Linear Programming and its Applications to Computer Science
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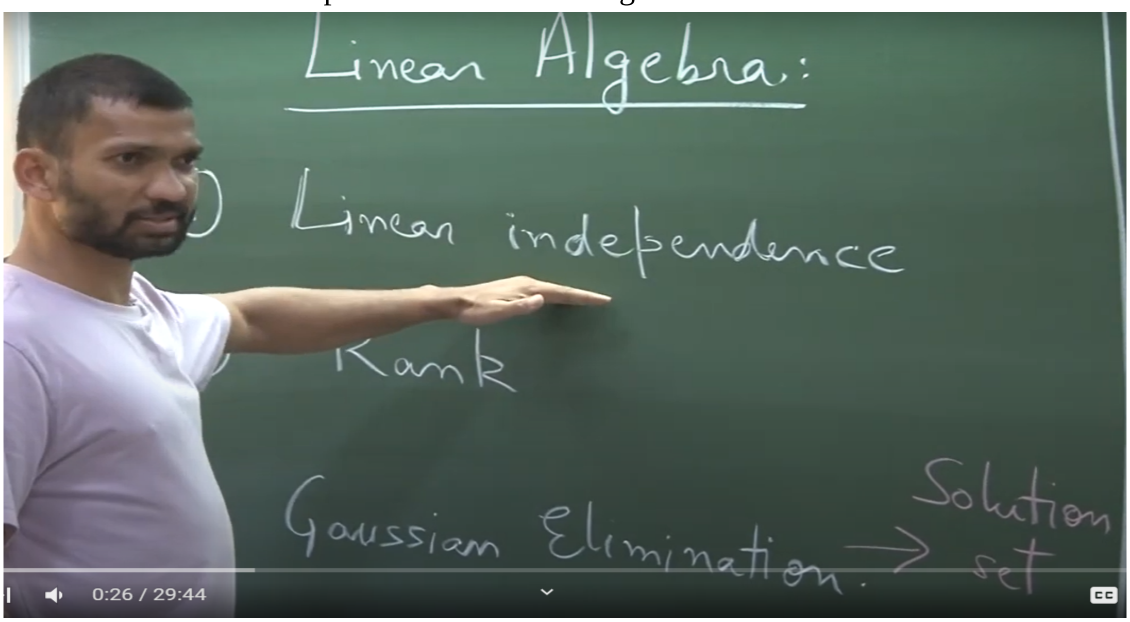
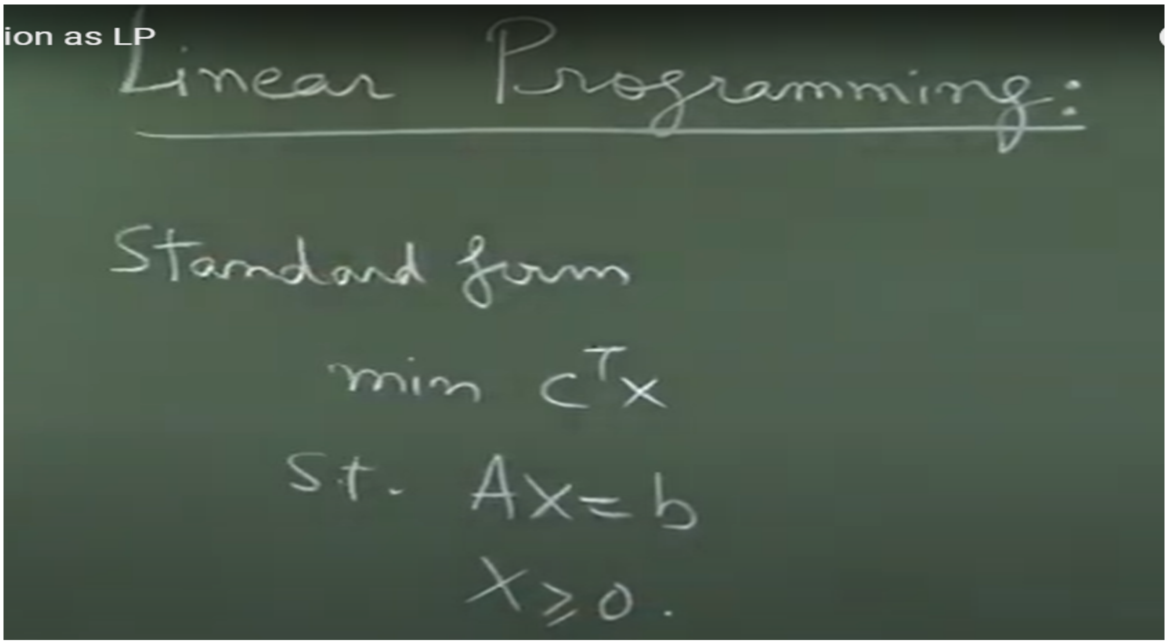
Lecture – 08
Resource Allocation as LP

Hello, welcome to the lecture on linear programming. We will start by recapping what we had done before. So, can you remind me, we were looking at linear algebra right and the important things which you want to remember is what linear independence is. We are given a bunch of vectors in a vector space, what is linear dependence, what is linear independence and all of you know it right that is fine. And then once we have this concept then we can talk about columns of a matrix as vectors and then we can consider their rank which is the number of linearly independent columns or the dimension of the span of those columns or the dimension of the span of the rows multiple ways we can say what is the rank of a matrix right. And then the third thing which kind of allows us to calculate or compute these things was Gaussian elimination.



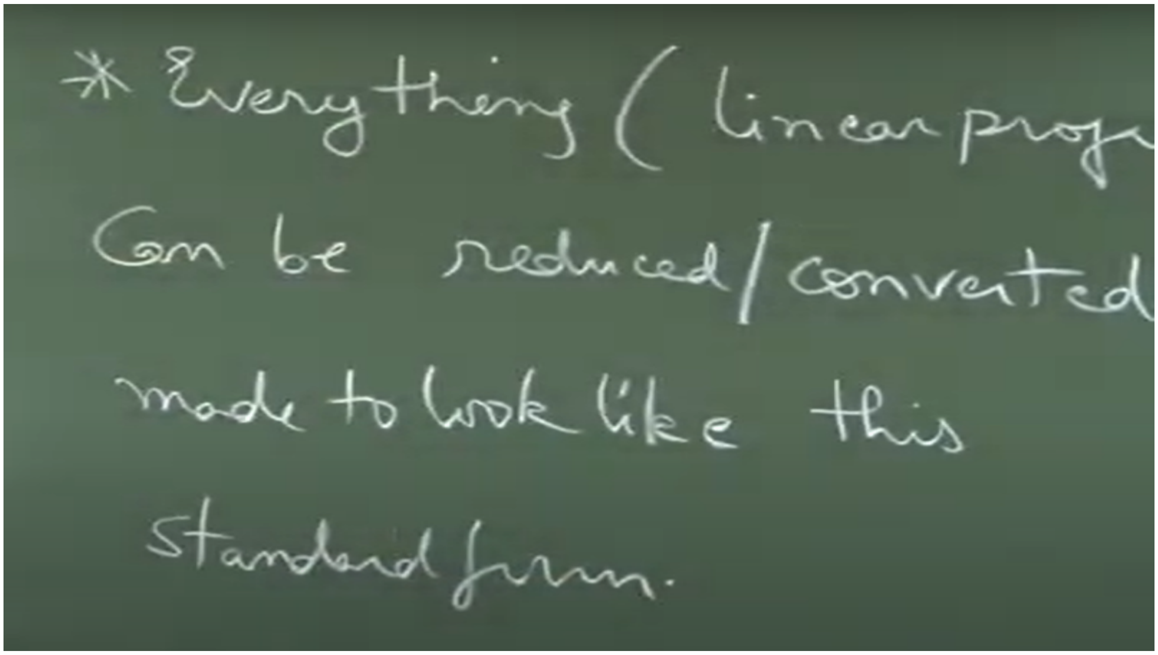
And I think this is this is a nice thing to know in general not just for this course because this is a very very strong tool it actually tells you how your solutions of Ax equal to b or Ax equal to 0 looks like right. So, what are the possibilities for Ax equal to b? There can be no solution when will that be the case when b is not in the column space or span of the columns of a right. Otherwise there could be a unique solution and unique solution will happen if rank is full rank is full rank is n. That means every vector can be uniquely represented as a linear combination of my column vectors right.

And then if the dimension span is less, but b is a vector then it turns out that I will have a large enough space which is going to be the solution. And that can actually be described by Gaussian elimination some of the variables will be free some of them others can be written down in terms of the free variables right. So, you think of how the space will look like? Can you tell me let us say in two dimensions let us say I have one degree of freedom then my solution space will be a point square line it would be a line right it would be a one dimensional space that is what I want to emphasize right it is not a set like a bounded set it is going to be a space ok good. Now I want to give you a slightly more detailed introduction of linear programming before we go further. We will kind of see that this linear algebra which we have seen is not good enough to give us a solution.

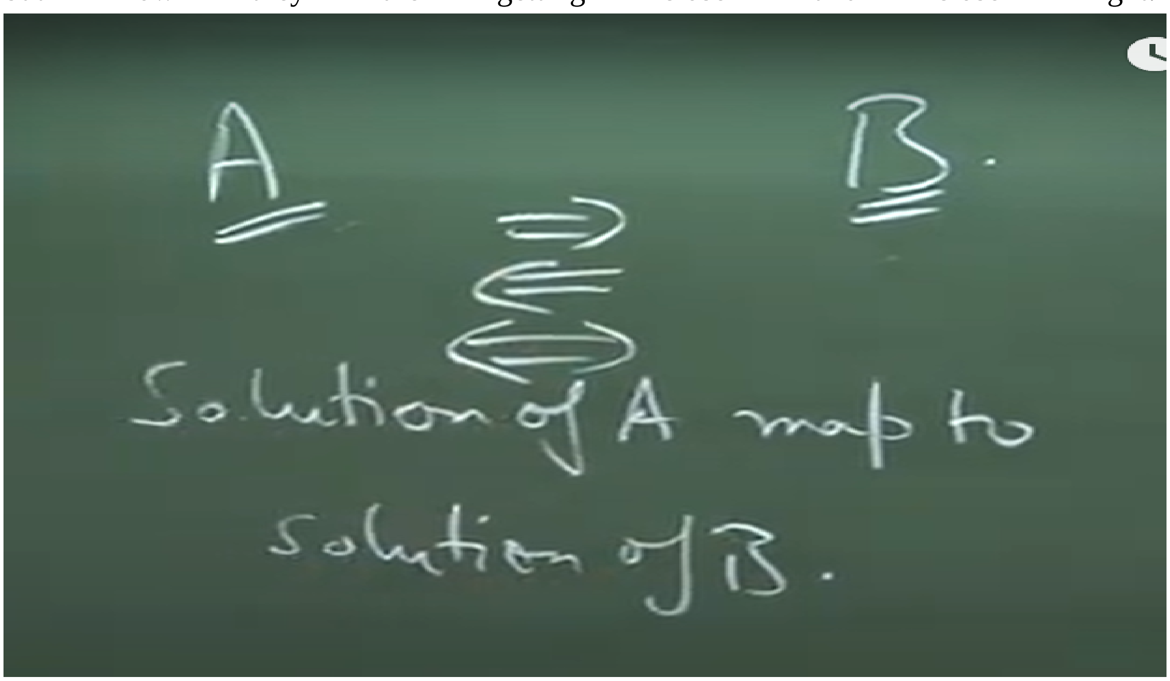


We already we need to know this, but this is not good enough for solving linear programs ok. But before that let us get acquainted with linear programming ok. Remember I told you the standard form is such that equality constraints and x greater than equal to 0 right. I told this and you accepted it, but now since you accepted I will ask a question what does it mean that this is the standard form. And now I need a mathematical definition right not the English definition of standard that I am familiar with.

So, I said standard form of linear program you said wonderful it is a standard form, but what did you get out of it? Everything can be reduced in terms. Everything you mean to say can be reduced depending on your preference correct. What does this reduced? You have two set of linear programs. Linear program A this is an optimization problem B this is another optimization problem. What does it mean that A can be reduced to B? Now see it is great we that is how mathematics works right you have some problem you formalize it.

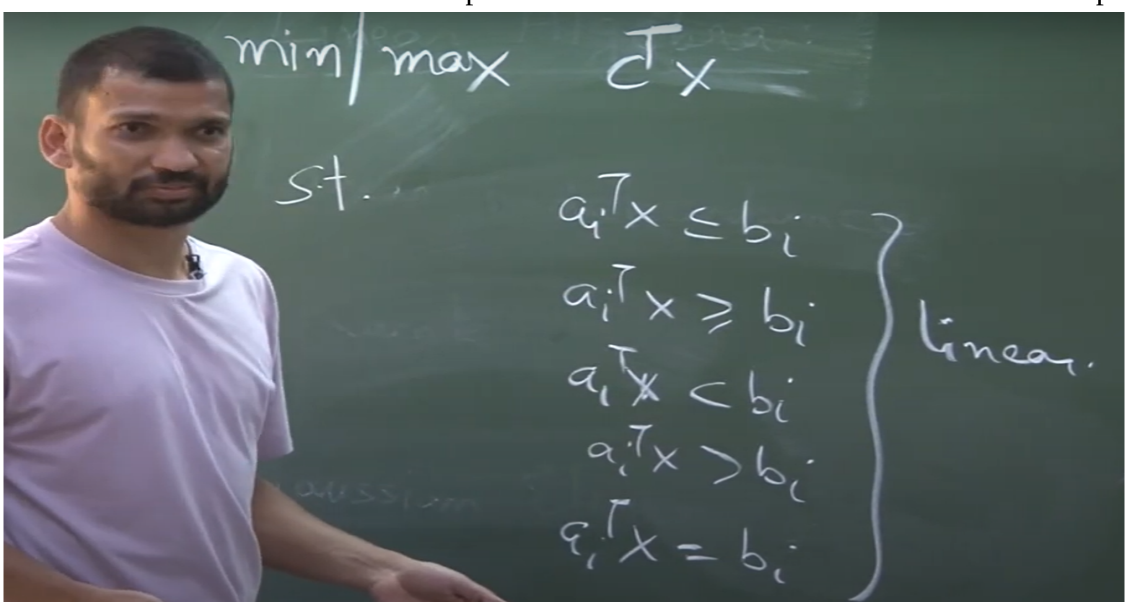


And then the next step is oh I can probably solve it like this then you have some intuition behind it. So, this is the intuition behind it, but then you need to formalize it because we are mathematicians right we are not biologist chemist or whoever you want to you know insult right. So since we are mathematician we need to formally make sense of this. So, it has to be something like solution of A map to solution of B and once again you are creating new terms and I will keep asking what map 2 means, but then we will describe, but now they are getting closer and closer right.



Map seems much more Mathematical as compared to there is a function or there is a process by which I can take a solution of A and convert it to solution of B. This was enough it was just a problem, but since this is an optimization problem we have to take care of the is it enough that solution of A is solution of B. There is a process let us say or a function, but is that enough? So, that will depend on whether you want the arrow to be this way or this way, but A also as an objective function. So, the solution of A should map to the solution of B and right because why are we doing all this? The reason is that instead of solving A we will just solve B. So, somehow the idea is the optimal solutions of A should map to optimal solutions of B and vice versa if you want to say that they are equivalent .

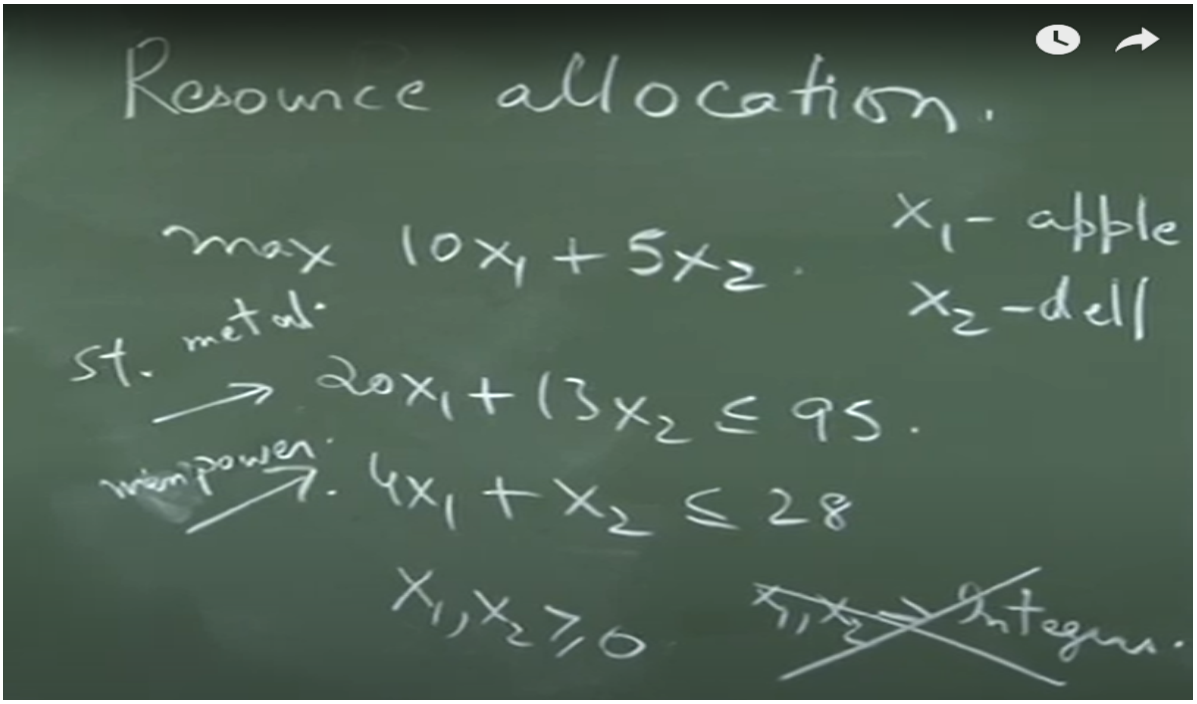
So, that is the plan that is what we want to kind of formally define and show that this is our standard form, but before I do this I need to talk about what could linear programming be right. So, linear programming is linear constraints and linear objective function right. So, it is min slash max C transpose x such that all these kind of constraints could have been possible all of them. So, they seem linear right and remember x greater than equal to 0 I can represent it as this right. So, these are the kind of possible linear constraints I can put and let us take some examples.



So, yes the first example once again would be resource allocation and now I write the linear program and you interpret it for me. So, let us say I have a linear program like this and x1 and x2 are the number of Dell computers and apple computers you can make right. So, what do you want x1 to be Dell computers or apple computers? 10 is higher than 5 right. So, any sane person will say that apple is priced higher than Dell right. So, this is the start x1 is the number of apple computers you can make x2 is the number of Dell computers you can make this is the profit you can make right.

And then how do you want to interpret this your interpretation let us see. So, this is not math this is just real world knowledge. So, what do you want let us say this constraint to be.

Sorry, which raw material? So, you have like 20 kgs of metal are required for apple and 13 kgs of metal are required for Dell and then you have total 95 kgs of metal. What about this? .



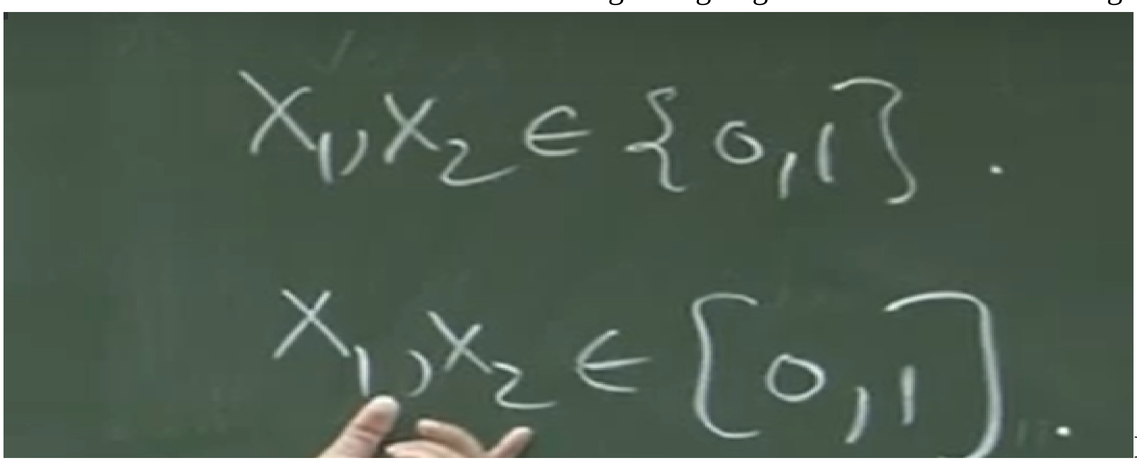
Great. So, I see that some people are not fan of apple computers here and given such problem they will only I have x2 greater than 0 and x1 equal to 0, but let us say some of us are business minded. So, fancy word manpower for child livers. So, then it requires 4 human beings because 1 for making it 3 for designing the logo on the top and then 1 for Dell and then we you know we have 28 people in total to work for us. So, if this is the case this is basically the resource allocation problem right.

. How is it different from what you saw in the first class mathematically? Remember what was the first that was the beer you were supposed to make right like you had 2 kind of dark and light. Can you think of this you know making apple computers and Dell computers not this linear program. How is it kind of different from that problem which is going to make lot of difference as a linear program? So, here I would want x1 and x2 to be variable. So, in a sense the real life problem is not a linear programming problem. So, I am not putting this constraint.

So, that is why this is almost an approximate version of the problem I wanted to solve. And the comment I want to make here is that this is not an easy constraint to put. This makes our life very difficult. There are integer linear programs, there are just integer programs and there are lots of heuristics, but we do not have any nice way to solve integer programs. And actually many of the very difficult problems as a computer scientist would call them NP hard problems can be formulated as integer programming problems.

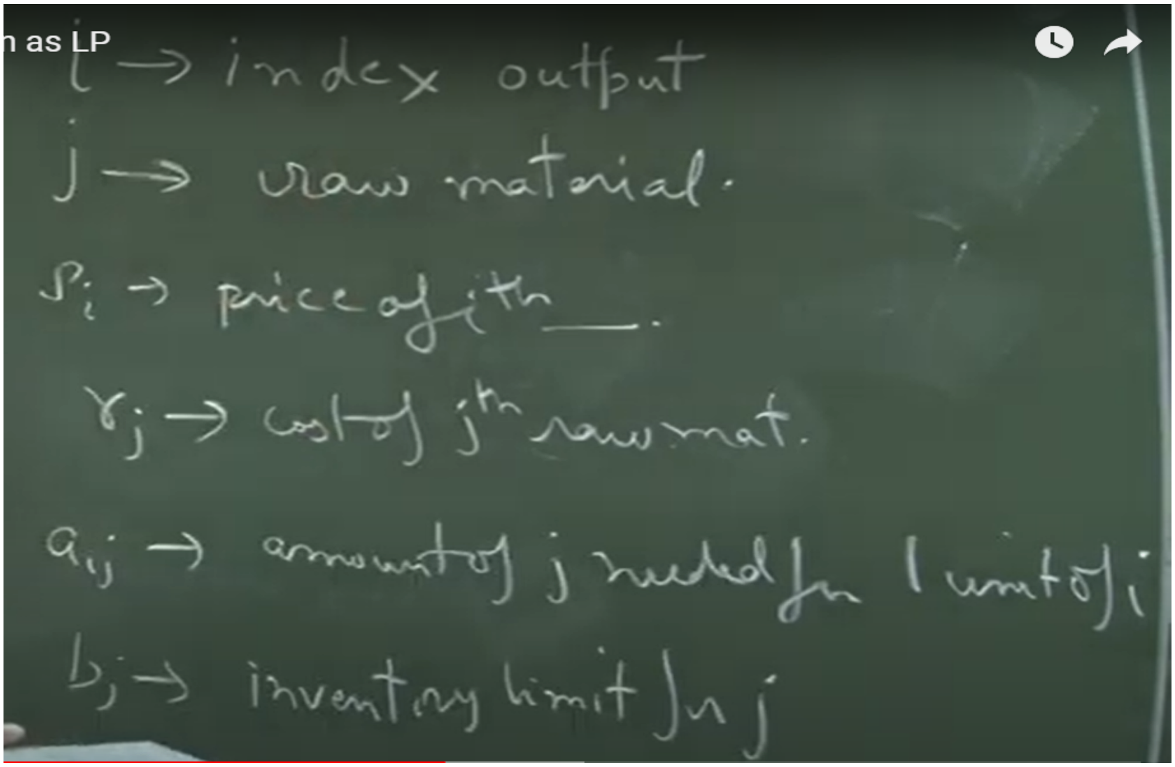
Ok, so, this is the first time which I am telling you I will probably tell you more anytime you have this kind of constraint in your optimization problem kind of in trouble. We will see in many cases we can still find an approximate solution. This would be one of the applications I will show to you, but in general this makes life difficult. So, what you are saying is that you solve this program your x1 would be let us say 13.5, x2 would be 12.1 and you will say that 13 and 12 is the answer, but what if 14 and 11 was the answer. What you have shown is that 13.5 and 12.1 you get the best solution of this, but it has not shown that the floor is the best possible. So, in that case your point would be you would say that oh it is at least approximately correct.

So, whatever is my optimal you are not far away from it. And this will probably work in this case, but many a times you will have such program and x1, x2 will just have a choice to be either 0 or 1. And then rounding is going to make a lot of right.



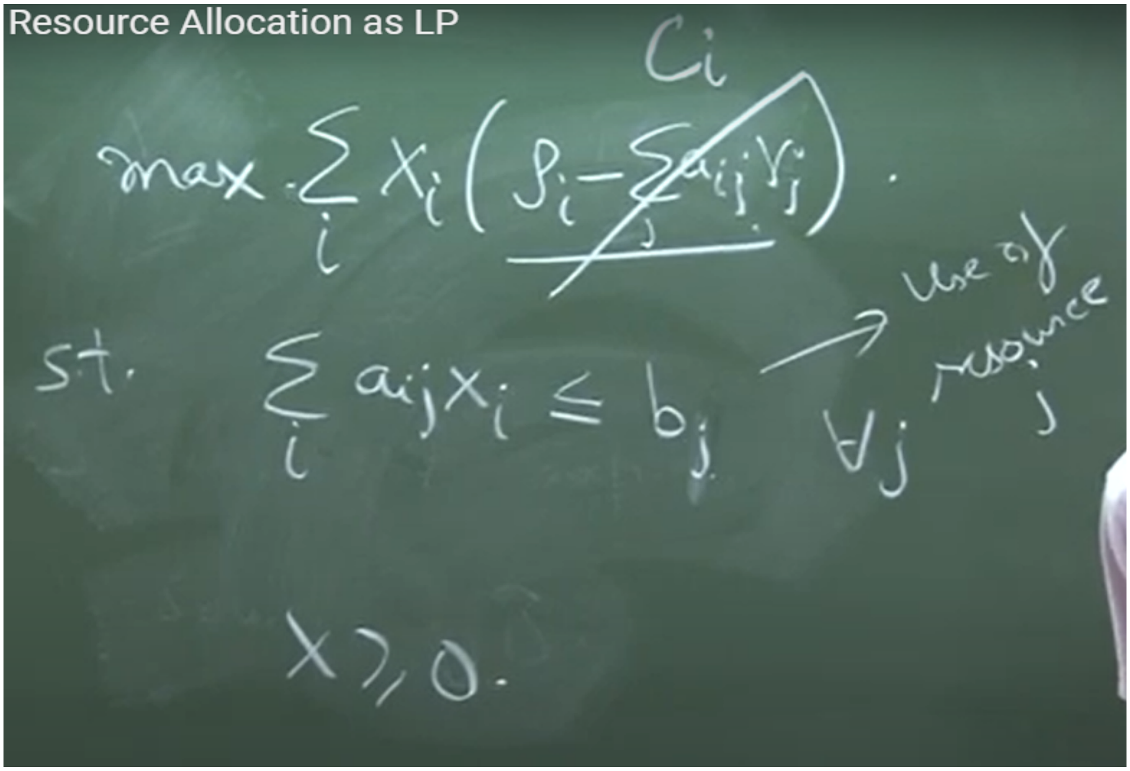
If your only choice is that you know think of this constraint as compared to this constraint. And suddenly rounding does not seem like a great option and most of these NP-hard problems I was talking about you have this kind of a constraint.

And then does this floor or ceiling or what we call sometimes rounding work or not that is a question in itself. So, for now we can assume that we can make 13.5 apple computers or 12.1 Dell computers ok and let us generalize this. What do I mean by generalizing this? You can have so this is just to take more and more examples of linear program that is why I am generalizing this.



I can say that I have this rho i as the market cost of yeah. So, let us say this will index my output. So, like in this case I will go for from 1 and 2, 1 is apple, 2 is Dell and j will be the raw material right will index the raw material. So, then rho i is the price of ith output, gamma j is the cost of jth raw material, a ij is amount of j needed for 1 unit of i and b j is the inventory limit for j right. Can you write the linear program now for the resource allocation problem. Min or max. It is not exactly the resource allocation problem which I had done before it has one extra thing right.

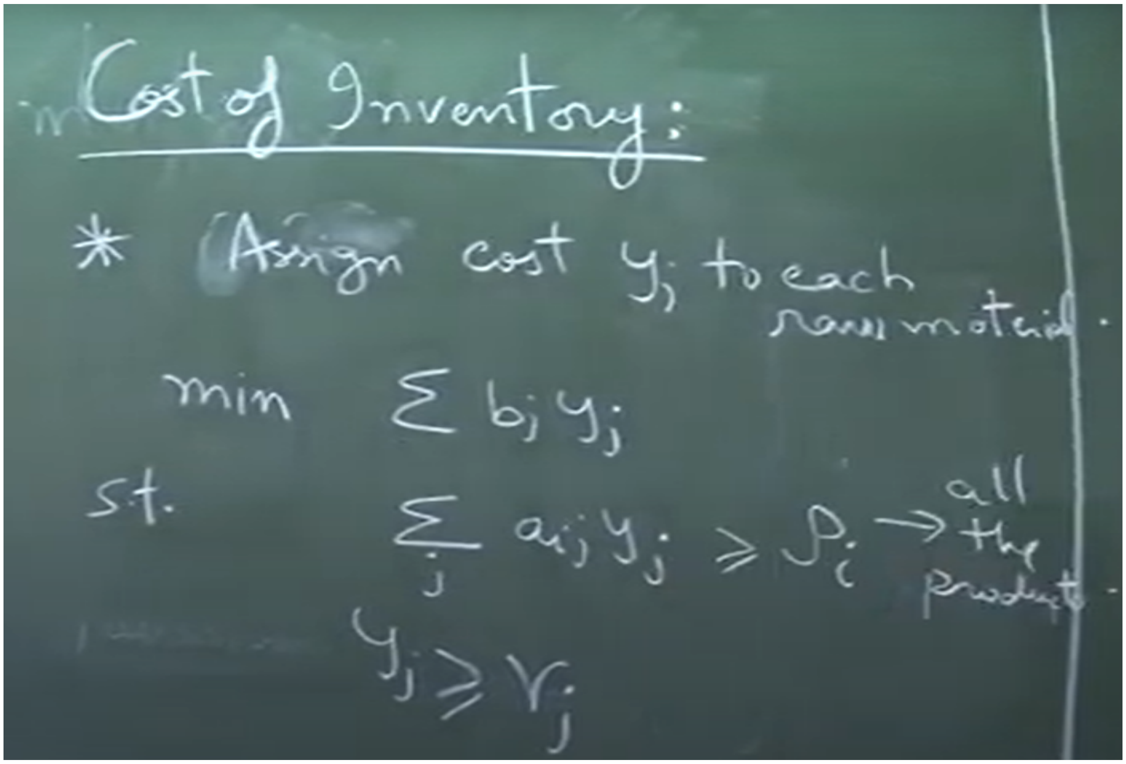
So, take care of it is not hard, but yeah what is extra? Yes . So, this is saying use of resource j. So, what will come here? Yeah that is rho i for me summation over.



Summation over. Correct. Right this is completely fine, and now this will be the first example of converting into an equivalent program which is the simplest thing I will just say this is Ci sounds good. Now you have the same data and it can be interpreted in a totally different way. What is the problem now? Cost of inventory. What do I mean by that? Suppose this information is given to you ok and as the manager of the inventory you want to assign a price to whatever you have and you want to minimize the total price ok.

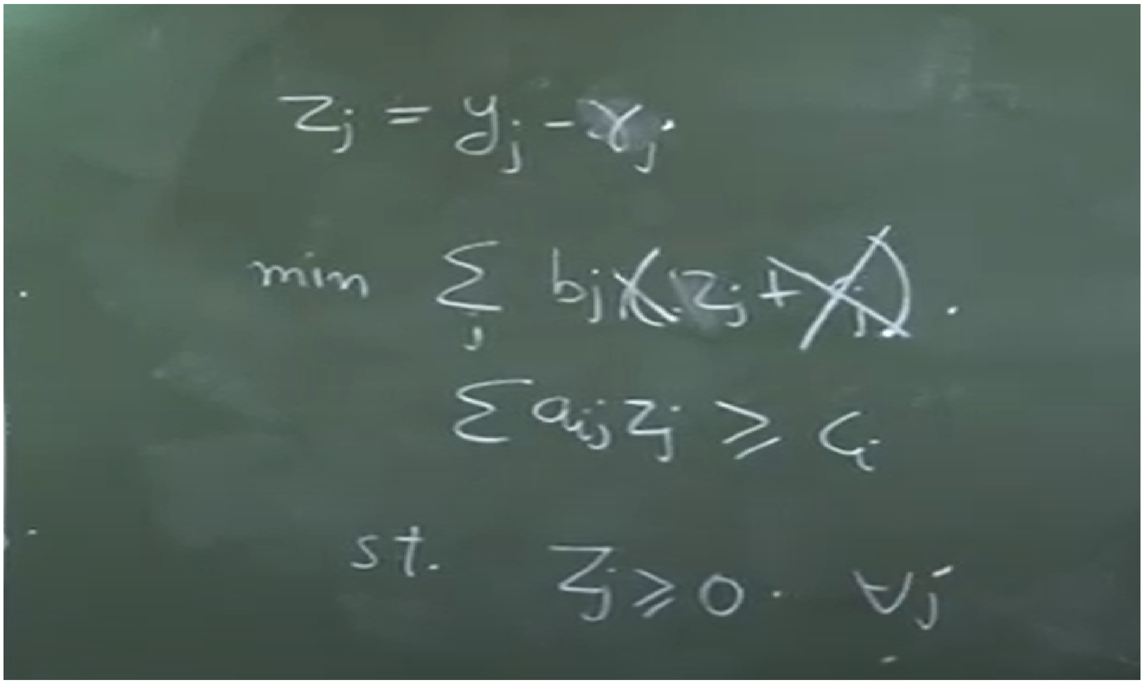
But if you just want to minimize it you will say everything is 0. But obviously if you make the price to be too small you should be willing to sell it in the market at that price ok. You understand? And then the competitor should not have any advantage by buying from you instead of the market. Clear? The minimizing the cost of inventory think of it as you are the manager and you want to say that for the other way to say it is for whatever production you want to do this is the minimum cost I am keeping this is the minimum inventory cost I am keeping you want to justify to your boss why do you have this much inventory. So you are saying that the its expected cost is actually the minimum that is one way to think about it.

So now if this is the case can you write the linear program corresponding to this problem. So again what do you want to do objective is to assign cost y j to each raw material right and when you do it what will be the cost of your inventory? If y j is the cost of each raw material of one unit you want to minimize sorry no there is no a i j here what is the cost of inventory if one unit the cost you are assigning as y j how much j do you have b j right bj yj ok. Such that can your cost be less than gamma j no if your cost is less than gamma j then competitor will buy directly the raw material from you instead of from the market. This is not a very interesting constraint what is the other interesting constraint what can competitor do they can buy the raw material from you and build the product i

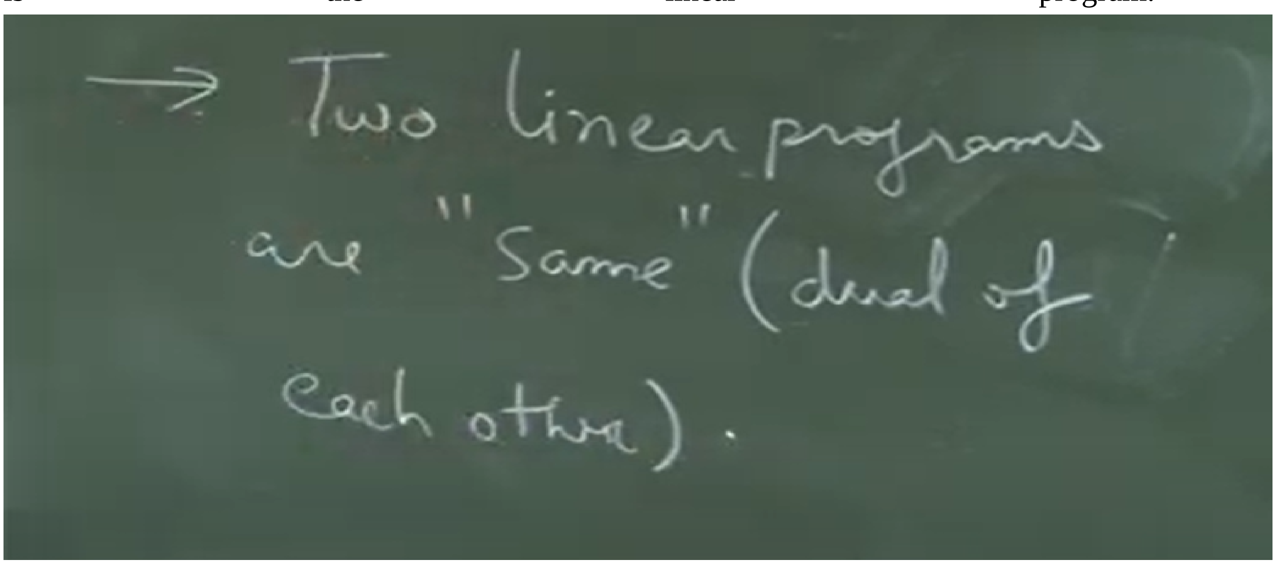


right. So what should be the constraint corresponding to that we should not make sure that we are giving added advantage to the competitor.

What should be the rate of output sorry yes good. So what is the mathematics what is the equation? Summation over j and this is therefore all the products. Ok And, again we can do a slightly non trivial equivalence here we can say that z j is y j minus gamma j. Remember C i was rho i minus the cost of making product i right.



So this was in some sense the profit ok. And even though I have not formally defined what reduced means what equivalence means you are still fine with the substitution. And actually should also be fine if I just cancel this term. Because optimizing this is same as optimizing this program with some constant attached to it. Clear? So one step is yeah this inventory value of inventory problem is not that natural. But if you take the leap then this is the linear program.



And so that that should be clear to you but this should be absolutely clear how this and this program are equivalent. In the sense at this point I would just say that if I wanted to solve this instead I will just solve this. And the advantage is that this is much closer to the standard form than before right. Gamma j is the market price and y j is the actual price which you suppose for you what is the cost of the material right. So you want to say that oh even the market it is 10 rupees per but for me its value is 12 rupees or something.

So that is why there are two different y j is the variable gamma j is the fixed market cost. Does that answer your question? And now you saw same data but two different linear programs. And to now advertise this course in few weeks you will see that it means is that the best possible profit you can make is going to be equal to the minimum possible cost of your inventory. And I am sure at this point you do not see right. This seems like two different problems but this is the in some sense the power of linear program we will see that how this optimization problem and this optimization problem are intrinsically related.

And we will mathematically be able to prove that yes the maximum profit you can make should in some sense be equal to the imagined cost of my inventory. And this will be shown but in few weeks. And again if someone asks me this is my favorite concept and I will keep highlighting it and you will see the power of duality again and again in this course.