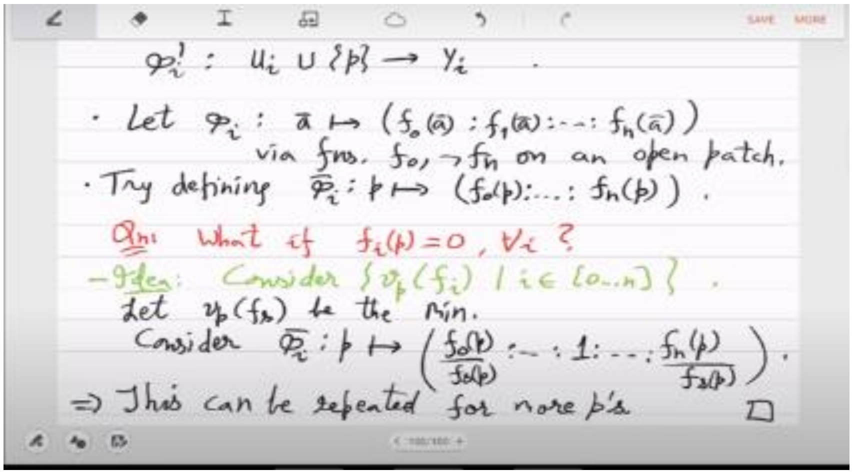
So, I am not going into the complexity of that. What you are asking is even more. I haven't even given the proper algorithm, so complexity will be more complicated. But I think it's, especially if you are given everything in two variables, you can do it fast.

It's only hard when you have multivariate presentation. I don't think there is any problem with bivariates. Yes. Yes. Why? I just added a free variable. Yeah. No, but if you just add z there. So, what do you want to do there then? what is the correct way? So, if you had $y^2 = x^3$, you go from here to $z y^2 = x^3$, that is already.

So, let us take the simple example. and now what what is the problem in this okay x is to 5 + x so you get this okay and because of Z you are saying the only new point you add is because of Z = 0 right so Z = 0 X = 0 Y = 1 that's the only new point that's a singular point how do you correct this ok so let me leave that as an exercise for now but this believe me is not the hard part So make this smooth projective. Yes, there might be some obstructions with the obvious homogenization like I did with And Madhavan isn't happy with that. But I mean you can believe that there will be some way to make this smooth affine into smooth projective.

So that's yi. Yeah, so now we have the setting. We have to now do the actual gluing process. How do I glue these yi? So let's say y1 and y2. How do I glue them into a single smooth projective curve? which is which should realize as promised C k the abstract of C k. So, what we will do is make this observation that C k - u i is finite that this is fine UI was open so complement is closed which is by definition finite and what I claim is that because of this finiteness this φ i uniquely extends I do not think I need uniqueness, φ i extends to φ i ‘ on the whole of CK.

So, I have a map φ i which is from ui to yi. And using the projective nature, I can actually prove. There is a geometric proof that any morphism to a projective curve from ui to yi, I can add another point in ui and still extend the map. It's extension of the morphism point by point. so why is that possible so ui to yi extends uniquely to φ i ‘ in ui i add another point let's say p and there will be a way to define φ i ‘ at the new point p so this actually is simply the following idea you take let φ i be mapping points to these functions f 1 to or f 0 maybe because it is projective n a bar.



Okay so here I will use the fact projective, so hopefully the idea will be clear why we needed a projective curve in the range. So the reason is that morphism has this property that you will wherever you see around the neighborhood of a point A bar, the image will be these fractions, there will be rational functions, so these are F0 to Fn.

Now you are introducing a completely new point P. Idea will be that you actually continue to use the same functions, f0 to fn. So on the new point, you continue to use these functions. That will be the obvious thing to do. you had function before so you can just evaluate those functions at the new point what could be the problem well the problem can be that this goes to a forbidden point which is the 0 all 0 coordinates right so p may be a common root of f0 to fn So this may fail.

In all other cases, it is fine. You have extended φ i to φ i bar. Only problem is that these could all be 0. Then what do you do? What's the solution then? Yeah, so at this point, we will use the fact that actually what p is is a valuation. CK was the abstract curve. So, the new point that you are adding is always a valuation. So, if you think in terms of valuation then what is happening is that FIP is simply the valuation of Fi.

I mean it is not simply, there is an associated valuation that you can study the valuation of Fi. so consider the valuation that this point defines of fi and what do you see so you see that these are integers you pick the minimum one let us say fk i = k gives you the minima And you normalize the above by fk. And what do you see has happened? Exactly. So now you have f0 / f, sorry, this is fk. I should not use k that is not good, let us say s.

So, f0 /fs dot dot somewhere you have a 1 and in the end you have fnp /fsp and this is in the projective coordinates. So, this is all ratio. Right, so I have now given you a valid point in the projective space. Right, so if I use the same functions then I will get the zero point which is illegal but then I realize that actually I have a valuation available which is kind of sorting the functions. So in the sorted functions you pick the minimum one and just make it one.

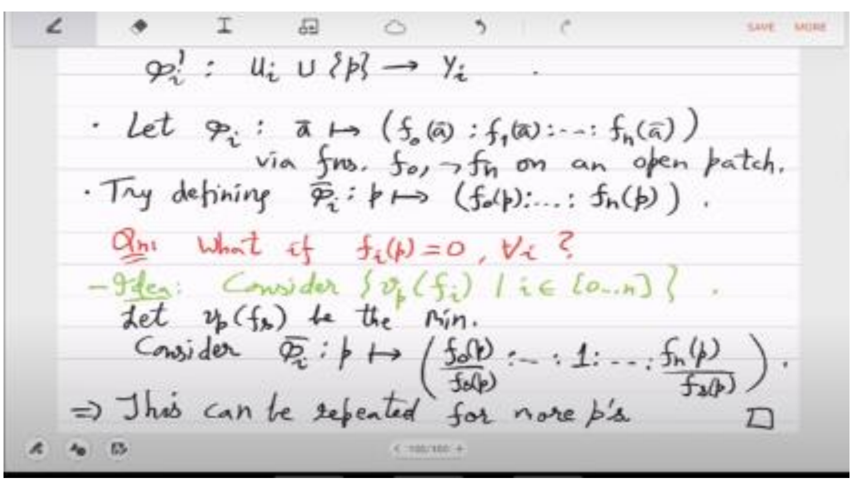
Okay, so this is why the projective space is so beautiful that it actually allows you to grow a morphism point by point. You can keep adding points as long as these points come from the valuation space of course. This argument needs that.

Right, so what we have done is, so now you can repeat this. This can be repeated for more p's. So that's the full proof.

It was important because you want to see why we go to projective space. It's not artificial. No, no, no. Vi was actual. Yi is also actual. Only you can think of the starting point as being abstract. ui is covering ck, so that was abstract. But subsequent things are, so φ i is what makes the abstract actual.

And we can actually grow φ i to the whole abstract curve. That is what the claim is saying. Here you use the fact that you are in projective space. Oh, how is it resolved? So, y was everything 0, FIP was 0. So, it means that, I mean say the point P is the 0 point.

So, it's like looking at the valuation by x1, think of x1 - 0. So, you are looking at how many powers of x1 / F0, F1, Fn. Yeah. So, say you had x1 , x1^2. So, if you substitute 0 there, you will get 0 , 0. But then you can realize that you can be clever, you can normalize it first by x1 and you get 1 , x1.



So, that's the beauty of projectivization. So, once you have this, you are pretty much done. with some more algebraic geometry that I will skip. So, what you do next is, now use these φ i bars to define the common morphism φ on the whole of CK. So what it will do is it will basically as promised do this Cartesian product. So where should a valuation go? So a valuation or a point which was abstract, you realize it into this tuple.

So, the initial idea was that you get these smooth models and then you take a Cartesian product to get one curve. So, this is how you will do it. You have an image of arbitrary v into yi, you just make this tuple. and define finally y to be the closure of the image of φ c k in projective space. big enough okay so in the some projective big M space a fine space projective big enough space you have these tuples you have these points which are actual points their set may not be closed yet so you close it and that will give you the the variety y which happens to be a curve.

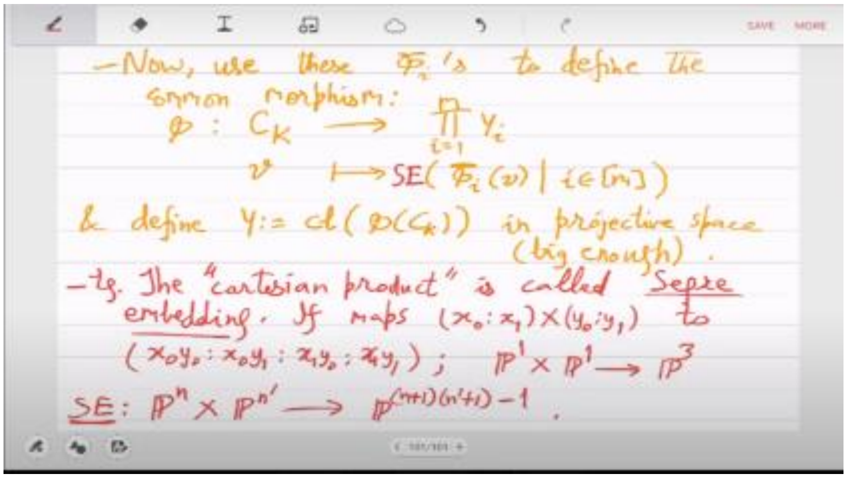
So, that is the actual curve. So, here I need to define the seg ray embedding. it maps, I think this… No, I think the notation I am using is not correct. This m tuple thing I do not want this. I think this has to be the Segre product of these points. I have to write here Segre. the segre embedding basically takes two projective points let's say x0 x1 y0 y1 and it maps it to how many coordinates four coordinates so you get x0 y0 So it's a map from the projective line cross itself to the projective three space.

So it blows up quite fast. So for M =2 , you will basically have two points in their respective projective spaces and you will have to then multiply them via this Hegra embedding. so you will get four coordinates in general it is if you have n projective projective n space projective n ‘ space you will get what exactly n + 1 n ‘ + 1 - 1 okay so this is the Zachary embedding What is the purpose of this? How do you define Cartesian product? If you want to say honestly that I have a point in a projective space, big space, what is that big space? So that big space is the following. You have a projective line, point in a projective line, another point in a projective line.

When you take Cartesian product, you actually go to projective three space. which is given by that all possible products. Yes, Segre was an Italian geometer who defined this. So, that Ck goes to product over i to where yi that you means that. Yes, so it's not just an m tuple, it's not just m copies, you actually have to take Segre product. So, Cartesian product in this category of curves, projective curves is actually this product, it's the Segram product.

Yeah, otherwise you can see that you can't generally define product because the issue is that x0, x1 are not two values, you are interested in the ratio. So, it's actually x0 /x1, y0 by y1, but then you don't know which one is 0, which one is non-zero. So, it's actually either x0 /x1 or x1 /x0 or both.

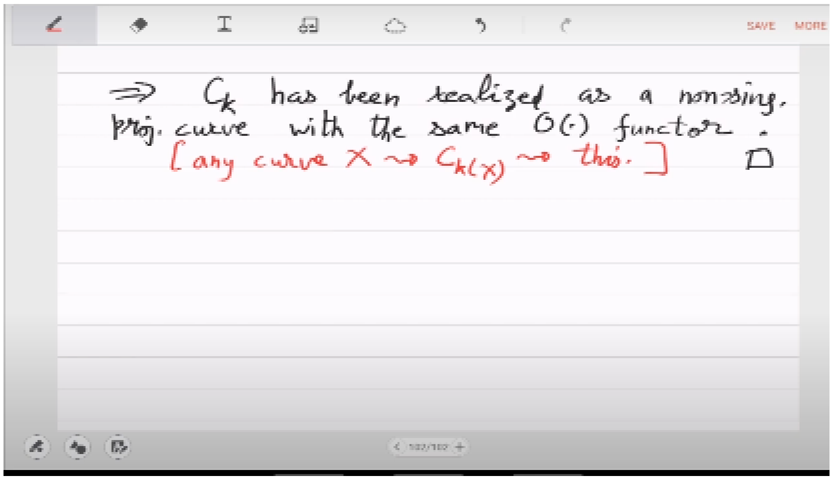
So, we are including all possible. Yeah. So, the only way to write this algebraically is to actually consider all products. Whichever works will give you the correct value and if all of them work, all of them will give you the correct value. up to ratios. So, you can I mean trust Segre he has thought about this centuries ago.



So, we just use that and yeah. So, then ultimately y is that big projective space where Segre lives and now where your curve is actualized. So, that is the realization. has been realized as or we use non-singular projective curve with the same O functor. Okay, so you shouldn't think of the function field because function field will be trivial.

But if you look at patches, then these patches are isomorphic to CK, those in CK. And in the patch, you have actually an isomorphism of functions. Okay, so this is an honest realization of the abstract curve.

And hence, you have made... any curve x that you started with, you come to Ck and from there you go to this. So, each of the things that we defined these complicated things they were actually necessary. For a general curve, because of singularity you need to look at abstract curves. So, all these DVRs and all these DVRs you resolve them into smooth projective curves and then you take their Segre product. So, this is I think completely and this is I think practically infeasible because you multiply every time.



So, even if it is finite ultimately the projective space you get is huge. So, you do not want to do this in practice. This is just for proving theorems. So, ultimately the zeta function the Riemann hypothesis that will prove It will be true for any curve because any curve is actually a smooth curve. And we'll prove Riemann hypothesis for smooth curves.